ON THE ARITHMETIC INEXPRESSIVENESS OF TERM REWRITING SYSTEMS

Sergey G. Vorobyov

Program Systems Institute of the USSR Academy of Sciences
Pereslavl-Zalessky, 152140, Soviet Union

Abstract

Unquantified Presburger arithmetic is proved to be non-axiomatizable by means of a canonical (i.e. noetherian and confluent) term rewriting system, if boolean connectives are not allowed in the left-hand sides of the rewrite rules. It is conjectured that the same is true if the number of boolean connectives in the left-hand sides of the rules is uniformly bounded by an arbitrary natural number.

1. Introduction

Term rewriting systems (TRSs for short), i.e sets of oriented equations, constitute an interesting model of correct nondeterministic computations with various applications. Uniquely terminating (or canonical) TRSs provide an effective search space-free decision mechanism for equational theories based on the normal form reduction [5, 2]. Different generalizations of TRSs such as equational TRSs [3] and conditional TRSs [4, 8] have been proposed. But do these generalizations suffice?

To make this precise we propound the following natural question about TRSs. What is the "absolute" expressive power of canonical TRSs (usual, equational or conditional)? More formally: does there exist a "well-structured" canonical system axiomatizing, say, the unquantified Presburger arithmetic or the discrete linear order? Arithmetic lies in the very basis of almost all formal systems of reasoning, so taking it as an "absolute standard" is valid. Well-structuredness may be thought of in different ways: e.g. as finiteness of TRS, boundedness of symbol occurences in the left-hand sides of the rewrite rules, easy feasibility of reductions, various complexity constraints, and so forth. Hence, the hierarchy of questions arises. Studying it may yield precise results on comparing the expressive power of different classes of TRSs.

We prove the first negative result of that sort. A TRS is called context-free iff boolean connectives are prohibited in the left-hand sides of its rules. While reducing with the context-free TRS it's unnecessary to take associative, commutative and distributive properties of logical connectives (which complicate matters further) into account. So, such TRSs may be thought of as "best possible". It turns out, however, that the unquantified Presburger arithmetic cannot be axiomatized by a canonical context-free TRS, usual or conditional, finite or infinite (in fact, this is true for any unquantified theory which is strong enough to argue about the validity of the "less or equal" relation between the numerals). We put forward the conjecture that the similar negative result remains true if the number of the boolean connectives occurences permitted in the left-hand sides of the rewrite rules is bounded.

Hence, arithmetic cannot be easily covered by TRSs (remind that the quantifier-free Presburger arithmetic may be decided by the integer simplex-method or by the SUP-INF method [6]). In [7] we propose a method of building-in arithmetic decision procedures into conditional context-free TRSs, which combines reduction steps, decision algorithm invocations and case splittings.

The paper is organized as follows. In Section 2 we briefly survey the basic notions of the TRSs theory (for the thorough treatment of TRSs see [2]). In Section 3 we make precise the notion of a theory axiomatized by means of the canonical reduction relation. This is done abstractly, i.e. independently of the way the reduction relation is generated. So, our result equally applies to the usual, equational or conditional TRSs, finite or infinite. In Section 4 we formulate and prove the main result, and Section 5 contains concluding remarks.

2. Preliminaries

Let $W(\Sigma, V)$ denote the set of terms built of variables from a set V and function symbols of a many-sorted signature Σ. A *reduction relation* $\xrightarrow{R}$ is a binary relation on $W(\Sigma, V)$, such that $t \xrightarrow{R} s$ implies $F(t) \xrightarrow{R} F(s)$ for any term $F(t)$, and $t \xrightarrow{R} s$ implies $\sigma(t) \xrightarrow{R} \sigma(s)$ for any substitution σ of terms for variables.

By $\xrightarrow{R}{}^{+}$, $\xrightarrow{R}{}^{*}$ and $\xrightarrow{R}{}^{-1}$ we denote the transitive closure, the reflexive-transitive closure and the inverse of $\xrightarrow{R}$ respectively. We say that a term t is in the *R-normal form* iff there doesn't exist a term s such that $t \xrightarrow{R} s$. A term t^* is called the *R-normal form of a term* t iff $t \xrightarrow{R} t^*$ and t^* is in the R-normal form.

Perhaps, the simplest method to generate reduction relations is by means of (unconditional) term rewriting systems. A *rewrite rule* is an oriented pair $l \dashrightarrow r$ such that $l, r \in W(\Sigma, V)$, l is different from a variable and every variable occuring in r occurs also in l. A *term rewriting system* (TRS for short) is a set of rewrite rules $R = \{ l_i \dashrightarrow r_i \}_{i \in I}$. The *reduction relation* $\xrightarrow{R}$ *generated* by a TRS R is defined as follows. Let a term t contain an occurence of a subterm s such that for some rule $l \dashrightarrow r \in R$ and substitution σ the term $\sigma(l)$ coincides with s. Then $t \xrightarrow{R} t'$ where t' is obtained from t by replacing the occurence of s with $\sigma(r)$. Similarly, reduction relations generated by equational and conditional TRSs may be defined, see the detailes in [3, 4, 8].

A reduction relation $\xrightarrow{R}$ is called

1) *noetherian*, iff there are no infinite chains of the form $t_0 \xrightarrow{R} t_1 \xrightarrow{R} t_2 \xrightarrow{R} \ldots$;

2) *confluent*, iff whenever $t \xrightarrow{R}{}^{*} t_1$ and $t \xrightarrow{R}{}^{*} t_2$ there exists a term s satisfying $t_i \xrightarrow{R}{}^{*} s$ for $i = 1, 2$;

3) *canonical*, iff it is both noetherian and confluent.

An *equational theory* (ET for short) is an arbitrary set of identities $T = \{ t_i = s_i \}_{i \in I}$, every identity being universally quantified. An ET T generates the binary relation $=_T$ on $W(\Sigma, V)$ defined as follows: $t =_T s$ iff the identity $t = s$ is the valid consequence of the theory T. A reduction relation $\xrightarrow{R}$ and an ET T are called *equivalent* iff $=_T$ coincides with $(\xrightarrow{R}{}^{*} \cup \xrightarrow{R}{}^{-1})^{*}$. When an ET T possesses an equivalent canonical reduction relation $\xrightarrow{R}$ then the relation $=_T$ is decidable by a simple normal form reduction: $t =_T s$ iff R-normal forms t^* and s^* of t and s are syntactically equal.

3. Context-free reduction relations

The aim of this section is to clarify the notion of a theory axiomatizable by means of the canonical reduction relation, and to define the class of context-free reduction relations admissible as axiomatizations. We also prove two simple structural lemmas justifying the naturalness of the imposed restrictions.

Definition 1. Let T be a theory (an arbitrary set of formulas). A reduction relation $\xrightarrow{R}$ is called

a) *T-complete*, iff $\Phi \in T$ implies $\Phi \xrightarrow{R}{}^{*} true$;

b) *T-consistent*, iff the inverse holds.

We say that a theory T is *axiomatizable* iff there exists a canonical reduction relation being T-complete and T-consistent. □

Example. Let an ET T possesses a canonical TRS R_T. Then T is axiomatizable via the reduction relation generated by

$$R'_T = R_T \cup \{ x = x \dashrightarrow true \}. \quad \square$$

Let's make two obvious observations about Definition 1. Firstly, it doesn't restrict the class of axiomatizable theories only to equational ones. Using a many-sorted language we may introduce logical connectives as functions on the boolean sort and formulas - as terms of the boolean sort. From now on we freely use the words "*an atomic formula*", "*a predicate symbol*" insead of "a term of the boolean sort with no boolean subterms" and "a function symbol of the boolean sort different from logical connectives". Secondly, the definition permits the degenerate cases of axiomatizations, since an arbitrary

logical theory T may be axiomatized via the reduction relation generated by the TRS $R_T = \{ \phi \dashrightarrow true \mid \phi \in T \}$. But what's the benefit of such axiomatizations?

Therefore, we must restrict the notion of an admissible axiomatization. In this section we introduce the class of context-free reduction relations and in Section 5 - the class of *n-context bounded* (for $n \in \omega$) reduction relations.

Before we give the precise definitions let's discuss one necessary condition of the T-completeness. The minimal requirement of any axiomatization is the provability of propositional tautologies, so we may consider that any reduction relation in question includes a canonical subrelation for the boolean algebra. Without the loss of generality we may think that this subrelation is generated by the following canonical equational rewrite system **BA**, see [1]:

$$
\begin{aligned}
x \vee y &\dashrightarrow x * y + x + y,\\
x \wedge y &\dashrightarrow x * y,\\
x \Rightarrow y &\dashrightarrow x * y + x + 1,\\
x \equiv y &\dashrightarrow x + y + 1,\\
\neg x &\dashrightarrow x + 1,\\
x + 0 &\dashrightarrow x,\\
x + x &\dashrightarrow 0,\\
x * 1 &\dashrightarrow x,\\
x * x &\dashrightarrow x,\\
x * 0 &\dashrightarrow 0,\\
x * (y + z) &\dashrightarrow x * y + x * z,
\end{aligned}
$$

where $\vee$, $\wedge$, $\Rightarrow$, $\equiv$, $\neg$ are the usual disjunction, conjunction, implication, equivalence, negation, $*$ and $+$ are the boolean ring multiplication and addition (exclusive or) assumed to be associative and commutative, and 0 and 1 are the ring zero and one respectively.

Convention. We suppose further on that any reduction relation is the union of *logical* $\xrightarrow{R}_{log}$ and *nonlogical* $\xrightarrow{R}_{nl}$ parts: $\xrightarrow{R} = \xrightarrow{R}_{log} \cup \xrightarrow{R}_{nl}$. We'll assume also that the logical part $\xrightarrow{R}_{log}$ satisfies two obvious requirements:

1) if $\alpha \xrightarrow{R}_{log} \beta$ then the formula $\alpha \equiv \beta$ is the propositional tautology;

2) if $\alpha \xrightarrow{R}_{log} \beta$ then every atom occuring in β occurs also in α. □

It's easy to see that both conditions are true for $\xrightarrow{BA}$.

Definition 2. A reduction relation $\xrightarrow{R}$ is called *context-free* (CF for short) iff for its nonlogical part $\xrightarrow{R}_{nl}$ the following two conditions hold:

1) for any formulas α_1, α_2 and binary logical connective $\blacksquare \in \{ \vee, \wedge, \Rightarrow, \equiv, \neg \}$

if $\alpha_1 \blacksquare \alpha_2 \xrightarrow{R}{}^{*}_{nl} \beta$

then β has the form $\beta_1 \blacksquare \beta_2$ and $\alpha_i \xrightarrow{R}{}^{*}_{nl} \beta_i$ for $i = 1, 2$;

2) for every formula α of the form $\neg\beta$

if $\alpha \xrightarrow{R}{}^{*}_{nl} \gamma$

then γ is of the form $\neg\delta$ and $\beta \xrightarrow{R}{}^{*}_{nl} \delta$. □

One obvious necessary condition for the context-freedom is, of course, the absence of boolean connectives in the left-hand sides of the rewrite rules generating the relation. The next two lemmas show that proofs with context-free reduction relations are easy feasible and the structure of the proofs is tranparent.

Normal form lemma. Let $\xrightarrow{R}$ be a context-free reduction relation and every atom occurence in a formula Φ is in the R-normal form. Then $\Phi \xrightarrow{R}{}^{*} \Psi$ implies $\Phi \xrightarrow{R}{}^{*}_{log} \Psi$. □

Proof. Suppose that $\Phi \xrightarrow{R}{}^{*}_{log} \Omega \xrightarrow{R}_{nl} \Gamma \xrightarrow{R}{}^{*} \Psi$. Then for some atom ρ occuring in Ω we have $\rho \xrightarrow{R}_{nl} X$ because $\xrightarrow{R}$ is context-free. By the second requirement for the logical part $\xrightarrow{R}_{log}$ of $\xrightarrow{R}$ we may conclude that ρ occurs also in the formula Φ, and is not reduced to the R-normal form. This is, however, the contradiction. □

Remark. The Normal form lemma is not true for non-CF reduction relations. Let $\xrightarrow{R}_{nl}$ is generated by the unique rewrite rule $\rho \wedge q \dashrightarrow true$. Then both ρ and q are in the R-normal forms, $\rho \wedge q \xrightarrow{R}{}^{*} true$, but it is wrong that $\rho \wedge q \xrightarrow{R}{}^{*}_{log} true$.

Midformula lemma. Let $\xrightarrow{R}$ be a canonical context-free reduction relation, and Φ be a non-atomic formula such that $\Phi \xrightarrow{R}{}^{*} true$. Then there exists *a middle formula* Ψ (being a tautology) such that $\Phi \xrightarrow{R}{}^{*}_{nl} \Psi \xrightarrow{R}{}^{+}_{log} true$. □

Remarks. This lemma states that each provable formula has the canonical form proof: first by applying only the non-logical rewrite rules, then by applying only the logical ones, both

parts of the proof are divided by the midformula. The example with $p \wedge q \longrightarrow true$ shows that the Midformula lemma does not hold for non-CF reduction relations.

Proof of the lemma. Suppose that Ψ is a $\xrightarrow{R}_{nl}$-normal form of Φ. By the canonicity condition we have $\Psi \xrightarrow{R}{}^{*} true$ and so by the Normal form lemma $\Psi \xrightarrow{R}{}^{*}_{log} true$. But Ψ must contain logical connectives, because $\xrightarrow{R}$ is context-free. So $\Psi \xrightarrow{R}{}^{+}_{log} true$. □

4. The main result

Let $\mathfrak{N} = \langle \Sigma, \omega \rangle$ be a model of a similarity type (signature) Σ and the set of natural numbers ω as the carrier. Suppose that

1) every element of $\mathfrak{N}$ is denoted by some constant term of Σ, i.e. $\forall n \in \omega \; \exists t \in W(\Sigma, \emptyset) \; t^{\mathfrak{N}} = n$, where $t^{\mathfrak{N}}$ denotes the valuation of the term t in $\mathfrak{N}$;

2) there exists a quantifier-free formula $\Phi(x, y)$ of Σ which numerically expresses the predicate "less or equal" between the integers, i.e. for any $n, m \in \omega$ and $t, s \in W(\Sigma, \emptyset)$ such that $t^{\mathfrak{N}} = n$, $s^{\mathfrak{N}} = m$

- if $n \leq m$ is true then $\mathfrak{N} \models \Phi(t, s)$,
- if $n \leq m$ is false then $\mathfrak{N} \models \neg \Phi(t, s)$.

Let $Th^{\forall}(\mathfrak{N})$ denote the quantifier-free (universal) fragment of the elementary theory of $\mathfrak{N}$, i.e. the set

$$Th^{\forall}(\mathfrak{N}) =_{df} =_{df} \{ \Psi \mid \mathfrak{N} \models \Psi \; , \; \Psi \; is \; unquantified \}.$$

T H E O R E M . *$Th^{\forall}(\mathfrak{N})$ is non-axiomatizable via a canonical context-free reduction relation.* □

The scenario of the proof. To simplify the notation we use $x \leq y$ instead of $\Phi(x, y)$ and 0, 1, 2, ... instead of the terms denoting zero, one, two, ... etc. The boolean ring zero and one will be denoted by the bold **0** and **1**.

The proof will be carried out in two stages. Let's suppose, on the contrary, that there exists a canonical context-free reduction relation $\xrightarrow{R}$ axiomatizing $Th^{\forall}(\mathfrak{N})$.

In the first stage we prove the existence of an atomic formula $P(x, y)$ being in the R-normal form and containing occurences of not less than two different variables x, y. Roughly speaking, this is true due to the non-represetability of $\leq$ predicate by means of monadic predicates.

In the second stage we prove that for each $k \in \omega$ $P(k, y) \xrightarrow{R}{}^{*} \mathbf{1}$. Since $\xrightarrow{R}$ is $Th^{\forall}(\mathfrak{N})$-consistent, all formulas $P(0, y)$, $P(1, y)$, $P(2, y)$,... are true in $\mathfrak{N}$. Hence, $P(x, y) \in Th^{\forall}(\mathfrak{N})$ (because the carrier of $\mathfrak{N}$ is ω). Therefore, by the assumption about the $Th^{\forall}(\mathfrak{N})$-completeness of $\xrightarrow{R}$ there must be $P(x, y) \xrightarrow{R}{}^{*} \mathbf{1}$, contradicting with the non-reducibility of $P(x, y)$.

Stage 1. We must prove that there exists an atomic formula $P(x, y)$ in the R-normal form depending on at least two variables. Assume, on the contrary, that there is no such a formula. Then all atomic formulas occuring in the R-normal form of $x \leq y$ depend on only one variable, either x, or y. Let $\langle \rho_1(x), \ldots, \rho_n(x) \rangle$ be the ordered tuple of such atoms depending only on x, and $\langle \rho_{n+1}(y), \ldots, \rho_{n+m}(y) \rangle$ - of atoms depending only on y. Let the functions $f_1 : \omega \longrightarrow \{ \mathbf{0}, \mathbf{1} \}^n$ and $f_2 : \omega \longrightarrow \{ \mathbf{0}, \mathbf{1} \}^m$ associate with each $k \in \omega$ the truth value distributions $\langle \rho_1^*(k), \ldots, \rho_n^*(k) \rangle$ and $\langle \rho_{n+1}^*(k), \ldots, \rho_{n+m}^*(k) \rangle$, where $\rho_j^*(k)$ denotes the R-normal form of $\rho_j(k)$. Note that any $\rho_j^*(k)$ is either **1** or **0** since $\rho_j(k)$ is true or false in $\mathfrak{N}$ by the assumption about the $Th^{\forall}(\mathfrak{N})$-completeness of $\xrightarrow{R}$. It's easy to see that for an arbitrary distribution function f_1 there exist the integers k_1, k, k_2 such that $k_1 < k < k_2$ and $f_1(k_1) = f_1(k_2)$, since in the infinite sequence $f_1(0), f_1(1), f_1(2), \ldots$ some element of the finite set $\{ \mathbf{0}, \mathbf{1} \}^n$ must occur repeatedly. Let's fix such numbers k_1, k, k_2.

Suppose that $\Psi(x, y)$ is the R-normal form of $x \leq y$. By the completeness and the consistency assumptions on $\xrightarrow{R}$ there must be $k_1 \leq k \xrightarrow{R}{}^{*} \mathbf{1}$ and $k_2 \leq k \xrightarrow{R}{}^{*} \mathbf{0}$. At the same time $k_i \leq k \xrightarrow{R}{}^{*} \Psi(k_i, k)$. So by the confluency of $\xrightarrow{R}$, $\Psi(k_1, k) \xrightarrow{R}{}^{*} \mathbf{1}$

and $\Psi(k_2, k) \xrightarrow{R}{}^{*} 0$. This is, however, the contradiction. Obviously, by the assumption, $\Psi(x, y)$ consists of atoms depending only on one variable, either x, or y. Therefore, the value of $\Psi(k_i, k)$ is uniquely determined by the pair of distributions $f_1(k_i)$ and $f_2(k)$. But by the choice of k_1 and k_2 we have $f_1(k_1) = f_1(k_2)$, so the R-normal forms of $\Psi(k_1, k)$ and $\Psi(k_2, k)$ must coincide. Q.E.D. □

Stage 2. Let $P(x, y)$ be an atom in the R-normal form which depends on two or more variables. We prove that for all $k \in \omega$

$$P(k, y) \xrightarrow{R}{}^{*} 1. \qquad (1)$$

Consider the formula

$$\Phi \equiv (x = k) \wedge P(x, y) \Rightarrow P(k, y)$$

which is valid in $\mathfrak{N}$ (as the equality substitution axiom). For the $Th^{\forall}(\mathfrak{N})$-comleteness of $\xrightarrow{R}$ it's necessary that $\Phi \xrightarrow{R}{}^{*} 1$ or

$$(x = k) * P(x, y) * (P(k, y) + 1) \xrightarrow{R}{}^{*} \xrightarrow{R}{}^{*} 0. \qquad (2)$$

Notice that we cannot conclude directly that (2) implies (1) because the boolean ring contains zero divisors, e.g. $q * (q + 1) = 0$ whereas q and $q + 1$ are not identically equal to zero.

Let's prove that the assumptions on $\xrightarrow{R}$ nevertheless give the possibility to derive (1) from (2). Suppose, on the contrary, that (2) and not-(1) are true. Then by the confluency and the context-freedom of $\xrightarrow{R}$ the Midformula lemma gives

$$(x=k)^* * P(x,y) * (P^*(k,y)+1) \xrightarrow{R}{}^{*}_{log} \xrightarrow{R}{}^{*}_{log} 0, \qquad (3)$$

where $(x = k)^*$ and $P(k, y)^*$ are the R-normal forms of $(x = k)$ and $P(k, y)$ respectively, and the R-irreducibility of $P(x, y)$ is taken into account.

The first requirement for the logical part $\xrightarrow{R}_{log}$ of $\xrightarrow{R}$ together with (3) imply that the left-hand side of (3) is unsatisfiable. Remind that by the not-(1) assumption $P(k, y)^*$ is not 1.

To obtain the desired contradiction let's construct an interpretation to satisfy the left-hand side of (3). Notice that $(x = k)^*$ cannot be 0, otherwise $(k = k)^*$ must be 0, contradicting the completeness of $\xrightarrow{R}$. Therefore there exists an interpretation $\mathfrak{I}$ satisfying the formula $(x = k)^*$. Inasmuch as $(x = k)^*$ consists of predicates depending only on x, we may vary arbitrarily the valuations of all other predicates in $\mathfrak{I}$, depending, say, on y or both on x and on y. It wouldn't violate the truth of $(x = k)^*$ in $\mathfrak{I}$. So we may suppose that $\mathfrak{I}$ satisfies the atom $P(x, y)$. It remains to show that $\mathfrak{I}$ may be transformed to satisfy $P(k, y)^* + 1$ in addition to $(x = k)^* * P(x, y)$. By the not-(1) assumption $P(k, y)^*$ is not 1. If $P(k, y)^*$ is 0, the proof is completed. So, assume $P(k, y)^*$ is neither identically true, nor false. Then there exists falsifying $P(k, y)^*$ assignment of the logical values to the atoms depending only on y and on no other variables. So, by the above argument we may reconstruct $\mathfrak{I}$ (not affecting the truthfold of $(x = k)^* * P(x, y)$ in $\mathfrak{I}$) in such a way that it will make $P(x, y)^*$ false. This concludes the construction of the interpretation $\mathfrak{I}$ satisfying the unsatisfiable left-hand side of (3). This contradiction completes the proof of the second stage and of the whole theorem. Q.E.D. □

5. Concluding remarks

Let's call a TRS R *n-context dependent* ($n \in \omega$) iff for every rule $l \dashrightarrow r$ of R the number of logical connectives occurences in its left-hand side l is less or equal to n. So, context-free TRSs are 0-context dependent. Similar definitions may be given for equational and conditional TRSs. We state the following

Conjecture. *$Th^{\forall}(\mathfrak{N})$ cannot be axiomatized via a canonical n-context dependent TRS for any $n \in \omega$.* □

Unfortunately, the Normal form and the Midformula lemmas or their analogues do not work in the general case. So, some fresh idea is necessary.

In conclusion we formulate one more negative result. Let $\Sigma_{\leq}$ be the signature consisting of the unique binary predicate symbol $\leq$, and $T^{\forall}_{PO}$ be the quantifier-free theory of the *partial order* of the signature $\Sigma_{\leq}$. Following [1], we say

that a rewrite rule l ---> r is the *N-rule* iff l is the boolean product (conjunction) of atoms. An arbitrary TRS is called the *N-TRS* iff it consists of the N-rules only. To reduce with a N-TRS it's necessary to keep the associativity and the commutativity of * in mind, but not the distributivity of * w.r.t. + . So, N-TRSs are simpler than TRSs of the general form. Note that an infinite N-TRS may not be n-context dependent for no $n \in \omega$.

THEOREM. *$T^{\forall}_{PO}$ cannot be axiomatized by a canonical N-TRS of the signature $\Sigma_{\leq}$.* □

We do not know, however, whether this result is stable w.r.t. the signature extentions.

Acknowledgements. *I am greatly indebted to Dr. Alexey Stolboushkin for invaluable comments made on the draft version of the paper. My special thanks are to Dr. Sergey Duzhin and to Dr. Grigory Schwarz for helpful discussions.*

REFERENCES

[1] Hsiang J. Refutational Theorem proving using term rewriting systems. - Artificial Intelligence, 1985, vol. 25, no. 2, pp. 255-300.

[2] Huet G., Oppen D.C. Equations and Rewrite Rules: A Survey. - In: Formal Language Theory: Perspectives and Open Problems. - New-York, Academic Press, 1980, pp. 349-406.

[3] Jouannaud J.-P. Confluent and Coherent Equational Term Rewriting Systems: Applications to Proofs in Abstract Data Types.- Lecture Notes in Computer Science, 1983, vol. 159, pp. 256-283.

[4] Kaplan S. Conditional Rewrite Rules. - Theoretical Computer Science, 1984, vol. 33, no. 2-3, pp. 175-193.

[5] Knuth D.E., Bendix P.B. Simple Word Problems in Universal Algebras. - In: Computational Problems in Universal Algebras. - Pergamon Press, 1970, pp. 263-297.

[6] Shostak R.E. On the SUP-INF Method for Proving Presburger Formulas. - Journal of the ACM, 1977, vol. 24, no. 4, pp. 529-543.

[7] Vorobyov S.G. On the Use of Conditional Term Rewriting Systems in Program Verification. - Programmirovanie, 1986, no. 4, pp. 3-14 (in Russian).

[8] Zhang H., Remy J.-L. Contextual Rewriting. - Lecture Notes in Computer Science, 1985, vol. 202, pp. 46-62.