Solving (Q)SAT Problems via Tree Decomposition and Dynamic Programming

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Joint work with

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Introduction

Tree Decomposition and Treewidth

By-product in the theory of graph minors due to Robertson and Seymour (1984); similar notions appeared even earlier (Bertelè and Brioschi, 1972; Halin, 1976).

Courcelle's Theorem (1990)
Any property of finite structures which is definable in MSO can be decided in time $O(f(k) \cdot n)$ where $n$ is the size of the structure and $k$ is its treewidth.
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... and, more recently ...

“Courcelle’s theorem […] should be regarded primarily as classification tool, whereas designing efficient dynamic programming routines on tree decompositions requires ’getting your hands dirty’ and constructing the algorithm explicitly.” (Cygan et al., 2015)
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Main Challenge

Can we turn the huge body of theoretical results on parameterized algorithms into systems that perform competitive in practice?
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Requirements
1. domain exhibits suitable instances
2. design of smart algorithms and well-engineered systems
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(Most) Competitive Arena: SAT problems
- QSAT – propositional logic with quantifiers
- #SAT – model counting
- WMC – weighted model counting
- PMC – projected model counting
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- QSAT – propositional logic with quantifiers (PSPACE-complete)
- #SAT – model counting (#P-complete)
- WMC – weighted model counting (#P-complete)
- PMC – projected model counting (#NP-complete)
Outline

1. Treewidth, Tree Decompositions and Dynamic Programming
2. Solving SAT via TD + DP
3. **dynQBF**: a QSAT solver based on BDDs
4. **gpusat**: a #SAT solver that runs on the GPU
5. Some new results for PMC
6. Conclusion and Outlook
Treewidth and Tree Decompositions
Treewidth

- Some graphs are more “tree-like” than others
- Treewidth measures “tree-likeness”:
  - Trees have treewidth 1
  - The higher the treewidth, the more complex the graph
- Often “easy on trees” implies “easy on tree-like graphs”
  - Many problems are fixed-parameter tractable w.r.t. treewidth $k$, i.e. can be decided in $O(2^k \cdot n)$
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  - That is, they become easy when putting a bound on the treewidth
- It works for many hard problems
- Real-world applications often have small treewidth
Example: Treewidth 3.
Treewidth (ctd.)


Treewidth is defined in terms of tree decompositions.
Definition

A tree decomposition is a tree obtained from an arbitrary graph s.t.

1. each vertex must occur in some \textit{bag}
2. for each edge, there is a bag containing both endpoints
3. tree is \textit{connected}: if \( v \) appears in bags of nodes \( t_0 \) and \( t_1 \), then \( v \) is also in the bag of each node on the path between \( t_0 \) and \( t_1 \)
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Tree Decompositions

Definition

- **width** of a decomposition: size of largest bag minus 1
- **treewidth** of an instance: minimum width over all its TDs

Finding Tree Decompositions

- Constructing a tree decomposition of minimal width intractable
  - but solvable in time $2^{O(w^3)} \cdot |V|$ [Bodlaender, 1996]
- In Practice:
  - generate a tree decomposition of reasonably low, but not necessarily minimal width using heuristics (e.g. MinFill)

- **htd**: https://github.com/mabseher/htd

Given a tree decomposition of input instance $\mathcal{I}$ of width $w$, one can solve the problem via **dynamic programming** in time $f(w) \cdot O(|\mathcal{I}|^c)$ for some computable function $f$ and constant $c$. 
Tree Decompositions

Dynamic Programming - Overall Schema
Tree Decompositions

Dynamic Programming - Overall Schema

1. Decompose graph

- Decompose graph
  - a
  - b
  - c
  - y
  - x

- b, c
- b, c
- b, x, c
- b, x, a
- b, c
- b, c
- b, c
- b, x, a
- b, x, c
- b, c, y
- b, c
Tree Decompositions

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2. Solve subproblems
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Dynamic Programming - Overall Schema

1. Decompose graph
2. Solve subproblems
3. Output solutions
Let’s Get Things Started: Solving SAT via TD and DP
DP algorithm for SAT [Samer & Szeider, 2010]

\[ \varphi = (\neg a \vee b \vee x) \land (a \vee b) \land (c \vee \neg x) \land (b \vee \neg c) \land (\neg b \vee \neg c \vee \neg y) \]
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$\text{Mod}(\varphi) = \{ \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, x\}, \{a, b, c, x\}, \{b, y\}, \{a, b, y\} \}$
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1. Decompose graph
2. Solve subproblems

```
\begin{array}{c|c|c}
  b & c & \text{Count} \\
  \hline
  1 & 0 & 1 \\
  1 & 1 & 1 \\
\end{array}
```

```
\begin{array}{c|c|c|c|c|c}
  b & x & c & \text{Count} \\
  \hline
  1 & 0 & 0 & 1 \\
  1 & 0 & 1 & 0 \\
  1 & 1 & 1 & 0 \\
\end{array}
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\begin{array}{c|c|c|c|c|c}
  b & x & a & \text{Count} \\
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\[ \begin{array}{ccc}
 b & c \\
 0 & 0 & 1 \\
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1. Decompose graph
2. Solve subproblems
3. Counting solutions
Put Things on Track: QSAT via TD and DP
QSAT

- Extension of propositional logic
- Compactly encode computationally hard problems (e.g., verification, planning, synthesis, ...)
- Satisfiability problem (QSAT) is PSPACE-complete
- Various techniques: search (DPLL, CDCL), expansion, resolution, CEGAR
- Annual QBF Competition (47 systems submitted in 2017)
Dynamic Programming for QSAT

\[ \Phi = \forall x, y \exists a, b, c \ \varphi, \text{ where} \]
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$\Phi = \forall x, y \exists a, b, c \varphi$, where

$\varphi = (\neg a \lor b \lor x) \land (a \lor b) \land (c \lor \neg x) \land (b \lor \neg c) \land (\neg b \lor \neg c \lor \neg y)$

Recall,

$$\text{Mod}(\varphi) = \{ \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, x\}, \{a, b, c, x\}, \{b, y\}, \{a, b, y\} \}$$

Hence, $\Phi$ invalid:

$$\varphi[x = 1, y = 1] \equiv (a \lor b) \land c \land (b \lor \neg c) \land (\neg b \lor \neg c) \equiv \bot$$
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Data Structure

Binary Decision Diagrams (BDDs)

- BDDs store Boolean functions as rooted DAG
- Reduced Ordered BDDs
  - Usually space-efficient (given a good variable ordering)
  - Canonical (equivalent formulae represented by same BDD)

 Nested Set of Formulae (NSF)

Innermost elements are BDDs and store parts of the QBF matrix

Example NSF:

```
{{{⊤}, {⊥}}, {{¬a ∨ b}, {⊥}}, {a ∧ b}}
```

with $k = 3$
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NSF: $\{\{\top, \bot\}\}, \{\{\neg a \lor b\}, \{\bot\}, \{a \land b\}\}$ with $k = 3$
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Run Time

Given QBF $Q_1 X_1 \ldots Q_k X_k \psi$ and a tree decomposition for $\psi$ of width $w$.

The algorithm determines the truth value of the QBF in time

$$O(2^{2^w \cdot \log_2 w \cdot (k+1)} \cdot |\psi|),$$

where the height of the tower of exponents is $k + 1$:

- the size of each BDD is at most $2^{w+1}$
- $k$ quantifier blocks
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where the height of the tower of exponents is \( k + 1 \):

- the size of each BDD is at most \( 2^{w+1} \)
- \( k \) quantifier blocks

- QSAT is fixed-parameter tractable for bounded treewidth and the number of quantifier alternations [Chen, 2004]
- QSAT is \emph{not} fixed-parameter tractable w.r.t. parameter treewidth only [Atserias and Oliva, 2014].
Towards Efficiency in Practice

Clause splitting

- Size of largest clause gives lower bound for width
- Splitting usually reduces the width (but increases the number of variables)

Dependency Schemes

- A dependency scheme $D$ is an overapproximation of full independence. (independence: reordering of quantifiers does not change satisfiability)
- If a variable is removed, we only need to split tables if a dependent variable is not yet fully processed

Feature-based tree decomposition selection

- Choose tree decomposition based on certain criteria (besides width)
- Promising: variable position, children of join nodes
Towards Efficiency in Practice

Intermediate unsatisfiability checks
- Reuse procedure for deciding the problem on NSFs obtained during bottom-up traversal
- If procedure returns $\perp$, the instance is unsatisfiable
- For $\top$, the QBF might still be unsatisfiable due to clauses that were not yet considered

Subset-based compression
- Check for subsets w.r.t. models represented by the BDDs and subsets w.r.t. nested sets
- Similar to subsumption checking [Biere, 2004]

Balance NSF and BDD size
- Delay splitting of removed variables (store them in a cache)
- Increases size of BDDs (no longer bounded by width)
- Apply heuristics to obtain optimal NSF and BDD size
The dynQBF System

System Specifics

- C++, open source
- Tree decomposition: **htd** library
- BDD management: CUDD
- Standard dependencies (optional): DepQBF

Core Features

- Deciding QSAT
- Partial certificates (outermost quantifier block)
  - Compact enumeration
  - Counting

https://github.com/gcharwat/dynqbf
Experiments: Setup

QBF solvers that participated in the 2016 QBF Evaluation

- Top-ranked in the competition
- Publicly available
- Without (explicit) tool-chained preprocessing

2016 QBF Evaluation instances (preprocessed with Bloqqer)

- 2-QBF track: 305 instances, 130 solved by Bloqqer
- PCNF track: 825 instances, 341 solved by Bloqqer

Run limitations and measurements

- Ranked by number of solved instances
- Timeout: 10 minutes; Memout: 16 GB
- Given solving time (in seconds) includes penalty of 10 minutes for non-solved instances
- Detailed analysis w.r.t. width of instances
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## Experiments: 2-QBF instances

<table>
<thead>
<tr>
<th>System</th>
<th>Solved</th>
<th>Time</th>
<th>SAT</th>
<th>UNSAT</th>
<th>Unique</th>
</tr>
</thead>
<tbody>
<tr>
<td>Qesto</td>
<td>236</td>
<td>50K</td>
<td>160</td>
<td>76</td>
<td>0</td>
</tr>
<tr>
<td>RAReQS</td>
<td>232</td>
<td>51K</td>
<td>161</td>
<td>71</td>
<td>1</td>
</tr>
<tr>
<td><strong>dynQBF</strong></td>
<td>221</td>
<td>53K</td>
<td>172</td>
<td>49</td>
<td>43</td>
</tr>
<tr>
<td>DepQBF</td>
<td>221</td>
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<tr>
<td>QSTS</td>
<td>220</td>
<td>58K</td>
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<td>58</td>
<td>2</td>
</tr>
<tr>
<td>CAQE</td>
<td>204</td>
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<td>202</td>
<td>66K</td>
<td>141</td>
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<tr>
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<td>151</td>
<td>95K</td>
<td>123</td>
<td>28</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table**: Data set: QBFEval’16 – 2-QBF track, preprocessed with Bloqger

### Uniquely solved instances
- **dynQBF**: fixpoint detection (43)
- **QSTS**: query (1), sorting networks (1)
### Experiments: 2-QBF instances

<table>
<thead>
<tr>
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<th>Time</th>
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<tr>
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</tr>
<tr>
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<td>33</td>
<td>34K</td>
</tr>
<tr>
<td>CAQE</td>
<td>28</td>
<td>36K</td>
</tr>
<tr>
<td>AReQS</td>
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<td>38K</td>
</tr>
<tr>
<td>QSTS</td>
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<td>40K</td>
</tr>
<tr>
<td>GhostQ (CEGAR)</td>
<td>9</td>
<td>47K</td>
</tr>
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<tr>
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<th>Time</th>
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<td>QSTS</td>
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<tr>
<td>Qesto</td>
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<td>CAQE</td>
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<tr>
<td>dynQBF</td>
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<tr>
<td>GhostQ (CEGAR)</td>
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Table: Data set: QBFEval’16 – 2-QBF track, 175 non-trivial instances after preprocessing
Experiments: PCNF instances

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<tr>
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<th>UNSAT</th>
<th>Unique</th>
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<td>134K</td>
<td>298</td>
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<td>144K</td>
<td>296</td>
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<td>QSTS</td>
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<tr>
<td>GhostQ (CEGAR)</td>
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<tr>
<td>dynQBF</td>
<td>494</td>
<td>203K</td>
<td>239</td>
<td>255</td>
<td>21</td>
</tr>
</tbody>
</table>

Table: Data set: QBFEval’16 – PCNF track, preprocessed with Bloqger

Uniquely solved instances
- dynQBF: fixpoint detection (11), ...
- RAReQS: dungeon/planning (3), emptyroom (3), ...
### Experiments: PCNF instances

<table>
<thead>
<tr>
<th></th>
<th>$w \leq 80$ (182 instances)</th>
<th>$w &gt; 80$ (302 instances)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>System</strong></td>
<td><strong>Solved</strong></td>
<td><strong>Time</strong></td>
</tr>
<tr>
<td>RAReQS</td>
<td>137</td>
<td>28K</td>
</tr>
<tr>
<td><strong>dynQBF</strong></td>
<td>134</td>
<td>32K</td>
</tr>
<tr>
<td>Qesto</td>
<td>129</td>
<td>34K</td>
</tr>
<tr>
<td>DepQBF</td>
<td>124</td>
<td>36K</td>
</tr>
<tr>
<td>QSTS</td>
<td>123</td>
<td>37K</td>
</tr>
<tr>
<td>CAQE</td>
<td>119</td>
<td>40K</td>
</tr>
<tr>
<td>GhostQ (CEGAR)</td>
<td>118</td>
<td>41K</td>
</tr>
</tbody>
</table>

**Table:** Data set: QBFEval’16 – PCNF track, 484 non-trivial instances after preprocessing
QSAT solving — Summary

A novel expansion-based approach for QBF solving

- Motivated by fixed-parameter tractability results
- Explicitly takes QBF structure into account
- Various optimizations towards feasibility in practice

QBF solver dynQBF

- Competitive on instances up to treewidth 80, and 2-QBF instances
- Many uniquely solved instances
- 2-QBF track: Ranked 8 (out of 29 participants) in QBFEval'17
- PCNF track: Ranked 13 (out of 30 participants) in QBFEval'17
QSAT solving — Summary

A novel expansion-based approach for QBF solving

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Massive Parallelisation: #SAT on the GPU
Given propositional formula $\varphi$, #SAT asks for the number of models

$$|\{M \subseteq \text{var}(\varphi) \mid M \models \varphi\}|$$

Applications in several domains, e.g.:
- Bayesian reasoning [Sang et al., 05]
- Infrastructure reliability [Meel et al., 17]

Traversal of entire search space required

Systems relying on different approaches exist
- Cachet, sharpSAT;
- ApproxMC, sts;
- countAntom;
- c2d, d4;
- ...
Recall: DP for #SAT

\[ \varphi = (\neg a \lor b \lor x) \land (a \lor b) \land (c \lor \neg x) \land (b \lor \neg c) \land (\neg b \lor \neg c \lor \neg y) \]

1. Decompose graph
2. Solve subproblems
3. Counting solutions

\begin{array}{ccc|c}
 b & c & y & \# \\
 0 & 0 & 0 & 1 \\
 0 & 0 & 1 & 1 \\
 1 & 0 & 0 & 1 \\
 1 & 0 & 1 & 1 \\
 1 & 1 & 0 & 1 \\
\end{array}

\begin{array}{cc|c}
 b & c & \# \\
 1 & 0 & 2 \\
 1 & 1 & 4 \\
\end{array}

\begin{array}{ccc|c}
 b & x & c & \# \\
 1 & 0 & 0 & 2 \\
 1 & 0 & 1 & 2 \\
 1 & 1 & 1 & 2 \\
\end{array}

\begin{array}{ccc|c}
 b & x & a & \# \\
 1 & 0 & 0 & 1 \\
 1 & 0 & 1 & 1 \\
 1 & 1 & 0 & 1 \\
 1 & 1 & 1 & 1 \\
 0 & 1 & 1 & 1 \\
\end{array}

30 / 45
DP on the GPU

How to parallelize DP?

1. Compute tables in parallel
   - No massive parallelization due to dependencies of child nodes

2. Compute rows in parallel
   - Since computation of specific rows is independent of other rows, this allows for massive parallelization
   - Used here!
DP on the GPU

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DP on the GPU

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⇒ Used here!
The gpuSAT System

System Specifics

- C++11
- Tree decomposition: htd library
- OpenCL
  - Vendor independent
  - C99
  - SIMT

Core Features

- Each potential row of a table runs in one thread of the GPU
- Tables need to be split for $w > 27$
- Precision can be toggled

https://github.com/Buddit/GPUSAT
Experiments

- Timeout: 900s
- Memory: 8GB
- 16 #SAT Solvers
- 3 WMC Solvers
- 5 random TDs per instance
Width Distribution of Instances

- Instances: 1091 WMC and 1456 #SAT
Instances: 1091 WMC and 1456 #SAT
- 54% ≤ width 30; 70% ≤ width 40
- Decomposition time below a second (max 2.5s)
#SAT solving — Summary

Towards DP on the GPU

- Distribute computation of tables among different computation units
- All threads have same instructions but start from different data
- Each row forms a potential pixel with the corresponding sum as its value

Prototype System **gpuSAT**

- Competitive up to width 30; solved instances up to width 45
- High Precision
- Easily extendible to WMC (also supported by gpuSAT)
Side Result: DP for PMC
Given propositional formula $\varphi$ and $P \subseteq \text{var}(\varphi)$, $\text{PMC}_P(\varphi)$ asks for the number of $P$-projected models

$$|\{M \cap P \mid M \subseteq \text{var}(\varphi), M \models \varphi\}|$$

**Extremal Cases**

- $P = \emptyset$ amounts to SAT
- $P = \text{var}(\varphi)$ amounts to #SAT

However, the problem is in general harder than #SAT ($\# \cdot \text{NP-complete}$ vs. #P-complete)
Towards Dynamic Programming for PMC

Theorem

Unless ETH fails, there is no algorithm for PMC running in time $2^{o(tw)} \cdot |\varphi|^c$. 

Proof. Unless ETH fails, $2$-QSAT cannot be solved [Lampis & Mitsou, 2017] in time $2^{o(tw)} \cdot |\varphi|^c$. Solve $\forall X. \exists Y. \varphi$ by checking whether PMC $X(\varphi) = 2^{|X|}$. 

39 / 45
Towards Dynamic Programming for PMC

**Theorem**

*Unless ETH fails, there is no algorithm for PMC running in time* \(2^{2^{o(tw)}} \cdot |\varphi|^c\).

**Proof.**

- Unless ETH fails, 2-QSAT can not be solved [Lampis & Mitsou, 2017] in time \(2^{2^{o(tw)}} \cdot |\varphi|^c\).
Towards Dynamic Programming for PMC

**Theorem**

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**Proof.**

- Unless ETH fails, 2-QSAT can not be solved [Lampis & Mitsou, 2017] in time $2^{2^o(tw)} \cdot |\varphi|^c$.

- Solve $\forall X. \exists Y. \varphi$ by checking whether $\text{PMC}_X(\varphi) = 2^{|X|}$. 
Dynamic Programming for PMC

Can we find a DP that (asymptotically) matches this lower bound.
Dynamic Programming for PMC

Can we find a DP that (asymptotically) matches this lower bound.
Yes ...

PMC in three Steps

1. Run algorithm for SAT
Dynamic Programming for PMC

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Yes ...

**PMC in three Steps**

1. Run algorithm for SAT
2. Purge non-solutions
Dynamic Programming for PMC

Can we find a DP that (asymptotically) matches this lower bound.

Yes ...

PMC in three Steps

1. Run algorithm for SAT
2. Purge non-solutions
3. Add projection counters for sets of rows
Dynamic Programming for PMC

\[ P = \{ x, y \} \]
\[ \varphi = (\neg a \lor b \lor x) \land (a \lor b) \land (c \lor \neg x) \land (b \lor \neg c) \land (\neg b \lor \neg c \lor \neg y) \]

\[ \text{Mod}(\varphi) = \{ \{ b \}, \{ a, b \}, \{ b, c \}, \{ a, b, c \}, \{ b, c, x \}, \{ a, b, c, x \}, \{ b, y \}, \{ a, b, y \} \} \]

\[ \text{PMod}_P(\varphi) = \{ \emptyset, \{ x \}, \{ y \} \} \]
\[ \text{PMC}_P(\varphi) = |\text{PMod}_P(\varphi)| = 3 \]
Dynamic Programming for PMC

\[ P = \{ x, y \} \]

\[ \varphi = (\neg a \lor b \lor x) \land (a \lor b) \land (c \lor \neg x) \land (b \lor \neg c) \land (\neg b \lor \neg c \lor \neg y) \]

1. DP for SAT
Dynamic Programming for PMC

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1. DP for SAT
2. Purge non-solutions
3. Solve PMC via \( \mathcal{P} \)
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![Dynamic Programming Diagram]

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<td>1</td>
</tr>
</tbody>
</table>
Dynamic Programming for PMC

\[ P = \{ x, y \} \]

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\[ P = \{x, y\} \]
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1. DP for SAT
2. Purge non-solutions
3. Solve PMC via \( P \)
PMC – Summary

Contribution

- Algorithm for PMC using treewidth with worst-case runtime $2^{2^{O(tw)}} \cdot |\varphi| \cdot \gamma(|\varphi|)$
  - Relies on inclusion/exclusion principle
  - In the worst-case we require $2^k$ counters for $k$ rows in a table
- Unless ETH fails, there is no algorithm for PMC running in time $2^{2^{o(tw)}} \cdot |\varphi|^c$. 
Conclusion and Outlook
Conclusion and Outlook

Lessons learned:

- DP on TD efficient in practice (at least if width not too high)
- However, engineering efforts required
  - smart data-structures help a lot (BDDs)
  - (simple) DP allows for massive parallelisation
  - shape of TD crucial
- How to tame high treewidth / performance bottlenecks?
  - width-reducing preprocessing
  - abstraction / hybrid solving
  - relaxed decompositions
  - other type of TD heuristics needed
  - [Maniu, Senellart, Jog; 2017]
  - [Jégou, Kanso, Terrioux; 2016]
  - lazy materialization of tables in DP
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Thanks for your attention ;)

45 / 45