Fsub with Recursive Types: "Types-As-Propositions" Interpretation in M. Rabin's S2S

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Abstract

Subtyping judgments of the polymorphic second-order typed λ-calculus $F_{\leq}$ extended by recursive types and different known inference rules for these types could be interpreted in **S2S**, M.Rabin's monadic second-order theory of two successor functions. On the one hand, this provides a comprehensible model of the parametric and inheritance polymorphisms over recursive types, on the other, proves that the corresponding subtyping theories are not essentially undecidable, i.e., possess consistent decidable extensions.

1. Introduction

In 1985 Cardelli and Wegner [CW85] introduced the language *Fun* based on an extension of Girard-Reynolds's second-order polymorphic λ-calculus, supporting both parametric ($\forall$-) and inheritance ($\leq$-) polymorphisms in an elegant and uniform way, by means of *bounded quantification.*

Since then a large amount of work has been done on understanding and elucidating formal properties of λ-systems underlying *Fun*. Not to repeat the whole story once again, we refer the reader to the excellent articles cited in this paper, written by the pioneers of the domain.

To simplify matters and for theoretical purposes the original *Fun* was purified by Bruce and Longo [BL90], Curien and Ghelli [CG92]. The

*This work was done when the author was at CRIN, *Centre de Recherche en Informatique de Nancy*, FRANCE. Preliminary version of this paper appeared as Technical Report CRIN–94–R–035, February, 1994 resulting system is now known as the *minimal bounded Fun*, or system $F_{\leq}$, see Appendix B.

The minimal bounded *Fun* possesses several funny properties, for example:

1. Subtyping relation between types in $F_{\leq}$, $\Gamma \vdash \sigma \leq \tau$, is *undecidable* [Pie92].
2. Different *weakenings* of the general subtyping rule for universally quantified types lead to decidable subsystems of $F_{\leq}$, as $F_{\leq}{}^{\top}$ [CP94].
3. Although undecidable, $F_{\leq}$ is *is not essentially undecidable*, i.e., possesses *consistent decidable extensions*, in fact, infinitely many different ones, see [Vor94a, Vor95]. So, trying to get a decidable system, one can *reinforce* $F_{\leq}$ instead of weakening it, as opposed to 2 above.

Amadio and Cardelli [AC90] were the first who systematically introduced recursive types and subtyping over these types into simply typed λ-calculus. Ghelli in [Ghe93] extended $F_{\leq}$ by the $(Unfold\text{-}L)$ and $(Unfold\text{-}R)$ rules for recursive types (see right below) and investigated connections of the extended system with the kernel $F_{\leq}$. Breazu-Tannen *et.al.* in [BTCCS91] suggested the coercion translation (sometimes called the *Penn translation*) of $F_{\leq}$ extended by recursive types into $F^{\top}$, Girard's system F equipped with the terminal object $\top$. They *did not*, however, extend $F_{\leq}$ by additional rules to compare recursive types. The question whether their coercion translation is compatible with these rules remains open.

This is the companion to the paper on existence of consistent decidable extensions of $F_{\leq}$ [Vor94a]. We consider extensions of $F_{\leq}$ by polymorphic recursive types and, correspondingly, by four known inference rules for these types:

- (μ_R) and (μ^+) rules [AC90] extended for polymorphic types;
- $(Unfold\text{-}L)$ and $(Unfold\text{-}R)$ rules [Ghe93, CP94] (see the next Section and Appendix C).

We extend the "Types-As-Propositions Interpretations" of $F_{\leq}$ in **S2S**, introduced in [Vor94a] to polymorphic recursive types and obtain the same kind of results as in [Vor94a]: all these extensions are not essentially undecidable.

Call an inference rule $\mathcal{R}$ *consistent* with an interpretation $\mathcal{I}$ (or, equivalently, $\mathcal{I}$ *compatible* with $\mathcal{R}$) iff $\mathcal{R}$ preserves validity with respect to $\mathcal{I}$. The latter means that the conclusion of $\mathcal{R}$ is valid whenever the premises are. For the "good" rules (μ_R) and (μ^+) we prove that they are consistent with every extended **S2S**-interpretation of $F_{\leq}$. As a result, all our **S2S**-interpretations are compatible simultaneously with $F_{\leq}$ plus (μ_R), plus (μ^+), plus all inference rules enlisted in Appendices C.1 and C.2, notably, with the two interesting Curry-style rules suggested in [Vor94a]. The latter rules permit one to compare universally quantified types with their particular cases, the feature absent in *Fun* and $F_{\leq}$.

On the other hand, we prove that the rules $(Unfold\text{-}L)$ and $(Unfold\text{-}R)$ are not, in general, consistent with the **S2S**-interpretations. First, it implies immediately that $(Unfold\text{-}L)$ and $(Unfold\text{-}R)$ are independent of the other rules of $F_{\leq} \cup \{(\mu_R),\ (\mu^+)\}$. Second, this gives another proof that $F_{\leq} \cup \{(Unfold\text{-}L),\ (Unfold\text{-}R)\}$ is not conservative over $F_{\leq}$, the result due to Ghelli [Ghe93].

The history of this paper is as follows. After reading [Vor94a] L. Cardelli noticed that there are certain similarities between the *Penn translation* of $F_{\leq}$ into $F^{\top}$ by coercions and the **S2S**-translations of $F_{\leq}$, so he asked me to clarify the point. One thing is clear, there are infinitely many different **S2S**-translations and all of them are decidable. Trying to understand further relations and studying [BTCCS91] I suddenly came across the obvious asymmetry of Penn translation (recursive types without possibility to compare them). At that time I could not interpret recursive types, so I was obliged to extend the **S2S**-interpretations to recursive types. Now extended **S2S**-interpretations capture recursive types and are compatible with inference rules for recursive types. Further connections with the Penn translation are under development and will be reported shortly.

An interesting problem about **S2S**-interpretations of $F_{\leq}$ was posed to me by B. Pierce: whether there exist **S2S**-interpretations with the *minimal subtyping property* [CG92] in the sense that every typable term has a *least type*. The problem became more important after Ghelli's remark[1] concerning the absence of the minimal typing property in the decidable subtyping system $F_{\leq}{}^{\top}$ considered in [CP94]. Partially positive solution of this problem is given in a series of papers [Vor94b, Vor94c, Vor94d].

The **S2S**-interpretations of $F_{\leq}$ are rather curious: they easily interpret types and subtyping judgments, but *not typing judgments*. Otherwise, typing in F would not be essentially undecidable. We will report on this issue shortly. Combining **S2S**-interpretations with typing rules (without recursive types) is described in [Vor94c, Vor94d].

[1] In the "types" mailing list, January, 1994

Anyway, despite certain drawbacks, **S2S**-interpretations perfectly fulfill their purposes. First, they explain intricate second-order subtyping over polymorphic types in terms of extremely simple and understandable thing[2], the theory **S2S**. Second, they are all *decidable*. Third, they help to understand and justify inference rules and their mutual relations, as well as the rôles of different restrictions on types and variables. At first glance, the Kelly-Klein or Poincaré models have little to do with non-Euclidian geometry, but they satisfy exactly all necessary properties.

The paper is organized as follows. In the next Section we introduce the subject of discussion. In Section 3 we define the "Types-As-Propositions" interpretations of $F_{\leq}$ extended by recursive types in **S2S**, and recall some their properties concerning non-recursive types proved in [Vor94a]. Next, in Section 4 we prove our main results on consistency of different inference rules with respect to extended **S2S**-interpretations. In Section 5 for beauty and symmetric purposes we make one more generalization of **S2S**-interpretations. We prove that at least one rule, namely the positive (μ^{+}) rule, is compatible with these generalized interpretations. Appendices include necessary definitions of **S2S**, $F_{\leq}$, and some its variations and extensions.

I am grateful to Benjamin Pierce, Luca Cardelli, and Jean-Paul Bodeveix for extremely stimulating remarks and questions. However, all mistakes and delusions remain on my responsibility. I also do not insist on solutions proposed here, I only analyze consequences of certain, perhaps controversial, hypotheses.

2. Subtyping Rules for Recursive Types

We consider the standard system $F_{\leq}$ (see Appendix B) with powerful rule to compare $\forall$-types:

$$\frac{\Gamma \vdash \tau_1 \leq \sigma_1 \qquad \Gamma, \alpha \leq \tau_1 \vdash \sigma_2 \leq \tau_2}{\Gamma \vdash (\forall \alpha \leq \sigma_1 . \sigma_2) \leq (\forall \alpha \leq \tau_1 . \tau_2)} \qquad (All)$$

enriched with type intersection $\sigma \cap \tau$ and recursive types $\mu\alpha.\sigma$.

Formally, we consider the system $F_{\leq}^{\mu\cap}$ with types defined by the following abstract grammar:

$$\mathbb{T} \equiv_{df} \mathbb{V} \mid \top \mid \mathbb{T} \to \mathbb{T} \mid \forall \mathbb{V} \leq \mathbb{T} . \mathbb{T} \mid \mathbb{T} \cap \mathbb{T} \mid \mu\mathbb{V}.\mathbb{T}$$

where all but two last cases refer to standard $F_{\leq}$-types. The semantics of $\sigma \cap \tau$ is the obvious type intersection, and the meaning of

[2] Which is not the case for PER models, or domain models, or even $F^{\top}$-coercions

$\mu\alpha.\sigma$ is: "*a solution to the equation* $\alpha = \sigma$", see [AC90] for motivations and systematic development. Type intersection poses no problem and we keep it just for completeness, it does not play any rôle in this paper. So, we almost "forget" it in the sequel.

There exist at least four rules for subtyping recursive types:

$$\frac{\Gamma,\ \alpha \leq \beta\ \vdash\ \sigma \leq \tau}{\Gamma\ \vdash\ \mu\alpha.\sigma \leq \mu\beta.\tau} \qquad (\mu_R)$$

(**Restriction:** with α only in σ, β only in τ; α, β not in Γ, [AC90])

$$\frac{\Gamma,\ \alpha \leq \top\ \vdash\ \sigma \leq \tau}{\Gamma\ \vdash\ \mu\alpha.\sigma \leq \mu\beta.\tau} \qquad (\mu^+)$$

(**Restriction:** α appears in σ, τ only positively, [AC90])

$$\frac{\Gamma\ \vdash\ \sigma[\mu\alpha.\sigma\ /\ \alpha] \leq \tau}{\Gamma\ \vdash\ \mu\alpha.\sigma \leq \tau} \qquad (\mathit{Unfold\text{-}L})$$

$$\frac{\Gamma\ \vdash\ \tau \leq \sigma[\mu\alpha.\sigma\ /\ \alpha]}{\Gamma\ \vdash\ \tau \leq \mu\alpha.\sigma} \qquad (\mathit{Unfold\text{-}R})$$

Remarks: The rule (μ_R) is introduced in [AC90] in simply typed λ-calculus framework. The rule (μ^+) is also discussed in [AC90] and in the conclusion to [BTCCS91]. In [BTCCS91] the only rule for comparing recursive types is reflexivity, but it is conjectured that the coercion semantics introduced there should be compatible with (μ^+). Rules $(\mathit{Unfold\text{-}L})$ and $(\mathit{Unfold\text{-}L})$ are introduced in [Ghe93] and are also dealt with in [CP94].

3. Types-As-Propositions Interpretations of $F_{\leq}^{\mu\cap}$ in S2S

We extend the main definition from [Vor94a, Vor95] to recursive types.

Definition 3.1 **($TAP_{S2S}^{F_{\leq}}(\Im^{\rightarrow}, \Im^{\forall})$-interpretation)** *Let* $\Im^{\rightarrow}$ *and* $\Im^{\forall}$ *be two arbitrary strings composed of function symbols* l *and* r *of* **S2S**. *Both may be equal to the empty string* ε.[3]

For an arbitrary $F_{\leq}^{\mu\cap}$*-type* ρ *the* Types-As-Propositions-Interpretation *of* ρ *in* **S2S** *with parameters* $\Im^{\rightarrow}$ *and* $\Im^{\forall}$*, the* $TAP_{S2S}^{F_{\leq}}(\Im^{\rightarrow}, \Im^{\forall})$-

[3]Note that $\Im^{\rightarrow}x$ and $\Im^{\forall}x$ are **S2S**-terms.

interpretation for short, is defined as an **S2S**-*formula* [4] $[\![\ \rho\ ]\!]_{\Im^\forall}^{\Im^\rightarrow}(x)$ *with the unique distinguished free object variable* x *by induction on the structure of the type* ρ:

1. $[\![\ \alpha\ ]\!]_{\Im^\forall}^{\Im^\rightarrow}(x) \quad \equiv_{df} \quad A(x)$

   (new predicate variable A *for each type variable* α*);*

2. $[\![\ \top\ ]\!]_{\Im^\forall}^{\Im^\rightarrow}(x) \quad \equiv_{df} \quad x = x;$

3. $[\![\ \sigma \rightarrow \tau\ ]\!]_{\Im^\forall}^{\Im^\rightarrow}(x) \quad \equiv_{df} \quad [\![\ \sigma\ ]\!]_{\Im^\forall}^{\Im^\rightarrow}(x) \supset [\![\ \tau\ ]\!]_{\Im^\forall}^{\Im^\rightarrow}(\Im^\rightarrow(x));$

4. $[\![\ \sigma \cap \tau\ ]\!]_{\Im^\forall}^{\Im^\rightarrow}(x) \quad \equiv_{df}$

   $$\forall^2 A \left\{ \left[[\![\ \sigma\ ]\!]_{\Im^\forall}^{\Im^\rightarrow}(x) \supset ([\![\ \tau\ ]\!]_{\Im^\forall}^{\Im^\rightarrow}(x) \supset A(x)) \right] \supset A(x) \right\},$$

   (A *is a new predicate variable not free in* $[\![\ \sigma\ ]\!]_{\Im^\forall}^{\Im^\rightarrow}(x)$, $[\![\ \tau\ ]\!]_{\Im^\forall}^{\Im^\rightarrow}(x)$*);*

5. $[\![\ \forall \alpha \leq \sigma\,.\,\tau\ ]\!]_{\Im^\forall}^{\Im^\rightarrow}(x) \quad \equiv_{df}$

   $$\forall^2 A \left\{ \forall^1 x \left(A(x) \supset [\![\ \sigma\ ]\!]_{\Im^\forall}^{\Im^\rightarrow}(x) \right) \supset [\![\ \tau\ ]\!]_{\Im^\forall}^{\Im^\rightarrow}(\Im^\forall(x)) \right\};$$

6. $[\![\ \mu\alpha.\sigma\ ]\!]_{\Im^\forall}^{\Im^\rightarrow}(x) \quad \equiv_{df} \quad \bigvee_{i=1}^{\infty} \left([\![\ \sigma\ ]\!]_{\Im^\forall}^{\Im^\rightarrow} \right)^i [\bot/A](x),$

   (where $\Phi^1[\bot/P] \equiv_{df} \Phi[\bot/P]$, $\Phi^{i+1}[\bot/P] \equiv_{df} \Phi[\Phi^i[\bot]/P]$*).*

The $TAP_{S2S}^{F_\leq}(\Im^\rightarrow, \Im^\forall)$-*interpretation is extended on subtyping judgments by:*

7. $[\![\ \sigma\ \leq\ \tau\ ]\!]_{\Im^\forall}^{\Im^\rightarrow} \quad \equiv_{df} \quad \forall^1 x\, ([\![\ \sigma\ ]\!]_{\Im^\forall}^{\Im^\rightarrow}(x) \supset [\![\ \tau\ ]\!]_{\Im^\forall}^{\Im^\rightarrow}(x));$

8. $[\![\ \alpha_1 \leq \sigma_1, \ldots, \alpha_n \leq \sigma_n\ \vdash\ \sigma \leq \tau\ ]\!]_{\Im^\forall}^{\Im^\rightarrow} \quad \equiv_{df}$

   $$[\![\ \alpha_1 \leq \sigma_1\ ]\!]_{\Im^\forall}^{\Im^\rightarrow}, \ldots, [\![\ \alpha_n \leq \sigma_n\ ]\!]_{\Im^\forall}^{\Im^\rightarrow} \models_{\mathbf{S2S}} [\![\ \sigma \leq \tau\ ]\!]_{\Im^\forall}^{\Im^\rightarrow}.$$

Define the $TAP_{S2S}^{F_\leq}(\Im^\rightarrow, \Im^\forall)$-*theory as:*

$$TAP_{S2S}^{F_\leq}(\Im^\rightarrow, \Im^\forall) \equiv_{df} \{\ \Gamma\ \vdash\ \sigma \leq \tau\ \mid\ [\![\ \Gamma\ \vdash\ \sigma \leq \tau\ ]\!]_{\Im^\forall}^{\Im^\rightarrow}\ \}$$

[4] Strictly speaking, in item #6 of this Definition we use infinite disjunctions, which are not allowed in **S2S**. We prove in the beginning of the next Section, see Lemma 4.4, that the infinite disjunctions we need are in fact **S2S**-definable

Example. Let $\sigma \ \equiv_{df} \ \mu\alpha \,.\, \beta \to \alpha$. Then

$$
\begin{array}{rcl}
[\![\, \sigma \,]\!]^{\Im^{\to}}_{\Im^{\forall}}(x) & = & \bigvee_{i=1}^{\infty} (Bx \supset A\Im^{\to} x)^i \, [\bot/A](x) \\
& = & (Bx \supset \bot) \ \vee \ (Bx \supset (B(\Im^{\to}(x)) \supset \bot)) \ \vee \\
& \vee & (Bx \supset (B(\Im^{\to}(x)) \supset (B(\Im^{\to 2}(x)) \supset \bot))) \ \vee \ \ldots
\end{array}
$$

Remarks.

1. Note the "shift" function parameters $\Im^{\to}$ for $\to$ and $\Im^{\forall}$ for $\forall$ in interpretations of functional and universal types. They allow to define infinitely many different interpretations.
2. In $TAP^{F_{\leq}}_{S2S}(\Im^{\to}, \Im^{\forall})$-interpretation we use just one-variable restricted fragment of **S2S**.
3. if $\Im^{\to} = \Im^{\forall} = \varepsilon$ then this fragment is also function-free.
4. x is the only free object variable of any $TAP^{F_{\leq}}_{S2S}(\Im^{\to}, \Im^{\forall})$-interpretation of any type.
5. Subtyping judgments are interpreted as statements about **S2S**-semantical consequence relation $\models_{\mathbf{S2S}}$ containing no free object variables at all.
6. Any $TAP^{F_{\leq}}_{S2S}(\Im^{\to}, \Im^{\forall})$ is decidable. □

Further we will freely say that a typing judgment is true or valid in (or with respect to) a $TAP^{F_{\leq}}_{S2S}(\Im^{\to}, \Im^{\forall})$-interpretation iff it belongs to the set (theory) $TAP^{F_{\leq}}_{S2S}(\Im^{\to}, \Im^{\forall})$.

Definition 3.2 (Theory, Consistency, Essential Undecidability) A subtyping theory *is an arbitrary set T of subtyping judgments closed with respect to $F_{\leq}$-inference rules, i.e., whenever both premises of an $F_{\leq}$-rule belong to T, the conclusion is also in T. A subtyping theory is* consistent *iff it does not coincide with the set of all subtyping judgments. If $T_1 \subseteq T_2$ are two subtyping theories then T_1 is a* subtheory *of T_2 and T_2 is an* extension *of T_1. A subtyping theory is* essentially undecidable *iff it does not possess consistent decidable extensions.* □

Remarks. If we had not stipulated the closure of subtyping theories with respect to the $F_{\leq}$-rules it would have allowed simple "consistent" decidable theories like

$$T_\top \equiv_{df} \{\Gamma \vdash \sigma \le \tau \mid \textit{for any } \sigma \textit{ and } \tau\} \setminus \{\Gamma \vdash \top \le \tau \mid \tau \ne \top\}$$

By definition, $T_\top$ proves almost everything, except $\top \le \tau$ for $\tau \ne \top$. For example, $T_\top$ proves $\alpha \le \beta$ for any two type variables α and β. Intuitively it should also prove any particular case of $\alpha \le \beta$, for example, $\top \le \top \to \top$. But this is not the case by definition of $T_\top$. The problem is that $T_\top$ is not closed with respect to simple inference rules, as substitution. So, demanding the closure of a theory with respect to something like substitution or even all the rules of $F_\le$ is natural. Anyway, without the closure condition, essentially undecidable theories do not exist, since one can always throw away just one formula to obtain a "consistent" decidable extension of an undecidable theory. □

The following important properties of $TAP_{S2S}^{F_\le}(\Im^\to, \Im^\forall)$-interpretations are proved in [Vor94a]:

Lemma 3.3 (Adequacy of $TAP_{S2S}^{F_\le}(\Im^\to, \Im^\forall)$-interpretations)

1. *All $F_\le$-axioms are true in every $TAP_{S2S}^{F_\le}(\Im^\to, \Im^\forall)$-interpretation :*

   $[\![((Refl))]\!]_{\Im^\forall}^{\Im^\to}$, $[\![((Top))]\!]_{\Im^\forall}^{\Im^\to}$, $[\![((TVar))]\!]_{\Im^\forall}^{\Im^\to}$ *are valid in* **S2S**;

2. *the $F_\le$-inference rules* preserve validity *with respect to every $TAP_{S2S}^{F_\le}(\Im^\to, \Im^\forall)$-interpretation , i.e., if both premises of a rule $(Trans)$, $(Arrow)$, or (All) are in a $TAP_{S2S}^{F_\le}(\Im^\to, \Im^\forall)$, then so is the conclusion of the rule;*

3. *the additional Curry-style quantifier rules $(Curry\text{-}\forall\text{-}elimination)$ and $(Curry\text{-}\forall\text{-}introduction)$ (see Appendix C.1)* preserve validity *with respect to every $TAP_{S2S}^{F_\le}(\Im^\to, \varepsilon)$-interpretation;*

4. *any $TAP_{S2S}^{F_\le}(\Im^\to, \Im^\forall)$ is a* consistent *decidable subtyping theory;*

5. *there exist infinitely many different $TAP_{S2S}^{F_\le}(\Im^\to, \Im^\forall)$-interpretations distinguishable by subtyping judgments;*

6. *there exist infinitely many $TAP_{S2S}^{F_\le}(\Im^\to, \Im^\forall)$-interpretations with infinitely many incomparable types.*

Corollary 3.4 ($F_\le$ possesses decidable extentions) *All $F_\le$-provable subtyping judgments are true with respect to any $TAP_{S2S}^{F_\le}(\Im^\to, \Im^\forall)$-interpretation . Henceforth, $F_\le$ is* not essentially undecidable*, i.e., there exist consistent decidable theories containing all $F_\le$-provable subtyping judgments.* □

4. Consistency of Rules for Recursive Types with $TAP^{F_{\leq}}_{S2S}(\Im^{\rightarrow}, \Im^{\forall})$-interpretations

The main result of the paper is the following

Theorem 4.1 *The system $F^{\mu\cap}_{\leq}$, i.e., $F_{\leq}$ extended by recursive types and two inference rules for polymorphic types (μ_R), (μ^+), is compatible with any $TAP^{F_{\leq}}_{S2S}(\Im^{\rightarrow}, \Im^{\forall})$-interpretation in the sense that all the rules of $F^{\mu\cap}_{\leq}$ preserve the validity with respect to any of these interpretations.*

In Sections 4.3 and 4.4 we prove that under additional assumption the same is true if we add rules $(Unfold\text{-}L)$ and $(Unfold\text{-}R)$ to $F^{\mu\cap}_{\leq}$.

Corollary 4.2 *Everything derivable in $F^{\mu\cap}_{\leq}$ is valid in every $TAP^{F_{\leq}}_{S2S}(\Im^{\rightarrow}, \Im^{\forall})$-interpretation . Therefore, $F^{\mu\cap}_{\leq}$ is not essentially undecidable, i.e., possesses consistent decidable extensions.* □

Combining with Lemma 3.3 above we get

Corollary 4.3 *The properties stated in the preceding Corollary hold for the system $F_{\leq}$ with recursive types extended by all the rules given in Appendices C.1 and C.2, plus the rules (μ_R), (μ^+).* □

The proof of Theorem 4.1 consists of the following Lemmas 4.4, 4.5, 4.7.

First of all we have to prove the correctness of the $TAP^{F_{\leq}}_{S2S}(\Im^{\rightarrow}, \Im^{\forall})$-interpretation definition, where in item #6 we used infinite disjunctions:

Lemma 4.4 (S2S-definability of $\bigvee_{i=1}^{\infty}$) *For any **S2S**-formula $\Phi[P]$ the formula*

$$\Psi_1 \equiv_{df} \bigvee_{i=1}^{\infty} \Phi^i[\perp]$$

*is **S2S**-definable by*

$$\Psi_2 \equiv_{df} \mu Q\,(\Phi[\perp/P] \subseteq Q \;\&\; \forall^2 Y(Y \subseteq Q \supset \Phi[Y/P] \subseteq Q))$$

*(where $\Phi \subseteq \Psi$ abbreviates $\forall x(\Phi(x) \supset \Psi(x))$, and $\mu Q\ldots$ expresses "minimum set (predicate) such that ...", expressible in **S2S**, see Appendix A).* □

Proof . $(\Psi_1 \Rightarrow \Psi_2)$. Let $\Psi_1(x_0)$ be true. Then, by definition of Ψ_1, there exists an n such that $\Phi^n[\perp](x_0)$ is true. As $\Phi^1[\perp] \equiv \Phi[\perp/P] \subseteq \Psi_2$ by definition of Ψ_2, we conclude by the same definition that $\Phi^2[\perp] \subseteq \Psi_2$. Continuing (by induction on n) we have $\Phi^n[\perp] \subseteq \Psi_2$ (for any n). Therefore, $\Psi_2(x_0)$ is also true.

$(\Psi_2 \Rightarrow \Psi_1)$. Let $\Psi_2(x_0)$ be true. Then by definition of μ, x_0 should belong to all sets (=monadic predicates) Q satisfying

$$\Phi[\perp/P] \subseteq Q \ \& \ \forall^2 Y(Y \subseteq Q \ \supset \ \Phi[Y/P] \subseteq Q)$$

Note, finally, that Ψ_1 satisfies this property. □

4.1. Consistency of the Amadio-Cardelli Rule (μ_R)

Lemma 4.5 (Consistency of (μ_R)) *The rule (μ_R) preserves validity with respect to any $TAP_{S2S}^{F_\leq}(\Im^\to, \Im^\forall)$-interpretation , i.e., if the premise of the rule is true in a $TAP_{S2S}^{F_\leq}(\Im^\to, \Im^\forall)$-interpretation then so is the conclusion of the rule.*

Combining with Lemma 3.3 we get:

Corollary 4.6 *The system $F_\leq \cup \{(\mu_R)\}$ is not essentially undecidable, i.e., possesses consistent decidable extensions.* □

Proof of Lemma 4.5. Suppose that the sequent interpreting the premise of (μ_R),

$$[\![\, \Gamma \,]\!]_{\Im^\forall}^{\Im^\to}, \forall^1 x(Ax \supset Bx) \ \models_{\mathbf{S2S}} \ \forall^1 x \left([\![\, \sigma \,]\!]_{\Im^\forall}^{\Im^\to}(x) \supset [\![\, \tau \,]\!]_{\Im^\forall}^{\Im^\to}(x) \right) \quad (1)$$

is valid. We have to prove that so is the sequent interpreting the conclusion of (μ_R):

$$[\![\, \Gamma \,]\!]_{\Im^\forall}^{\Im^\to} \ \models_{\mathbf{S2S}} \ \forall^1 x \left(\bigvee_{i=1}^{\infty} \left([\![\, \sigma \,]\!]_{\Im^\forall}^{\Im^\to} \right)^i [\perp/A](x) \supset \bigvee_{i=1}^{\infty} \left([\![\, \tau \,]\!]_{\Im^\forall}^{\Im^\to} \right)^i [\perp/B](x) \right).$$

Suppose, that the latter is false, i.e., for some x_0:

$$[\![\, \Gamma \,]\!]_{\Im^\forall}^{\Im^\to} \ \models_{\mathbf{S2S}} \ \bigvee_{i=1}^{\infty} \left([\![\, \sigma \,]\!]_{\Im^\forall}^{\Im^\to} \right)^i [\perp/A](x_0) \quad \text{is true, but} \quad (2)$$

$$[\![\, \Gamma \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}} \models_{\mathbf{S2S}} \bigvee_{i=1}^{\infty} \left([\![\, \tau \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}} \right)^{i} [\perp/B](x_0) \quad \text{is false.} \tag{3}$$

(2) means that for some $n_0 \geq 1$

$$[\![\, \Gamma \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}} \models_{\mathbf{S2S}} \left([\![\, \sigma \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}} \right)^{n_0} [\perp/A](x_0) \quad \text{is true.} \tag{4}$$

From (1) substituting $\perp$ for A and B (recall that α, β do not appear in Γ) we derive the validity of:

$$[\![\, \Gamma \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}} \models_{\mathbf{S2S}} \forall^1 x \left(\left([\![\, \sigma \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}} \right)^{1} [\perp/A](x_0) \supset \left([\![\, \tau \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}} \right)^{1} [\perp/B](x_0) \right) \tag{5}$$

Substituting $\left([\![\, \sigma \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}} \right)^{1} [\perp/A](x_0)$ for A and $\left([\![\, \tau \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}} \right)^{1} [\perp/B](x_0)$ for B in (1) (it is **crucial** here that A appears only in $[\![\, \sigma \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}}$, B only in $[\![\, \tau \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}}$, neither A, nor B appears in $[\![\, \Gamma \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}}$, see the (μ_R) restriction) we get:

$$[\![\, \Gamma \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}}, \forall^1 x \left(\left([\![\, \sigma \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}} \right)^{1} [\perp/A](x_0) \supset \left([\![\, \tau \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}} \right)^{1} [\perp/B](x_0) \right) \models_{\mathbf{S2S}}$$

$$\models_{\mathbf{S2S}} \forall^1 x \left(\left([\![\, \sigma \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}} \right)^{2} [\perp/A](x_0) \supset \left([\![\, \tau \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}} \right)^{2} [\perp/B](x_0) \right) \quad \text{is true.}$$

By (5), the last hypothesis in the above could be omitted, giving:

$$[\![\, \Gamma \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}} \models_{\mathbf{S2S}} \forall^1 x \left(\left([\![\, \sigma \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}} \right)^{2} [\perp/A](x_0) \supset \left([\![\, \tau \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}} \right)^{2} [\perp/B](x_0) \right) \tag{6}$$

Proceeding this way (by induction) we obtain for each $n \geq 1$:

$$[\![\, \Gamma \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}} \models_{\mathbf{S2S}} \forall^1 x \left(\left([\![\, \sigma \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}} \right)^{n} [\perp/A](x_0) \supset \left([\![\, \tau \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}} \right)^{n} [\perp/B](x_0) \right) \tag{7}$$

which together with (4) gives:

$$[\![\, \Gamma \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}} \models_{\mathbf{S2S}} \left([\![\, \tau \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}} \right)^{n} [\perp/B](x_0) \quad \text{is true,}$$

which contradicts (3). $\square$

4.2. Consistency of the Positive Rule (μ^+)

Lemma 4.7 (Consistency of (μ^+)) *The rule (μ^+) preserves validity with respect to every $TAP^{F_\leq}_{S2S}(\Im^{\rightarrow}, \Im^{\forall})$-interpretation , i.e., if the premise of the rule is true in a $TAP^{F_\leq}_{S2S}(\Im^{\rightarrow}, \Im^{\forall})$-interpretation then so is the conclusion of the rule.* □

Combining with the above we get the following

Corollary 4.8 *The systems $F_\leq \cup \{(\mu^+)\}$ and $F_\leq \cup \{(\mu_R), (\mu^+)\}$ are not essentially undecidable, i.e., possess consistent decidable extensions.* □

Proof of Lemma 4.7. Suppose that the sequent corresponding to the $TAP^{F_\leq}_{S2S}(\Im^{\rightarrow}, \Im^{\forall})$-interpretation of the premise of the rule (μ^+),

$$[\![\, \Gamma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}, \forall^1 x(Ax \supset \top) \models_{\mathbf{S2S}} \forall^1 x \left([\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}(x) \supset [\![\, \tau \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}(x)\right) \tag{8}$$

is valid. We have to prove that so is the sequent corresponding to the conclusion of the rule (μ^+):

$$[\![\, \Gamma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \models_{\mathbf{S2S}} \forall^1 x \left(\bigvee_{i=1}^{\infty} \left([\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}\right)^i [\bot/A](x) \supset \bigvee_{i=1}^{\infty} \left([\![\, \tau \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}\right)^i [\bot/A](x)\right) \tag{9}$$

Instead of (9) we will prove the following **more strong** claim: for each $i \geq 1$ it is true that

$$[\![\, \Gamma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \models_{\mathbf{S2S}} \forall^1 x \left(\left([\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}\right)^i [\bot/A](x) \supset \left([\![\, \tau \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}\right)^i [\bot/A](x)\right). \tag{10}$$

The base case $i = 1$, follows from (8) for $A \equiv \bot$, because $\left([\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}\right)^1 [\bot/A](x) \equiv_{df} [\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} [\bot/A](x)$,
$\left([\![\, \tau \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}\right)^1 [\bot/A](x) \equiv_{df} [\![\, \tau \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} [\bot/A](x)$, and α does not appear in Γ by the definition of the context.

Since

$$\left([\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}\right)^{i+1} [\bot/A](x) \equiv_{df} [\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \left[\left([\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}\right)^i [\bot/A](x)/A\right](x),$$

we get, by induction hypothesis, due to positiveness of A in $[\![\,\sigma\,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}$ and $[\![\,\tau\,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}$, and by (8) above:

$$[\![\,\Gamma\,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \models_{\mathbf{S2S}} \left([\![\,\sigma\,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}\right)^{i+1}[\bot/A](x) \quad \equiv_{df}$$

$$\equiv_{df} \;\; [\![\,\sigma\,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}\left[\left([\![\,\sigma\,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}\right)^{i}[\bot/A](x)/A\right](x) \;\; \supset$$

$$\supset \;\; [\![\,\sigma\,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}\left[\left([\![\,\tau\,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}\right)^{i}[\bot/A](x)/A\right](x) \;\; \supset$$

$$\supset \;\; [\![\,\tau\,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}\left[\left([\![\,\tau\,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}\right)^{i}[\bot/A](x)/A\right](x) \quad \equiv_{df}$$

$$\equiv_{df} \quad \left([\![\,\tau\,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}\right)^{i+1}[\bot/A](x),$$

which finishes the inductive proof. □

4.3. Restricted Consistency of $(Unfold\text{-}R)$

In general, the rules $(Unfold\text{-}R)$ and $(Unfold\text{-}L)$ are **not consistent** with $TAP^{F_\leq}_{S2S}(\Im^{\rightarrow},\Im^{\forall})$-interpretations . To understand the phenomenon we prove their consistency under additional assumptions. Simultaneously we establish independence of $(Unfold\text{-}R)$ and $(Unfold\text{-}L)$ from the other rules of $F_\leq \,\cup\, \{(\mu_R),\ (\mu^+)\}$.

Lemma 4.9 (Consistency of $(Unfold\text{-}R)$.) *Provided one of the Conditions 1 or 2 below is verified, the rule $(Unfold\text{-}R)$ preserves validity with respect to any $TAP^{F_\leq}_{S2S}(\Im^{\rightarrow},\Im^{\forall})$-interpretation , i.e., if the premise of the rule is true in a $TAP^{F_\leq}_{S2S}(\Im^{\rightarrow},\Im^{\forall})$-interpretation then so is the conclusion of the rule.* □

Proof of Lemma 4.9. Let the $TAP^{F_\leq}_{S2S}(\Im^{\rightarrow},\Im^{\forall})$-interpretation of the $(Unfold\text{-}R)$ premise be valid and conclusion not, i.e., for some x_0:

$$[\![\,\Gamma\,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \models_{\mathbf{S2S}} \forall^1 x\left([\![\,\tau\,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}(x) \supset [\![\,\sigma[\mu\alpha.\sigma/\alpha]\,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}(x)\right) \quad \text{is valid,} \tag{11}$$

$$[\![\, \Gamma \,]\!]_{\Im^\forall}^{\Im^\rightarrow} \models_{\mathbf{S2S}} [\![\, \tau \,]\!]_{\Im^\forall}^{\Im^\rightarrow}(x_0) \quad \text{is valid,} \tag{12}$$

$$[\![\, \Gamma \,]\!]_{\Im^\forall}^{\Im^\rightarrow} \models_{\mathbf{S2S}} [\![\, \mu\alpha.\sigma \,]\!]_{\Im^\forall}^{\Im^\rightarrow}(x_0) \quad \text{is NOT valid.} \tag{13}$$

From (11) and (12) we conclude:

$$[\![\, \Gamma \,]\!]_{\Im^\forall}^{\Im^\rightarrow} \models_{\mathbf{S2S}} [\![\, \sigma[\mu\alpha.\sigma/\alpha] \,]\!]_{\Im^\forall}^{\Im^\rightarrow}(x_0) \quad \text{is valid,} \tag{14}$$

which can be rewritten (**attention!**) as:

$$[\![\, \Gamma \,]\!]_{\Im^\forall}^{\Im^\rightarrow} \models_{\mathbf{S2S}} [\![\, \sigma \,]\!]_{\Im^\forall}^{\Im^\rightarrow} \Big[\, [\![\, \mu\alpha.\sigma \,]\!]_{\Im^\forall}^{\Im^\rightarrow}(x_0) \,/\, A \,\Big](x_0) \quad \text{is valid.} \tag{15}$$

By (13), $[\![\, \mu\alpha.\sigma \,]\!]_{\Im^\forall}^{\Im^\rightarrow}(x_0)$ is false, so the latter could be presented as:

$$[\![\, \Gamma \,]\!]_{\Im^\forall}^{\Im^\rightarrow} \models_{\mathbf{S2S}} [\![\, \sigma \,]\!]_{\Im^\forall}^{\Im^\rightarrow}[\, \perp / A \,](x_0) \quad \text{is valid,}$$

which immediately contradicts (13), because the $TAP_{S2S}^{F_\leq}(\Im^\rightarrow, \Im^\forall)$-interpretation of $\mu\alpha.\sigma$,

$$[\![\, \mu\alpha.\sigma \,]\!]_{\Im^\forall}^{\Im^\rightarrow}(x_0) \;\equiv_{df}\; \bigvee_{i=1}^{\infty} \left([\![\, \sigma \,]\!]_{\Im^\forall}^{\Im^\rightarrow} \right)^i [\perp/A](x_0)$$

contains $[\![\, \sigma \,]\!]_{\Im^\forall}^{\Im^\rightarrow}[\, \perp / A \,](x_0)$ as a disjunct. □

Analysis. In general, the passage from (14) to (15) in the above proof is **incorrect**. In fact, due to the shift functions $\Im^\rightarrow$ and $\Im^\forall$, the $TAP_{S2S}^{F_\leq}(\Im^\rightarrow, \Im^\forall)$-interpretation of σ, $[\![\, \sigma \,]\!]_{\Im^\forall}^{\Im^\rightarrow}(x)$ may contain together with usual occurrences Ax also "shifted" occurrences, like $A(\Im^\rightarrow x)$. But if we replace in $[\![\, \sigma \,]\!]_{\Im^\forall}^{\Im^\rightarrow}$ **all** the occurrences of A, both normal and shifted ones by the same $[\![\, \mu\alpha.\sigma \,]\!]_{\Im^\forall}^{\Im^\rightarrow}(x_0)$ as in (15), it will no more be equivalent to (15). Indeed, $A(\Im^\rightarrow x)$ should be correctly replaced by $[\![\, \mu\alpha.\sigma \,]\!]_{\Im^\forall}^{\Im^\rightarrow}(\Im^\rightarrow x_0)$. That is why Lemma 4.9 is not correct in general.

The obvious correction consists in imposing additional restrictions:

Condition 1. $\Im^\rightarrow = \Im^\forall = \varepsilon$ and applications of ($Unfold$-R) are restricted to recursive types $\mu\alpha \,.\, \sigma$ where α does not appear in a bound ρ of a subexpression $\forall\beta \leq \rho \,.\, \tau$ of σ. □

or else

Condition 2. Applications of ($Unfold$-R) are restricted to recursive types $\mu\alpha \,.\, \sigma$ where α does not appear: 1) neither in a bound ρ, nor in a body τ of a subexpression $\forall\beta \leq \rho \,.\, \tau$ of σ; 2) in a right scope of $\to$ in σ. $\square$

It is easy to verify that either condition provides correctness of the passage (14)$\Rightarrow$(15).

4.4. Even More Restricted Consistency of $(Unfold\text{-}L)$

Lemma 4.10 (Consistency of $(Unfold\text{-}L)$**)** *Provided one of the Conditions 1 or 2 above and the Condition 3 below hold, the rule* $(Unfold\text{-}L)$ *preserves validity with respect to any* $TAP^{F_{\leq}}_{S2S}(\Im^{\to}, \Im^{\forall})$*-interpretation , i.e., if the premise of the rule is true in a* $TAP^{F_{\leq}}_{S2S}(\Im^{\to}, \Im^{\forall})$*-interpretation then so is the conclusion of the rule.* $\square$

Proof of Lemma 4.10. Let the $TAP^{F_{\leq}}_{S2S}(\Im^{\to}, \Im^{\forall})$-interpretation of the $(Unfold\text{-}L)$ premise be valid and conclusion not, i.e., for some x_0 and for some particular interpretations $B_1, \ldots, B_n$ of second-order variables interpreting free type variables $\beta_1, \ldots, \beta_n$ in Γ, $\mu\alpha.\sigma$, τ we have:

$$[\![\, \Gamma \,]\!]^{\Im^{\to}}_{\Im^{\forall}} \models_{\mathbf{S2S}} \forall^1 x \left([\![\, \sigma[\mu\alpha.\sigma/\alpha] \,]\!]^{\Im^{\to}}_{\Im^{\forall}}(x) \supset [\![\, \tau \,]\!]^{\Im^{\to}}_{\Im^{\forall}}(x) \right) \quad \text{is valid,} \tag{16}$$

$$[\![\, \Gamma \,]\!]^{\Im^{\to}}_{\Im^{\forall}} \models_{\mathbf{S2S}} [\![\, \mu\alpha.\sigma \,]\!]^{\Im^{\to}}_{\Im^{\forall}}(x_0) \quad \text{is valid,} \tag{17}$$

$$[\![\, \Gamma \,]\!]^{\Im^{\to}}_{\Im^{\forall}} \models_{\mathbf{S2S}} [\![\, \tau \,]\!]^{\Im^{\to}}_{\Im^{\forall}}(x_0) \quad \text{is NOT valid.} \tag{18}$$

From (18) and (16) by contraposition we conclude:

$$[\![\, \Gamma \,]\!]^{\Im^{\to}}_{\Im^{\forall}} \models_{\mathbf{S2S}} [\![\, \sigma[\mu\alpha.\sigma/\alpha] \,]\!]^{\Im^{\to}}_{\Im^{\forall}}(x_0) \quad \text{is NOT valid,}$$

which can be rewritten (due to Condition 1 or 2) as:

$$[\![\, \Gamma \,]\!]^{\Im^{\to}}_{\Im^{\forall}} \models_{\mathbf{S2S}} [\![\, \sigma \,]\!]^{\Im^{\to}}_{\Im^{\forall}} \left[[\![\, \mu\alpha.\sigma \,]\!]^{\Im^{\to}}_{\Im^{\forall}}(x_0) \,/\, A \right](x_0) \quad \text{is NOT valid.}$$

By (17), $[\![\, \mu\alpha.\sigma \,]\!]^{\Im^{\to}}_{\Im^{\forall}}(x_0)$ is true, so the latter could be presented as:

$$[\![\, \Gamma \,]\!]^{\Im^{\to}}_{\Im^{\forall}} \models_{\mathbf{S2S}} [\![\, \sigma \,]\!]^{\Im^{\to}}_{\Im^{\forall}} [\, \top \,/\, A \,](x_0) \quad \text{is NOT valid.} \tag{19}$$

From this by Condition 3 we conclude:

$$[\![\, \Gamma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \models_{\mathbf{S2S}} [\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} [\, \bot \,/\, A \,]\, (x_0) \quad \text{is NOT valid.} \tag{20}$$

Note that $[\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} [\, \bot \,/\, A \,]\, (x_0)$ does not contain any free variable, i.e., equals simply $\bot$.

Therefore, by definition, the $TAP^{F_{\leq}}_{S2S}(\Im^{\rightarrow}, \Im^{\forall})$-interpretation of $\mu\alpha.\sigma$,

$$[\![\, \mu\alpha.\sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} (x_0) \ \equiv_{df} \ \bigvee_{i=1}^{\infty} \left([\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \right)^i [\bot/A](x_0)$$

must be also $\bot$. But this contradicts (17). □

To make the transition (19)$\Rightarrow$(20) correct, we impose the following

Condition 3. A type $\sigma \equiv \sigma[\alpha, \beta_1, \ldots, \beta_n]$ considered as a boolean function $\Psi \equiv \Phi[A, B_1, ..., B_n]$ satisfies the property:

$$\Phi[\top, B_1, ..., B_n] = \bot \ \Rightarrow \ \Phi[\bot, B_1, ..., B_n] = \bot, \ \textit{for each fixed } B_i \in \{\bot, \top\}$$

(By contraposition it is the same thing as saying that $\Phi[A, B_1, ..., B_n]$ is *monotonic* on A). □

Example. The type $\sigma_1 \equiv (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \delta)$ satisfies the above property. In fact, for $\Phi_1[A, B, D] \equiv (A \supset B) \supset (B \supset D)$ we have $\Phi_1[\top, \top, \bot] = \Phi_1[\bot, \top, \bot] = \bot$.

The type $\sigma_2 \equiv (\beta \rightarrow \alpha) \rightarrow (\beta \rightarrow \delta)$ does not satisfy Condition 3, since for $\Phi_2[A, B, D] \equiv (B \supset A) \supset (B \supset D)$ we have $\Phi_2[\top, \top, \bot] = \bot$, but $\Phi_2[\bot, \top, \bot] = \top$.

Consider now the type σ_3 obtained from σ_2 by substituting $\top$ for β and and $\bot$ for δ, i.e., $\sigma_3 = (\top \rightarrow \alpha) \rightarrow (\top \rightarrow \bot)$. Let us see where the proof of Lemma 4.10 fails for this type. Suppose, that the $TAP^{F_{\leq}}_{S2S}(\Im^{\rightarrow}, \Im^{\forall})$-interpretation of

$$\sigma_3[\mu\alpha.\sigma_3/\alpha] \equiv (\top \rightarrow \mu\alpha.\sigma_3) \rightarrow (\top \rightarrow \bot) \ \leq \ (\top \rightarrow \top) \rightarrow (\top \rightarrow \bot) \equiv \tau$$

is true (corresponding to (16)). This is indeed the case since $[\![\, \mu\alpha.\sigma_3 \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} (x) = \top$ by definition. We thus have, in particular, $[\![\, \mu\alpha.\sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} (x_0) = \top$ (which corresponds to (17)). But $[\![\, \tau \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} (x) = \bot$. So, (18) is also satisfied without contradiction. Therefore, the rule (*Unfold-L*) does not in general work correctly for the types with recursion over negative type variables. □

We thus established, quite occasionally the following

Corollary 4.11 *The rules (Unfold-R) and (Unfold-L) are independent of the other rules of* $F_{\leq} \cup \{(\mu_R), (\mu^+)\}$. □

5. Rule (μ^+): Generalizing $TAP_{S2S}^{F_{\leq}}(\Im^{\rightarrow}, \Im^{\forall})$-interpretation by Additional Parameter

Analyzing the proof of Lemma 4.7 on consistency of the rule (μ^+) we remark that the stronger result is proved (instead of (9) we proved more strong (10)). This gives possibility to generalize the definition of $TAP_{S2S}^{F_{\leq}}(\Im^{\rightarrow}, \Im^{\forall})$-interpretation for recursive types in the case of the (μ^+) rule. Just as we have a parameter $\Im^{\rightarrow}$ for the $\rightarrow$ type constructor, and a parameter $\Im^{\forall}$ for the $\forall$ type constructor, we now introduce an additional parameter $\Im^{\mu}$ for the μ type constructor.

Definition 5.1 (μ-Generalized $TAP_{S2S}^{F_{\leq}}(\Im^{\rightarrow}, \Im^{\forall})$-interpretation) *Let $\Im^{\mu}$ be an arbitrary string composed of function symbols l and r of* **S2S**. *The* μ-generalized $TAP_{S2S}^{F_{\leq}}(\Im^{\rightarrow}, \Im^{\forall})$-interpretation *is defined exactly the same way as a usual $TAP_{S2S}^{F_{\leq}}(\Im^{\rightarrow}, \Im^{\forall})$-interpretation in Definition 3.1, except for item #6, which is replaced by:*

$$6'.\quad [\![\, \mu\alpha.\sigma \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}}(x) \quad \equiv_{df} \quad \bigvee_{i=1}^{\infty} \left([\![\, \sigma \,]\!]_{\Im^{\forall}}^{\Im^{\rightarrow}} \right)^{i} [\bot/A](\Im^{\mu i} x).$$

With this definition $TAP_{S2S}^{F_{\leq}}(\Im^{\rightarrow}, \Im^{\forall})$-interpretations become more symmetric, and all three type constructors $\rightarrow$, $\forall$, μ become the first-class equal-opportunity entities. In a sense, μ-generalized $TAP_{S2S}^{F_{\leq}}(\Im^{\rightarrow}, \Im^{\forall})$-interpretations have three degrees of liberty, corresponding to all type constructors. The rule (μ^+) is compatible with any μ-generalized $TAP_{S2S}^{F_{\leq}}(\Im^{\rightarrow}, \Im^{\forall})$-interpretation :

Lemma 5.2 (Consistency of (μ^+) with respect to μ-Generalized $TAP_{S2S}^{F_{\leq}}(\Im^{\rightarrow}, \Im^{\forall})$-interpretations). *The rule (μ^+) preserves validity with respect to any μ-generalized $TAP_{S2S}^{F_{\leq}}(\Im^{\rightarrow}, \Im^{\forall})$-interpretation , i.e., if the premise of the rule is true in such $TAP_{S2S}^{F_{\leq}}(\Im^{\rightarrow}, \Im^{\forall})$-interpretation then so is its conclusion.* □

We do not yet know whether the same is true or false for other three rules considered in the paper.

Proof of Lemma 5.2. Suppose that the sequent corresponding to the μ-generalized $TAP_{S2S}^{F_{\leq}}(\Im^{\rightarrow}, \Im^{\forall})$-interpretation of the premise of the rule

$$[\![\, \Gamma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}, \forall^1 x(Ax \supset \top) \models_{\mathbf{S2S}} \forall^1 x \left([\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}(x) \supset [\![\, \tau \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}(x) \right) \quad (21)$$

is valid. We have to prove that so is the sequent corresponding to the conclusion of the rule:

$$[\![\, \Gamma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \models_{\mathbf{S2S}}$$

$$\forall^1 x \left(\bigvee_{i=1}^{\infty} \left([\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \right)^i [\perp/A](\Im^{\mu i} x) \supset \bigvee_{i=1}^{\infty} \left([\![\, \tau \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \right)^i [\perp/A](\Im^{\mu i} x) \right) \quad (22)$$

Recall that α appears in σ, τ only positively, hence, the predicate variable A will appear only positively in $\bigvee_{i=1}^{\infty} \left([\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \right)^i [\perp/A](\Im^{\mu i} x)$ and $\bigvee_{i=1}^{\infty} \left([\![\, \tau \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \right)^i [\perp/B](\Im^{\mu i} x)$.

Instead of (22) we will prove the following more strong claim: for each $i \geq 1$ it is true that

$$[\![\, \Gamma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \models_{\mathbf{S2S}} \forall^1 x \left(\left([\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \right)^i [\perp/A](\Im^{\mu i} x) \supset \left([\![\, \tau \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \right)^i [\perp/A](\Im^{\mu i} x) \right) \quad (23)$$

The base case $i = 1$, follows from (21), because $\left([\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \right)^1 [\perp/A](\Im^{\mu} x) \equiv_{df} [\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} [\perp/A](\Im^{\mu} x)$, $\left([\![\, \tau \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \right)^1 [\perp/A](\Im^{\mu} x) \equiv_{df} [\![\, \tau \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} [\perp/A](\Im^{\mu} x)$, and α does not appear in Γ by the definition of the context.

Since

$$\left([\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \right)^{i+1} [\perp/A](\Im^{\mu i+1} x) \equiv_{df}$$

$$[\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \left[\left([\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \right)^i [\perp/A](x)/A \right] (\Im^{\mu i+1} x),$$

we get, by induction hypothesis and due to positiveness of A in $[\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}$ and $[\![\, \tau \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}}$:

$$[\![\, \Gamma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \models_{\mathbf{S2S}} \left([\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \right)^{i+1} [\perp/A](\Im^{\mu i+1} x) \equiv_{df}$$

$$\equiv_{df} [\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \left[\left([\![\, \sigma \,]\!]^{\Im^{\rightarrow}}_{\Im^{\forall}} \right)^i [\perp/A](x)/A \right] (\Im^{\mu i+1} x) \supset$$

$$\supset \; [\![\sigma]\!]^{\Im\rightarrow}_{\Im\forall} \left[\left([\![\tau]\!]^{\Im\rightarrow}_{\Im\forall} \right)^{i} [\bot/A](x)/A \right] (\Im^{\mu i+1} x) \; \supset$$

$$\supset \; [\![\tau]\!]^{\Im\rightarrow}_{\Im\forall} \left[\left([\![\tau]\!]^{\Im\rightarrow}_{\Im\forall} \right)^{i} [\bot/A](x)/A \right] (\Im^{\mu i+1} x) \quad \equiv_{df}$$

$$\equiv_{df} \quad \left([\![\tau]\!]^{\Im\rightarrow}_{\Im\forall} \right)^{i+1} [\bot/A](\Im^{\mu i+1} x)$$

this finishes the inductive proof. We also have to prove that the infinite disjunction in $\#6'$ of Definition 5.1 is **S2S**-definable. But this is just a *mutatis mutandis* repetition of Lemma 4.5. □

A. Theory S2S

Definition A.1 (S2S syntax) *Let:*

- $V \;\equiv_{df}\; \{x, x_0, x_1 \ldots\}$ *be a set of* object variables *(denoted by small Latin letters);*
- $\mathcal{V} \;\equiv_{df}\; \{X, A, A_0, \ldots\}$ *be a set of* monadic (i.e., of arity one) predicate variables *(denoted by capital Latin letters);*
- $($, $)$, $=$, l, r, $\supset$, $\forall^1$, *and* $\forall^2$ *be symbols for parentheses, object equality, two unary functions, implication, first- and second-order quantification respectively.*

The minimal **S2S** *is defined by the following abstract grammar:*

$$\mathbf{S2S} \;\equiv_{df}\; \mathcal{T} = \mathcal{T} \mid \mathcal{V}\,\mathcal{T} \mid (\mathbf{S2S} \supset \mathbf{S2S}) \mid \forall^1 V\,(\mathbf{S2S}) \mid \forall^2 \mathcal{V}(\mathbf{S2S})$$

where the set of object terms is generated by: $\mathcal{T} \;\equiv_{df}\; V \mid l\,\mathcal{T} \mid r\,\mathcal{T}$. □

This simply means that: 1) terms are built of object variables using unary functions l and r; 2) atoms are either object equalities or predicate variables applied to terms; 3) **S2S** is closed with respect to implication formation, as well as first- and second-order quantification. Note that omitting parentheses in terms is unambiguous: $Alrlllrx$.

Definition A.2 (S2S semantics) *Consider the infinite binary tree T^{∞} with root Λ and branches labeled by l and r (with l for "left son" and r for "right son").*

Interpret:

- *object variables and object terms as nodes of T^∞ in such a way that if t is interpreted as a node n, then $l(t)$ is interpreted as the left son of n and $r(t)$ as the right one;*
- *predicate variables as subsets of nodes of T^∞, and an atomic formula $A(t)$ as "t belongs to the set interpreting A";*
- *$=$, $\supset$, $\forall^1 x$, and $\forall^2 A$ as usual object equality between nodes, implication, first-order, and second-order quantification over nodes and sets of nodes respectively.*

The theory $Th(\mathbf{S2S})$ is defined semantically, as the set of all **S2S** *formulas true under above interpretation in T^∞. The model consequence relation $\{\Phi_i\}_{i\in I} \models_{\mathbf{S2S}} \Psi$ (an* **S2S***-formula Ψ is true provided all Φ_i are true) is defined in a standard way.* □

All other logical connectives and existential quantifiers are definable in $Th(\mathbf{S2S})$ from $\supset$, $\forall^1$, and $\forall^2$. For example, $\top$ or $true$ is defined by $x = x$, and $\perp$ or $false$ by $lx = x$.

We reserve capital Greek letters Φ, Ψ for **S2S**-formulas. Writing $\Phi[A]$ we stress the fact that the formula Φ may contain occurrences of the predicate variable A. By $\Phi[\Psi/A]$ we denote the result of substitution of Ψ instead of all occurrences of A in Φ. As unary predicates correspond to sets, to we can freely write $R \subseteq S$ to abbreviate $\forall^1 x(Rx \supset Sx)$.

Expressing minimal sets. The theory **S2S** is sufficiently powerful to express the second-order predicate term $\mu P\,.\,\Phi[P]$, meaning: "the minimal set (predicate) P such that ...". Let $\Phi[P]$ be an **S2S**-formula possibly containing occurrences of a predicate variable P. Then the property $\Psi[\mu P\,.\,\Phi[P]/Q]$ is expressed without μ as:

$$\forall^2 R\left(\Phi[R/P]\,\&\,\forall^2 S(\Phi[S/P] \supset R \subseteq S)\ \supset\ \Psi[R/Q]\right).$$

The main fact about **S2S** we need is the following

Theorem A.3 (M. Rabin) *$Th(\mathbf{S2S})$ is decidable.* □

More on **S2S** could be found in [Rab77].

B. System $F_\leq$

For complete and exact reference see, e.g., [CG92, Pie92]. We just briefly remind the essential, retaining the notation of [Pie92]

Definition B.1 (Types) *The set of* $F_{\leq}$-types *is defined by the following abstract grammar:*

$$\mathbb{T} \equiv_{df} \mathbb{V} \mid \top \mid \mathbb{T} \to \mathbb{T} \mid \forall \mathbb{V} \leq \mathbb{T} \,.\, \mathbb{T}$$

where:

1. $\mathbb{V}$ *is a set of* type variables *later denoted by metavariables* α, β *from the beginning of Greek alphabet;*
2. $\top$ *is the maximal type majorating any other type,* $\sigma \leq \top$*;*
3. $\to$ *is the functional type constructor,* $\sigma \to \tau$ *is the type of functions with domain of type* σ *and codomain of type* τ*;*
4. $\forall\alpha \leq \rho \,.\, \tau$ *is a polymorphic type, i.e., a function assigning to each subtype* σ *of* ρ, $\sigma \leq \rho$, *the type* $\tau[\sigma/\alpha]$ *obtained from* τ *by substituting* σ *instead of* α *(with usual non-clashing preconditions on free variables). In* $\forall\alpha \leq \rho \,.\, \tau$ *the bound* ρ *does not contain* α *free.*

We reserve letters τ, σ, ρ *from the end of the Greek alphabet to denote* $F_{\leq}$*-types. By* $\forall\beta.\tau$ *we abbreviate* $\forall\beta \leq \top.\tau$ □

The type quantification $\forall\alpha \leq \rho \,.\, \tau$ introduced in the above definition is called *bounded quantification* since it enables one to impose upper bounds on possible type substitutions into polymorphic types. Despite its ambiguous name it is not restrictive (as, say, usual bounded quantification in arithmetic), because the upper bound $\top$ imposes no constraints at all.

Definition B.2 (Contexts) *An* $F_{\leq}$-context *is an ordered sequence* $\alpha_1 \leq \sigma_1, \ldots, \alpha_n \leq \sigma_n$ *of* $\leq$*-relations between type variables and* $F_{\leq}$*-types such that:*

1. *all* α_i *are different type variables, and*
2. *for each* i *all free type variables of* σ_i *are among* $\{\alpha_1, \ldots, \alpha_{n-1}\}$, *i.e.,* $FV(\sigma_i) \subseteq \{\alpha_1, \ldots, \alpha_{n-1}\}$.

Contexts will be denoted by capital Greek Γ. $Dom(\Gamma)$ *is the set of type variables appearing to the left of* $\leq$ *in* Γ. *We write* $\Gamma(\alpha) = \sigma$ *if* Γ *contains* $\alpha \leq \sigma$. *If* $\Gamma(\alpha) = \sigma$ *then* σ *is called a* bound *of* α *in* Γ. □

Definition B.3 (Subtyping Judgments) *An* $F_{\leq}$-subtyping judgment *is a figure of the form:*

$$\Gamma \vdash \sigma \leq \tau$$

where $FV(\sigma) \cup FV(\tau) \subseteq Dom(\Gamma)$. □

The intuitive semantics of a judgment $\Gamma \vdash \sigma \leq \tau$ is: σ is a subtype of τ provided that all α_i mentioned in the context Γ are subtypes of their respective bounds σ_i.

Definition B.4 ($F_{\leq}$ Axioms and Inference Rules) *$F_{\leq}$ is the system of 3 axioms and 3 inference rules, shown in Figure 1, which formalize "true" subtyping judgments*

$$\Gamma \vdash \tau \leq \tau \qquad (Refl)$$

$$\Gamma \vdash \tau \leq \top \qquad (Top)$$

$$\Gamma \vdash \alpha \leq \Gamma(\alpha) \qquad (TVar)$$

$$\frac{\Gamma \vdash \tau_1 \leq \tau_2 \; \Gamma \vdash \tau_2 \leq \tau_3}{\Gamma \vdash \tau_1 \leq \tau_3} \qquad (Trans)$$

$$\frac{\Gamma \vdash \tau_1 \leq \sigma_1 \; \Gamma \vdash \sigma_2 \leq \tau_2}{\Gamma \vdash \sigma_1 \to \sigma_2 \leq \tau_1 \to \tau_2} \qquad (Arrow)$$

$$\frac{\Gamma \vdash \tau_1 \leq \sigma_1 \; \Gamma, \alpha \leq \tau_1 \vdash \sigma_2 \leq \tau_2}{\Gamma \vdash (\forall \alpha \leq \sigma_1 . \sigma_2) \leq (\forall \alpha \leq \tau_1 . \tau_2)} \qquad (All)$$

Figure 1: $F_{\leq}$-axioms and inference rules

Let $\vdash_{F_{\leq}}$ *denote the least three-place relation* $\Gamma \vdash \sigma \leq \tau$ *containing all particular cases of the* $F_{\leq}$*-axioms and closed with respect to the* $F_{\leq}$ *inference rules.* □

The interesting facts about $F_{\leq}$ are the following

Theorem B.5 (Undecidability of $F_{\leq}$, Pierce [Pie92]) $\vdash_{F_{\leq}}$ *is undecidable.* □

Theorem B.6 ($\vdash_{F_{\leq}}$ Is Not Essentially Undecidable, [Vor94a])
There exist infinitely many different nontrivial decidable extensions of $\vdash_{F_{\leq}}$. □

C. Extensions of $F_{\leq}$

C.1. More Quantification

$$\frac{\Gamma \vdash \tau_1 \leq \sigma_1 \; \Gamma, \alpha \leq \top \vdash \sigma_2 \leq \tau_2}{\Gamma \vdash (\forall\alpha \leq \sigma_1 . \sigma_2) \leq (\forall\alpha \leq \tau_1 . \tau_2)} \qquad (\textit{All-Top})$$

$$\frac{\Gamma, \alpha \leq \rho \vdash \sigma_2 \leq \tau_2}{\Gamma \vdash (\forall\alpha \leq \rho . \sigma_2) \leq (\forall\alpha \leq \rho . \tau_2)} \qquad (\textit{All-Fun})$$

$$\frac{\Gamma \vdash \sigma \leq \rho}{\Gamma \vdash (\forall\alpha \leq \rho . \tau) \leq \tau[\sigma/\alpha]} \qquad (\textit{Curry-}\forall\textit{-elimination})$$

(**Restriction:** ρ does not contain free occurrences of α)

$$\frac{\Gamma \vdash \sigma \leq \tau}{\Gamma \vdash \sigma \leq (\forall\alpha \leq \rho . \tau)} \qquad (\textit{Curry-}\forall\textit{-introduction})$$

(**Restriction:** Γ, σ contain no free occurrences of α)

C.2. Type Intersection

$$\Gamma \vdash \sigma \cap \tau \leq \sigma \qquad \Gamma \vdash \sigma \cap \tau \leq \tau \qquad (\cap \text{ - } \leq)$$

$$\frac{\Gamma \vdash \rho \leq \sigma \qquad \Gamma \vdash \rho \leq \tau}{\Gamma \vdash \rho \leq \sigma \cap \tau} \qquad (\leq \text{ - } \cap)$$

C.3. Recursive Types

$$\frac{\Gamma,\ \alpha \leq \beta\ \vdash\ \sigma \leq \tau}{\Gamma\ \vdash\ \mu\alpha.\sigma \leq \mu\beta.\tau} \qquad (\mu_R)$$

(**Restriction:** with α only in σ, β only in τ; α, β not in Γ, [AC90])

$$\frac{\Gamma,\ \alpha \leq \top\ \vdash\ \sigma \leq \tau}{\Gamma\ \vdash\ \mu\alpha.\sigma \leq \mu\beta.\tau} \qquad (\mu^+)$$

(**Restriction:** α appears in σ, τ only positively, [AC90])

$$\frac{\Gamma\ \vdash\ \sigma[\mu\alpha.\sigma\ /\ \alpha] \leq \tau}{\Gamma\ \vdash\ \mu\alpha.\sigma \leq \tau} \qquad (\mathit{Unfold\text{-}L})$$

$$\frac{\Gamma\ \vdash\ \tau \leq \sigma[\mu\alpha.\sigma\ /\ \alpha]}{\Gamma\ \vdash\ \tau \leq \mu\alpha.\sigma} \qquad (\mathit{Unfold\text{-}R})$$

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Note. All papers of the author are available through

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