Set- and Graph-theoretic Investigations in Abstract Argumentation

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by
Christof Spanring

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Abstract

Abstract argumentation roots to similar parts in philosophy, linguistics and artificial intelligence. The core (syntactic) notions of argument and attack are commonly visualized via digraphs, as nodes and directed edges, respectively. Semantic evaluation functions then provide a meaning of acceptance (i.e. acceptable sets of arguments also called extensions) for any such abstract argumentation structure.

In this thesis, for the very first time, we tackle the questions of acceptance and conflict from a graph- and set-theoretic point of view. We elaborate on the interspace between syntactic conflict/independence (defined by attack structure) and their semantic counterparts (defined by joint acceptance of arguments). Graph theory regards the filters and techniques we use to, respectively, categorize and describe abstract argumentation structures. Set theory regards the issues we have to deal with particularly for non-finite argument sets.

For argumentation in the arbitrarily infinite case this thesis can and should be seen as reference work. For the matter of conflicts in abstract argumentation we further provide a solid base and formal framework for future research. All in all, this is a mathematicians view on abstract argumentation, deepening the field of conception and widening the angle of applicability.

Acknowledgements

A PhD thesis is a very encompassing work. As a detailed book and after years of research it involves too many people to give a detailed listing with accurate credits. I do believe that encounters of any kind shape our thinking, and our thinking shapes our work, and, circling back, our work shapes our interactions with others. I thus formally thank every single person I have ever personally met or been in contact with in this life (or another if you so believe). I thank my teachers, my role models, my idols, my guides. I thank every institution that has supported me, every physical or mental being that has inspired me. I wholeheartedly thank the countless moments I spent wondering whether my desire for abstraction was really that a good idea. I thank the music and the light, inside and out.

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Contents

Hamlet, Act III, Scene I.

Hamlet: To be, or not to be, that is the question:
Whether ’tis Nobler in the mind to suffer
The Slings and Arrows of outrageous Fortune,
Or to take Arms against a Sea of troubles,
And by opposing end them: to die, to sleep
No more; and by a sleep, to say we end
The Heart-ache, and the thousand Natural shocks
That Flesh is heir to? ’Tis a consummation
Devoutly to be wished. To die, to sleep,
To sleep, perchance to Dream; aye, there’s the rub,
For in that sleep of death, what dreams may come,
When we have shuffled off this mortal coil,
Must give us pause. There’s the respect
That makes Calamity of so long life:
For who would bear the Whips and Scorns of time,
The Oppressor’s wrong, the proud man’s Contumely,
The pangs of despised Love, the Law’s delay,
The insolence of Office, and the Spurns
That patient merit of the unworthy takes,
When he himself might his Quietus make
With a bare Bodkin? Who would Fardels bear,
To grunt and sweat under a weary life,
But that the dread of something after death,
The undiscovered Country, from whose bourn
No Traveller returns, Puzzles the will,
And makes us rather bear those ills we have,
Than fly to others that we know not of.
Thus Conscience does make Cowards of us all,
And thus the Native hue of Resolution
Is sicklied o’er, with the pale cast of Thought,
And enterprises of great pitch and moment,
With this regard their Currents turn awry,
And lose the name of Action. Soft you now,
The fair Ophelia? Nymph, in thy Orisons
Be all my sins remembered

William Shakespeare [Wik16j]

Chapter 1

A less formal Introduction into Abstract Argumentation

This chapter is dedicated to discussing and introducing the main concepts of this thesis in an informal way. We give motivation and explanation without going into too much detail. All of the necessary formal definitions will follow in the subsequent chapters of Part I. We continue by briefly sketching a formal context of abstract argumentation.

What is abstract argumentation about? Abstract argumentation as first formalized by Phan Minh Dung in [Dun95] is exactly what its naming indicates: a formal abstraction of the intuitive concept of argumentation. To understand the meaning of this naming we therefore have to understand both words. Argumentation, the second but characterizing part is a mode of communication aimed at establishing justification of statements. For its clarity and comprehensiveness we cite the following.

> ***Argumentation** is a verbal, social, and rational activity aimed at convincing a reasonable critic of the acceptability of a standpoint by putting forward a constellation of propositions justifying or refuting the proposition expressed in the standpoint.* (Eemeren and Grootendorst [vEG04])

Further considering the abstract part means that we try to get rid of the contextual meaning of arguments and put forward a system designed for application in various fields that deal with argumentation but only roughly resemble each other. To this end Dung emphasized a central notion of acceptability of arguments, referred to as semantics. His only tools for developing abstract argumentation frameworks are abstract arguments and abstract attacks as a directed two-valued relation between such arguments. Acceptability and hence semantics do not consider intrinsic meaning of arguments but rather evaluate argumentation frameworks solely based on these abstract notions.

Research often starts with abstraction. We take a couple of real life problems, observe similarities, define a common base and derive conclusions for each different area. For instance numbers, as the abstraction of the concept of counting, can be used to compute such diverse things as an income tax, the statics of a throne room, the chances of win vs. defeat in a multi-army battle, or the necessary amount of water for harvesting an acre of wheat. Research usually goes a step further and investigates abstraction on its own without initially caring about useful application. This can for instance be seen in the development of prime number theories, which were a popular mathematical topic for centuries before finding profound application in encryption methods with the uprising of computers.

An overview of abstract argumentation, particularly argumentation in artificial intelligence can be found in [BD07]. An introduction to argumentation theory from a more philosophical point of view can be found in [Wal09]. A historical outline of various branches of argumentation can be found in [vEGH96]. In [Tou03] Toulmin gives an analysis of arguments in philosophical terms and in [Ham70] Hamblin interprets argumentation as a two-player game.

As far as applications are concerned argumentation in general and abstract argumentation in particular have found purposes in vastly different areas. We have work on negotiation [APM00, ADM07, DTT08] and incorporation of objective evaluation in law and legal reasoning [BCD05, BCPS09]. There are developments in multi-agent systems and game theory [MRPM10, KAK^{+}11, Mod09], there is work on decision making and recommendation [Cer11, CMS07], and finally argumentation can not only be implemented via machine learning, but the argumentation process itself can help machines to learn [MŽB07].

The remainder of this introductory chapter is organised as follows: First, in Section 1.1, we present a practical motivation to illustrate where arguments might come from and what (abstract) argumentation can be used for. Then, in Section 1.2, we elaborate informally on the theoretical structures used. In Section 1.3 we give an outline of the remaining parts and chapters of this thesis, relating our results to our prior publications.

1.1 Hamlet, a practical example

For the purpose of this introduction we face one essential challenge in that in this thesis we do not investigate applications of immediate practical value. That is we consider research questions that naturally arise when looking at abstract argumentation structures. There is a downside to this approach. Since we do not present solutions for distinct practical problems, any examples we discuss are artificial in nature. The immediate upside of this obstacle is that it allows us to separately construct examples to our liking. As running example for this chapter we thus choose Hamlet's famous soliloquy as presented on Page 1.

In formal argumentation we deal with abstract arguments and abstract attacks between arguments, structures we call argumentation frameworks. These basic structures can be evaluated with so-called semantics, which are mappings from argumentation frameworks to sets of arguments. Semantics as "*study of meaning*" [Wik16g] can be defined as rules that regulate how a formal language is to be interpreted. The intended meaning of such semantics in the field of argumentation is that the resulting sets of arguments are acceptable or justified under

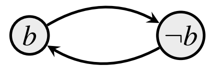


Figure 1.1: "*To be, or not to be*", visualized as an argumentation framework, b: "*to be*" and $\neg b$: "*not to be*", cf. Example 1.1.

some objective conditions. The origins of such formal argumentation frameworks might for instance be natural language arguments, logic programs, fact-finding databases, or moves in multi-player games. The intrinsic nature of all of these origins is a non-monotonic one: at each point in time we can add further arguments, further formulas, reveal further details of a board game, and so on; and by doing so influence the already established justification states. Further, by investigating argumentation frameworks and justifiable sets of arguments we can use insights to suggest tactics for modifications of the origins. Apparently such modifications can be seen as manipulation. And manipulation often comes with a moral taint. However also detection of manipulation and thus tracking of moral ambiguities require in depth knowledge of manipulative possibilities.

Example 1.1. To be, or not to be: that is Hamlet's opening question and that is the question we are going to investigate for the remainder of this section. For an interpretation of meaning it can be seen as juxtaposition of the states of being alive and being dead. From a philosophical point of view it can also be interpreted as a comparison of passivity and activity. In terms of argumentation it simply poses two statements that contradict each other: "*to be*" and "*not to be*". In abstract argumentation terms we can assign arguments to these statements and call them b and $\neg b$. Further the contradiction between these statements can be implemented as a bidirectional attack, i.e. b attacks $\neg b$ and $\neg b$ attacks b. Voilà, we receive a first argumentation framework $F_0 = (A_0, R_0)$ with argument set $A_0 = \{b, \neg b\}$ and attack set $R_0 = \{(b, \neg b), (\neg b, b)\}$. Such argumentation frameworks often are visualized as graphs, where arguments are represented by circles (with argument names inside) and attacks are represented as arrows (where the attacked argument is placed at the tip of the arrow), cf. Figure 1.1.

Accepting argument sets: For this first framework, without further information intuitively both arguments appear to be equally justifiable. However it seems reasonable that they are not justifiable in conjunction, i.e. either we accept b or we accept $\neg b$ (or we accept neither). For the remainder of this chapter we will use an argumentation semantics later on referred to as stable semantics, which for a set S of arguments to be acceptable requires

1. no two arguments in S attack each other;
2. and for each and every argument b that is not a member of S we want some member a of S to attack b.

The intuition behind these conditions is that in case we propose a series of statements we do not want to contradict ourselves (1), and that we want to be able to have a counter argument for

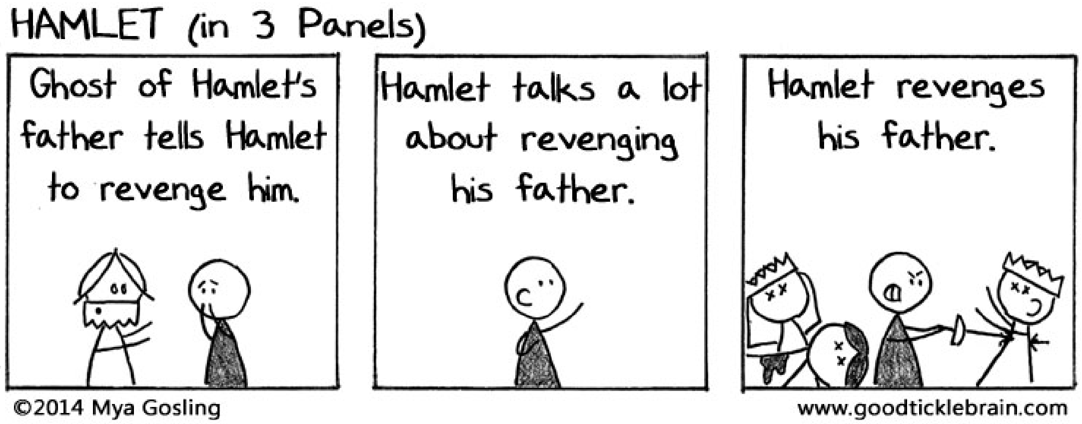


Figure 1.2: William Shakespeare's play Hamlet as visualized by Mya L. Gosling `http://goodticklebrain.com/home/2014/2/6/three-panel-plays-part-4`

any other argumentation strategy (2). The acceptable sets of arguments for F_0 thus are $\{b\}$ and $\{\neg b\}$. Now that we have established this notion of acceptance we can continue by harvesting Hamlet's soliloquy for further arguments and attacks.

Observe that this thesis is not a work on harvesting arguments from natural language. Further, words and sentences in general are of ambiguous nature and only very rare cases are likely to provide unique interpretations regarding their meaning. Our interpretation of Hamlet's soliloquy is thus to be seen as an illustration of how argumentation frameworks might be constructed, a motivation for how we handle abstract argumentation structures. In no means we claim that our interpretation as such is "*correct*" or "*justified*".

As is the case with natural language, lots of words and sentences do not provide any interpretation in our strict sense of argumentation. For instance, if Hamlet says "*Whether 'tis Nobler in mind to suffer The Slings and Arrows of outrageous Fortune*", it does not contribute to our abstract argumentation knowledge base as such. This statement merely is a more elaborate description of "*to be*", the statement we already named as b. However be aware that such details might lead to attacks from or to future arguments. And such details might even turn into arguments themselves, for instance if it becomes necessary for the argument to decide if "*to be*" actually involves suffering or not. However, in the case of Hamlet's soliloquy we will assume that this is his personal definition and not doubt his definition any further.

Context is of importance when acquiring abstract argumentation structures from natural language examples. The chosen text is taken from William Shakespeare's play Hamlet, which tells the story of a young man (Hamlet) whose father recently passed away. Before the start of the play Hamlet's uncle, Claudius the new king, and Hamlet's mother Gertrude married; and then there is also the love story between Hamlet and Ophelia, which is set before the play but continues interfering with characters and story line. The drama runs its course when Hamlet's father appears as a ghost (visible only to Hamlet) and claims to have been murdered by Claudius, exacts and is granted revenge. Hamlet decides to "*put an antic disposition on*" (to act as if he turned mad) to hide his plans. Act II can be seen as development and unfolding of the story, where further light is also shed on Polonius, Ophelia's scheming father. At the beginning of Act III (where the soliloquy under investigation takes place) Ophelia is used as bait by Polonius and Claudius to gather more information on the nature of Hamlet's madness. Only at the end of the soliloquy Hamlet seems to recognize Ophelia who is about to return his love letters. Meanwhile Polonius and Claudius hide behind a curtain and watch, which is interesting as in a similar situation later in the play Hamlet erroneously stabs and kills Polonius. The overall story arc of Shakespeare's play is compactly sketched in Figure 1.2. It should be noted though that most of the main characters (including Hamlet himself) die till the curtains are drawn.

Example 1.2 (Argumentation Structure of Hamlet's Soliloquy)**.** Now, the soliloquy can be seen as a philosophical discourse on the why of not die. It also reflects the moral dilemma Hamlet is dealing with over the course of all five acts. It seems morally justified to avenge his father by killing his uncle, but killing itself is a moral no go. Morality (especially the religious kind) also forbids suicide. Hence b, the act of not acting could be characterized as purgatory of inactivity. Consequently, Hamlet attributes b with "*a Sea of troubles*" in Line 4, "*the thousand Natural shocks That Flesh is heir to*" in Line 7 and lists a few samples in Line 15 "*Scorns of time, The Oppressor's wrong, the proud man's Contumely, The pangs of despised Love, the Law's delay ...* ", indicating that there is a sheer infinite amount of reasons (i.e. arguments) against b. We refer to these reasons as $t_1, t_2 \dots$. However in Line 10 we read "*to Dream; aye there's the rub*", which is described as "*the dread of something after death*" in Line 23. The interpretation is that we just don't know, uncertainty (further on referred to as u) about the "*what if*" remains, it is identified as humanities number one reason for b and hence argument against $\neg b$. Finally, one can recognize this uncertainty and doubt the solution called suicide, or as Hamlet puts it in Line 24, "*The undiscovered Country ... Puzzles the will*". As p we hence define this last argument attacking all previous reasons against b, namely $t_1, t_2 \dots$. Hamlet goes on to say "*Thus Conscience does make Cowards of us all*" in Line 28, which seems to further elaborate on p, but does not substantially influence any other of the established arguments. We depict the so far gathered (infinite) argumentation structure in Figure 1.3.

Following up on Example 1.1, we now have an argumentation framework $F_1 = (A_1, R_1)$ with $A_1 = A_0 \cup \{u, p, t_1, t_2, t_3 \dots\}$ and $R_1 = R_0 \cup \{(u, \neg b)\} \cup \{(p, t_i), (t_i, b) \mid i \in \{1, 2, 3 \dots\}\}$. As far as acceptable arguments are concerned this changes quite something. As u is not attacked it should be accepted and defeats $\neg b$. Similarly p should be accepted, defeats t_i and hence b is defended and should thus be accepted. For F_0 from Figure 1.1, the abstract argumentation interpretation of "*to be, or not to be*", the conclusion is that both statements b and $\neg b$ are equally justified. At this point in time (after Act III, Scene I) the conclusion is that b is justified, while $\neg b$ is not.

Hence following the story line of Hamlet the play, Hamlet is torn between passivity and activity, between life and death. In the scene of interest, on the surface he ponders whether

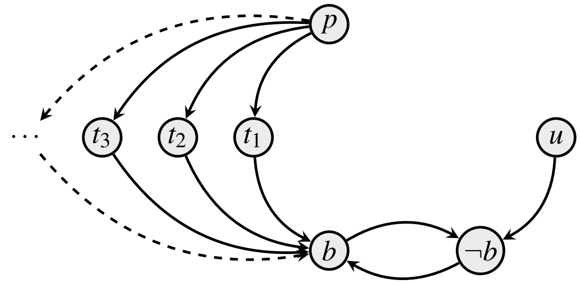


Figure 1.3: "*To be, or not to be*", Hamlet's soliloquy from Page 1 visualized as an argumentation framework. We augment Figure 1.1 with p: puzzles, u: uncertainty, t_i: troubles of life, cf. Example 1.2

suicide might be a solution to the troubles of life. He identifies staying alive with inaction and suicide with action, which in the context of the play however can also be translated as: inaction as playing along with the people surrounding him and action as avenging his father. The conclusion we draw from the soliloquy is that at this point he tends to favour life, inaction and distrusting the ghost (or even his very own mind and senses). Indeed it will take him a long journey of betrayal, scheming and not particularly good human (i.e. gone wrong) decisions to finally avenge his father.

The given abstract argumentation framework F_1 contains an infinite amount of arguments, which in the current state are rejected and thus not considered relevant. It seems natural to consider frameworks with an infinite amount of arguments. Although in each instance of time and humanly perceivable knowledge there can be only a finite amount of generating information, it does not take much fantasy to imagine an unbounded range of similarly constructed arguments. Already in [Dun95] Dung elaborately discusses infinite argumentation frameworks and points out the relation with logic programming, where naturally a finite knowledge base leads to an infinite amount of related formulas. ASPIC+, as very well presented in [MP14], was intended to be a more natural approach of generating abstract argumentation frameworks. There we have structured arguments, which essentially are (defeasible) proofs of statements. The generation of arguments and attacks in this field of research is intended to be automatic. Remarkably, such freedom immediately yields generation of infinitely many arguments with enclosed attack relations. In [BCDG13] the authors introduce and discuss automata for infinite argumentation structures. The idea is that regular languages (and in a certain way also natural languages) provide an unrestricted domain of possible words and sentences, yet build them with only finitely many rules and finite alphabets. Hence there is considerable work on infinite argumentation structures. In this thesis we extend the area of interest to cover also frameworks where notably we do not know how they were built.

To further elaborate on the art of argumentation we will extend the running example a bit. In Example 1.2 we pointed out that Hamlet refers to conscience that takes away humanities courage. In the context of the play we do not explicitly know how much content of the soliloquy

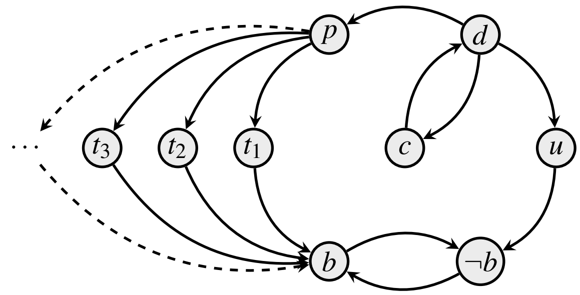


Figure 1.4: "*To be, or not to be*", Example 1.2, Figure 1.3 augmented by d: determination and c: conscience, as discussed in Example 1.3.

Claudius and Polonius are able to perceive, nor can we know what they might make of it. After all, it is an emotional speech and Hamlet is an emotionally driven young man. Questions as the discussed are probably asked by almost every human in their early adulthood and forgotten later. For the sake of illustration we assume that Claudius and Polonius are argumentation experts and instinctively analyze the speech for the abstraction of arguments and attacks to arrive at the same conclusions as we did in Example 1.2.

Example 1.3. Now we have Polonius and Claudius about to act. Instinctively they feel that Hamlet's speech is of great importance. After all, it evolved to become the probably most famous line from theater nowadays. We can further assume that for their private reasons they do not mean Hamlet well. Polonius might be in fear of losing his beloved daughter, Claudius might be in fear of losing his beloved crown or even life. They interpret Hamlet's "*to be, or not to be*" literally and thus belief that Hamlet was close to suicide but for now life won the argument.

Now Polonius and Claudius are about to get involved in Hamlet's mindset. They discuss matters and arrive at the conclusion that Hamlet is simply missing d: determination. Determination to jointly solve his indecisiveness u and the problem of his puzzled mind p. Hence they might decide that in order to get rid of their fear and thus Hamlet they could introduce d into Hamlet's argumentation scheme. Whether they consider his stance on c: "*conscience*" we do not know. What we do know however is that Hamlet implicitly stated that c is in contradiction with d: "*Conscience does make Cowards of us all*", Line 28.

How does one add determination to the mindset of a person? Claudius apparently decides to send Hamlet on a diplomatic mission to England, hoping that his troubled mind comes to senses. After adding determination to the framework F_1 from Example 1.2 the new framework $F_2 = (A_2, R_2)$ appears as $A_2 = A_1 \cup \{c,d\}$ and $R_2 = R_1 \cup \{(d,p),(d,u),(c,d),(d,c)\}$, cf. Figure 1.4.

After doing the math we now get that the set $\{d, \neg b, t_1, t_2 \ldots\}$ becomes justified. However due to c this is not the only justified set, we still have $\{c, p, u, b\}$. Hence once again, there are justifications for b as well as for $\neg b$. In the elders defence, manipulation of argumentation structures is not an easy task. Unbeknownst arguments might arise and influence the outcome significantly. Still, in F_1 there was no justification for $\neg b$, in F_2 there is, so the introduction of determination could also be regarded as successful manipulation. Now it is again up to Hamlet to decide whether "*to be, or not to be*" is the wiser solution.

Manipulation of argumentation structures for the purpose of achieving certain justifiable sets can be seen as an art form in itself. One needs to consider given necessities, such as established arguments and attacks. One needs to consider the common knowledge base of the argumentation participants. One needs to consider the desired outcome. And most of all one needs to consider the interplay of knowledge base, framework and outcome. For instance regardless of how good we have become at manipulating argumentation structures, in the given case we will not be able to convince Hamlet of a simultaneous realization of b and $\neg b$. Hence there is no modification such that sets that contain both arguments b and $\neg b$ are justified.

We would need to read or watch the full play for being able to see the full amount of interaction between Hamlet and Claudius. In short, there is rather inapt human interaction from both parts, intentional as well as unintentional. For the sake of this section we focus on manipulation. In this thesis we will discuss how to use the concept of conflict for active manipulation of argumentation structures. One must not forget that Hamlet is a theater play, written by William Shakespeare. As author Shakespeare has personal interest in developing a dramatic and entertaining story. In this respect manipulation is by far easier to handle. If Shakespeare manipulates the characters of his play into portraying certain consistent story lines he is in almost full control of involved arguments. Hence manipulation must also be seen as a tool for creators. Not only is this point of view less controversial from an ethical perspective, we believe it also far better captures the actual possibilities of manipulation.

Example 1.4. Now, without factually knowing what Shakespeare has schemed for his play Hamlet, we take a look at the story development and manipulation in regards of previous examples. The big question is "*To be, or not to be?*", where only Shakespeare fully knows whether it is about suicide or whether it is about "*To act, or not to act?*", and in the sense of Hamlet's ghostly father about "*To revenge, or not to revenge?*".

Hamlet and Claudius persistently play tricks on each other. Hamlet stages a play in the play telling the story of a brother-murderer to get to Claudius. Claudius tries to get rid of Hamlet, first for Hamlet's good by bringing him to his senses on a diplomatic mission, then for his bad by scheming for his termination abroad (on that very diplomatic mission). As Hamlet never arrives in England both approaches are without success. What Claudius succeeds at however is angering Hamlet. As the play nears its end, Hamlet becomes more and more enraged. When Gertrude drinks the poisoned wine (schemed for Hamlet) and a poisoned blade mortally wounds Hamlet as well as Ophelia's brother Laertes, the latter finally reveals Claudius' scheme, r: *rage* takes over and Hamlet kills Claudius.

And in the killing of Claudius in rage by Hamlet, Shakespeare's scheme unfolds in front of our witnessing eyes. Rage is the missing argument for the soliloquy and we can finally complete the resulting argumentation framework. We identify the missing attacks from r to p and u, as well as from d to r and hence starting from the framework F_2 from Example 1.3 we

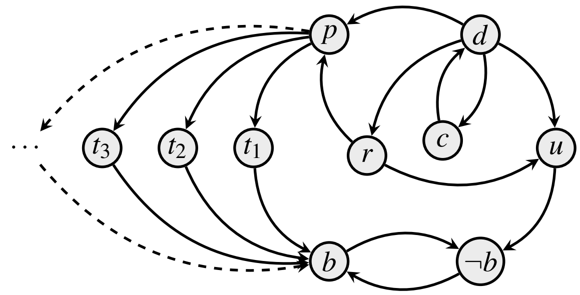


Figure 1.5: "*To be, or not to be*", Example 1.3, Figure 1.5 augmented by conflict manipulation by r: rage, as discussed in Example 1.4.

construct the framework $F_3 = (A_3, R_3)$ where $A_3 = A_2 \cup \{r\}$ and $R_3 = R_2 \cup \{(r,p),(r,u),(d,r)\}$, cf. Figure 1.5.

For F_3 and justification of arguments we first have to decide whether we accept d or c, the other arguments follow immediately and we hence receive the two justifiable sets $\{d,\neg b,t_1,t_2\ldots\}$ and $\{c,r,\neg b,t_1,t_2\ldots\}$. So for the resulting argumentation framework F_3 finally, among b and $\neg b$ only $\neg b$ is justified. If we look closely, the purpose of r in this light is to introduce an implicit conflict between c and b.

Conclusively Hamlet delivers both, revenge and his own death. In the light of Shakespeare's scheming we get that the attempts of resolving Hamlet's bewilderment by Claudius actually result in activating r: *rage* and thus are key to enabling Hamlet to finally act. In case Claudius did consider Hamlet's soliloquy and tried to manipulate him, he made the mistake of interpreting it solely as an argument about suicide, while for Hamlet's actual mindset the art of acting apparently included killing his uncle and avenging his father. The plot of "*Hamlet*" the play hence can be seen as a masterpiece of manipulation by Shakespeare.

1.2 The Argumentation Process

In Section 1.1 we discussed a case study of argumentation. We investigated the interplay of language and the argumentation process. In this section we will investigate the argumentation process on a more abstract level.

Let us start with discussing the argumentation work flow as depicted in Figure 1.6. We have fields related to the general field of argumentation or states of argumentation data in circles and arrows connecting these states representing numbered transitions in between. Observe that dashed lines represent the conventional approach while dotted lines are less commonly considered. In the following we discuss the diagram in detail and relate it to Section 1.1, the running example of Hamlet's soliloquy "*to be, or not to be*".

Initially we start with some knowledge base, for instance a live discussion on the radio [MRH16], or a legal case [PS15], or clinical evidence [HW15]. In the case of Hamlet this is the soliloquy from Page 1, the text we identified as a natural language argumentation process

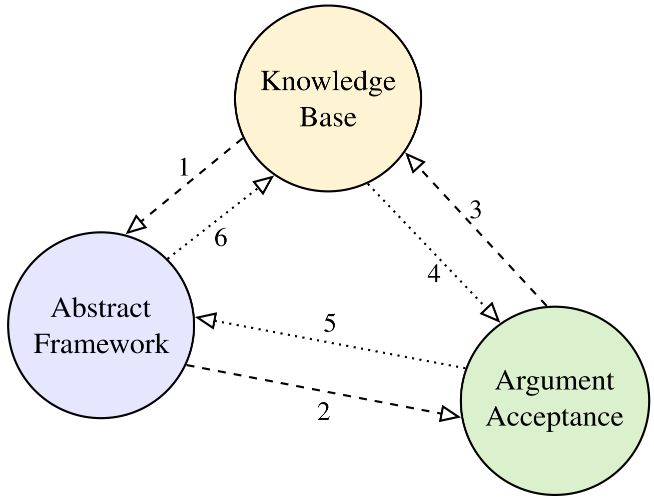


Figure 1.6: A schematic argumentation work flow.

delivered by one person (either Hamlet on stage, or Shakespeare as the original author). For practical purposes we might also start with a question, such as who wins an argument, whether the defendant is guilty, or which treatment of a clinical patient the system recommends. For Hamlet the question we focus on is "*to be, or not to be*", his own initial question as discussed in Example 1.1.

We then derive (transition 1) an abstract argumentation framework. In general we will need to make sure that this transition (and any other) are accurate. For most real world applications this means that we need to be able to refer to established standards and a witnessing community of sorts. In the case of Hamlet this transition as well as the presentation of the resulting framework takes place in Example 1.2. As emphasized we can not give an objective view of argumentation on the soliloquy, not least because we merely use such examples for illustration in this thesis. Accurateness in this context thus mainly means that to the best of our knowledge and ability we fully harvest the soliloquy and interpret all available relevant arguments.

From this argumentation framework we compute (transition 2) justification states, the acceptability of arguments. This means that we obtain scenarios that justify acceptance of arguments that can be considered plausible. In discussions (and in legal cases) we will in general receive a mixture of arguments from all participating parties for any such scenario. For a medical database we would hope for association of symptoms with explanations, diagnosis and treatments. In the case of Hamlet we reach the conclusion that "*to be*" is justified, while "*not to be*" is not.

Finally we use the justification states to draw conclusions (transition 3) for the real world application. In a discussion on the radio an opinion research institute might claim a winner or a suitable compromise, in a legal case ideally we will have a decision or maybe even better an objective overview of the case, and for the clinical evidence similarly we expect qualified persons to be able to give medical advice to their patients based on the acquired data. Hamlet, after the soliloquy, is in a state of mind and being that prevents him from committing suicide. In another interpretation he is not yet ready to trust his father's ghost and kill his uncle.

Naturally argumentation processes are rarely complete, additional data, additional arguments can occur. If we have to adjust the given knowledge base, we will also have to adjust the abstract framework and thus the justification states and the conclusions. Ideally such adjustments would not interfere with prior conclusions. However, due to the non-monotonic nature of argumentation, interference is part of the concept, i.e. changes to the knowledge base will result in possibly vastly different interpretations. The benefit of argumentation though is that we model arguments, hence for reasonable implementations we should expect the modifications of the argumentation framework to be of monotonic nature. It is though debatable whether established arguments might be able to vanish, i.e. be forgotten after some time.

If we intend to manipulate data, to influence the audience of the argumentation process, we might use the knowledge base to interact (transition 4) with desired justification states and derive (transition 5) a suiting argumentation framework the other way around. This time for manipulated data to evaluate back (transition 6) into the real world application.

For a radio show, manipulation would mean for instance live evaluation and target-oriented intervention by a moderator or discussion participant. In law court manipulation can be regarded as adaption of the strategy of any involved lawyer. In the medical case manipulation might be considered by pharmaceutical companies to tweak their medical studies to better match a broader range of common symptoms. In a more sinister way companies could also design such automated assistants to highlight their own product's benefits over the one from their competitors. For Hamlet we discuss the ups and downs of manipulation in Examples 1.3 and 1.4. In particular in Example 1.3 we observe that manipulation is not necessarily armed against setback from previous knowledge. In Example 1.4 we learn that the only way of ensuring a desired outcome of some manipulation is to author the full knowledge base, like Shakespeare did with his play. Totalitarian regimes, populists and dictators alike, tend to strive for this undistributed authorship by suppressing opposing opinions. History however shows that such attempts never work for too long a time.

In the context of this thesis we are interested in transitions 2 and 5 only. In Part II we consider some arbitrary and possibly infinite Abstract Argumentation Structure as given and discuss acceptance questions in the light of Zermelo Fraenkel set theory. Remarkably, this has not been investigated before in the literature. In Part III we put emphasis on acceptable sets of arguments, or to be precise induced conflicts by any such collection of sets. We then discuss implications for the formal structure as well as structural modifications forcing desired acceptance/conflict outcomes.

1.3 Structure & Publications

Who is to blame for progress and advances in science and society? The one to ask the question or the one to give the answer? Sadly in our society (maybe for the rare exception of Hilton's famous list) we attribute solutions with the one giving the answer rather than the one posing the question. In this sense we harvest the credit for providing solutions while most research questions we started with did originate from someone or somewhere else. The scientific community relies on input of others, on collaboration and joint work. Still, even in joint work we may rightfully attribute parts to one or the other of the researchers. In the common sense, all research presented in this thesis was conducted by the author of this thesis unless otherwise indicated. We now briefly outline the remainder of this thesis.

Part I **(Abstract Argumentation)** is a formal presentation of abstract argumentation from a mathematical point of view. Some results and definitions there, for instance ZFC or the Bourbaki-Witt theorem, regard well-established principles. Other results are our insights we consider important but too trivial to be called a contribution. Triviality applies to our observations (Theorem 5.17, first published in [Spa16b]) on collapse vs. crash/interference/contamination. We also consider as trivial our observation (Example 4.23, published in [BS17]) of the original definition of cf2 semantics not being well-defined in the infinite case. Our intended contribution of Chapters 2 to 5 thus is to provide necessary information and relevant definitions in a structured way. Although some of it might be seen as relevant scientific work, we feel no need to brag about trivial matter.

- In Chapter 2 **(Mathematical Foundations)** we introduce and discuss set and graph theory as the basis of our investigations further on. This chapter is intended as a refresher as well as an initiation for the intricacies and most important results regarding ZFC, the axiom of choice and graph theory.

- In Chapter 3 **(Syntax of Abstract Argumentation)** we relate directed graph theory with abstract argumentation and discuss relevant results that are of structural nature.

- In Chapter 4 **(Semantics of Abstract Argumentation)** we expand the area of interest to argumentation semantics, our chosen methods of evaluation when given abstract argumentation structures.

- We consider syntax, as the graph theoretic foundation of abstract argumentation; and semantics as a mode of giving meaning to syntactic structures. This distinction is mostly for didactic reasons and allows us to further on derive natural structuring. In Chapter 5 **(Preliminary Properties of Abstract Argumentation Semantics)** we discuss the interspace between syntax and semantics.

Given the formal tools from Part I, we identify two main areas of interest, two main research questions of this thesis. How do semantics behave in the general infinite case? And what is the relation between semantic and syntactic conflicts?

In Part II **(Infinite Argumentation Structures)** we discuss the case of arbitrarily infinite argumentation frameworks. As highlighted in Chapter 2, intuition developed for finite examples in general does not carry over to the infinite case.

- In Chapter 6 **(Existence of Extensions and Set Theoretic Principles)**, built on our research from [Spa14], we elaborate on the trip wires we may encounter in the realm of set theory with respect to abstract argumentation.

- In Chapter 7 **(Collapse)**, built on our research from [Spa15c], we discuss our findings regarding the collapse of abstract argumentation semantics. That is, we give examples of argumentation structures not providing any acceptable set of arguments and relations of such occurrences in respect of different semantics.

- In chapter 8 **(Perfection)**, built on our research from [Spa16b] and [BS16], we discuss the opposite question, i.e. cases where argumentation structures and any substructure do not collapse.

Thus, in Part II we investigate questions of acceptance and the behaviour of semantics in the general infinite case. Acceptance is most prominently considered for single arguments or sets of arguments. What if we are interested in relations between such acceptable sets? What if we are interested in pairs of (sets of) arguments?

In Part III **(Conflict and Expressiveness)** we take a closer look at pairs of arguments and their acceptance relations. In particular we are interested in conflicts, the case where argument (sets) are not jointly acceptable. The dual question is that of independence, the case where argument (sets) are (pairwise) jointly acceptable. We differentiate between explicit and implicit conflicts, that is conflicts with and without structural counterpart. Expressiveness, as the second keyword in this part, regards extension sets. Extension sets are commonly used to describe semantic evaluation and thus the output of meaning given some argumentation structure. In particular this means that we might be dealing with equivalence classes of argumentation structures, defined via equal semantic evaluation.

- In Chapter 9 **(Necessity)** we focus on the conflicts with structural counterparts. That is, we relate and discuss conflicts as defined by extension sets with the syntactic structures of arguments and attacks. Research from this chapter has been discussed at various venues (for instance [Spa16a, Spa16c]), but not been published yet.

- Chapter 10 **(Purity)**, partially published in our work from [LSW15, BDL$^+$16, DSLW16], then deals with conflicts without structural counterparts. To this end we discuss argumentation structures with explicit conflicts only, extension sets that do not allow such explicit realizations and semantic conflicts that are never realized syntactically.

Finally in Chapter 11 **(Conclusions)** we connect the dots and present possible future research directions.

Part I

Abstract Argumentation

Chapter 2

Mathematical Foundations

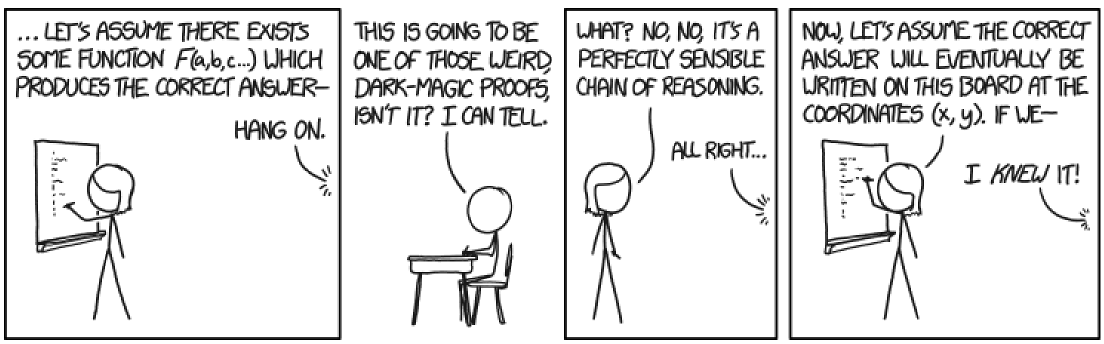


Figure 2.1: Next, let's assume the decision of whether to take the Axiom of Choice is made by a deterministic process ... `https://xkcd.com/1724/`

In this chapter, for the purpose of self-containedness and to make sure the terms we make use of later on in the thesis are not ambiguous in nature, we present and discuss the mathematical foundations of abstract argumentation. Figure 2.1 illustrates that mathematical work, proofs and definitions might be seen as miraculous magic without actual evidence. This chapter is dedicated to explaining the relevant magic for this thesis, we sincerely hope that we do a better job at this than the character in the comic strip.

Abstract argumentation has repeatedly been approached from a computational point of view [Dun95, Dun07, BB10, DW10], with the particular limits and aims of computability, i.e. finite or finitary sets and approaches that are motivated to a great deal by benefits for computations. In some respect this can be seen as naive argumentation, i.e. in the sense of physics before relativity theory or more accurately set theory before Cantor. This is a reasonable approach, as there might not be any need for argument systems without structural restrictions and for sure we can not compute such systems with contemporary technology.

In Section 2.1 we mainly present and discuss examples highlighting peculiarities of possibly infinite sets. In Section 2.2 we provide methods, i.e. the well established theory surrounding ZFC, to formally deal with such infinite sets. In Section 2.3 we discuss implementation of set theoretic methods for mathematics and formal computer science and further introduce the axiom of choice. In Section 2.4 we switch to another formal matter, graphs and digraphs, which can (and in this thesis will) be used as formal basis for abstract argumentation as well.

2.1 Naive Set Theory

At the very base of formal foundations of both, mathematics and computer science, resides propositional logic, the rules by which we apply logical operations. It was shown to be complete [Gen35] and consistent [Göd30], but does not provide quantification and thus allows sentences only over finite domains. Basically, if we add quantification over variables to propositional logic we get predicate logic. Kurt Gödel [Göd31] showed that predicate logic, arithmetics and any mathematical theory that allow an intuitive definition of the natural numbers either lose completeness or consistency. This means that if we deal with statements over an infinite domain (also see [Ruc04] for a human perspective) we might either end up with a theory that is not consistent and hence proves everything, or it is not complete and hence does not know about the truth content of some sentences.

On first glance Gödel's kind of artificial constructions do not seem to pose a threat to the belief that computational systems can eventually derive every useful theorem. However, for instance the for computer scientists well known Halting Problem [Tur37] and as later discussed the controversy around the Axiom of Choice [Jec73] (and thus Zorn's Lemma) are examples that show otherwise.

Since we will be dealing with potentially infinite structures one might of course also wonder whether there should not be a rather straightforward definition for the foundational theories regarding the matter. We continue with a collection of classical examples to highlight the potentially problematic issues at hand.

The following example (attributed to David Hilbert) is an illustration of why the use of infinite numbers greatly differs from the use of finite numbers. In particular we immediately lose some expressiveness regarding comparability. In the finite case given two natural numbers $a = b$ we get $a < b+1$, in the infinite case it is difficult to speak about equality in the first place.

Example 2.1 (Hilbert's Hotel [Gam67])**.** Suppose there is a hotel with an infinite number of rooms. Of course this hotel is able to host an infinite number of guests. Further assume that the hotel is filled, i.e. each room hosts a guest. Now a new guest knocks on the door and the concierge seems to have to tell him that they are fully booked.

But no, there is another solution. The concierge proceeds by moving the guest from the first room to the second room, the guest from the second room to the third room, and so on. After finishing (we don't go into technical details regarding the physical process here) this reassignment, each old guest is still assigned to some room and the first room is empty. Which means that the concierge can assign the first room to the new guest.

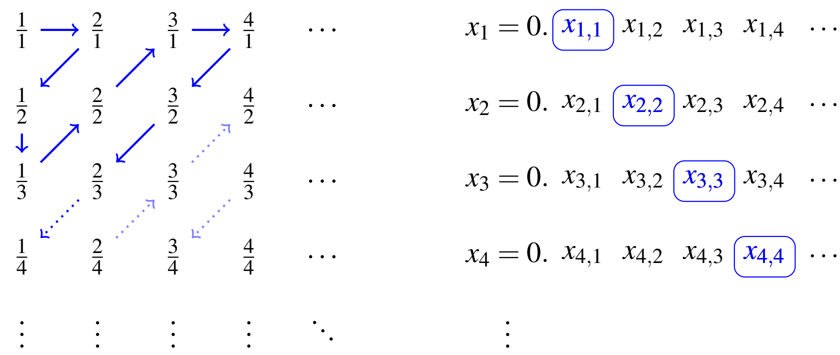
By induction we get that the same procedure works for any finite number of new guests. How about an infinite amount of new guests?

Assume the rooms to be numbered as $r_1, r_2, \ldots$, the old guests as $o_1, o_2, \ldots$ and the new guests as $n_1, n_2, \ldots$. We attempt to empty every second room, or every odd-numbered room to be precise. Let us first send o_1 to r_2 to empty r_1. Next we need to reassign old guests o_2 and o_3, which we do by sending them to rooms o_4 and o_6. Next we need to reassign the old guests o_4, o_5, o_6, o_7. Intuitively this should work, the formula however is even more simple. For any number i we simply assign the old guest o_i to the room r_{2i} and the new guest n_i to the room r_{2i-1}. Since even and odd numbers do not overlap with this scheme we can assign a room to each of the infinitely many new guests while keeping all the old guests in house. Similarly we can extend this approach to host new guests from infinitely many buses with infinitely many occupants each.

One natural question to ask regarding the previous example is whether there actually are different kinds of infinity, whether there are sets of new guests Hilbert's Hotel is not able to provide rooms for. In the following example we collect two proofs attributed to Georg Cantor, the first one giving hope that infinity might still be "just" another kind of number, the second immediately crushing this hope. We will use the established term *cardinality* to refer to the size of sets and hence a measurement for infinities.

Example 2.2 (Cantor's Diagonal Arguments [Can92])**.** It can be observed, that the cardinalities of natural and integer numbers are the same, i.e. there is a bijection between them. Since we use natural numbers to count objects, we refer to sets that have the same cardinality as the natural numbers as countable sets. In the following we investigate the sets of, first, rational numbers ($\mathbb{Q}$: numbers x that can be expressed as fractions $x = \frac{p}{q}$ of an integer p and a non-zero natural number q) and, second, real numbers ($\mathbb{R}$: numbers x that can be expressed as a countable sequence $x = \pm x_0 x_1 \ldots x_m . x_{m+1} \ldots$ of digits of some finite base $x_i \in \{0, 1, \ldots n\}$).

- $\mathbb{Q}$ **is countable**: w.l.o.g., we consider only positive rational numbers $x \in \mathbb{Q}^+$, i.e. numbers $x \in \mathbb{Q}$ with $0 < x$ and show that there is an injective mapping $\mathbb{Q}^+ \to \mathbb{N}$. We choose as presentation of $\mathbb{Q}^+$ a table with entries $\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \ldots; \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \ldots; \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \ldots; \ldots$. This a table has a starting point at the top left, but ending points neither to the right nor to the bottom, cf. Figure 2.2a. Each positive rational number is listed in this table an infinite amount of times, e.g. $\frac{1}{1} = \frac{2}{2} = \frac{3}{3} = \cdots$. For the required mapping we count in a diagonal way (hence the name) by assigning each fraction in the table a natural number. As illustrated in Figure 2.2a we assign $\frac{1}{1} \mapsto 1$; then as downward diagonal $\frac{2}{1} \mapsto 2$, $\frac{1}{2} \mapsto 3$; then as upward diagonal $\frac{1}{3} \mapsto 4$, $\frac{2}{2} \mapsto 5$, $\frac{3}{1} \mapsto 6$; then downward $\frac{4}{1} \mapsto 7$ and so on.
- $\mathbb{R}$ **is uncountable**: For the real numbers first observe that any chosen base, for instance representation as decimal numbers (with digits ranging from 0 to 9) allows for ambiguity. We can not distinguish between $1.0000\ldots$ and $0.9999\ldots$ as these two numbers do not have a gap in between. Such numbers however are rational and this ambiguity hence concerns only a countable amount of decimal numbers.



(a) $|\mathbb{N}| = |\mathbb{Q}|$ (b) $|\mathbb{N}| < |\mathbb{R}|$

Figure 2.2: Cantor's diagonal proofs illustrated, cf. Example 2.2.

W.l.o.g., we consider only decimal numbers x of the restricted range $0 < x < 1$ and assume a countable listing $(x_i)_{i\in\mathbb{N}}$ of each such number as given. As illustrated in Figure 2.2b this listing can be written as $x_1 = 0.x_{1,1}x_{1,2}x_{1,3}\ldots$; $x_2 = 0.x_{2,1}x_{2,2},x_{2,3}\ldots$; $x_3 = 0.x_{3,1}x_{3,2}x_{3,3}\ldots$; $\ldots$, where $x_{i,j} \in \{0,1,\ldots,9\}$. It remains to show that there are $0 < y < 1$ such that $y \notin (x_i)_{i\in\mathbb{N}}$. As illustrated in Figure 2.2b the given listing $(x_i)_{i\in\mathbb{N}}$ implicitly defines a number y different from all x_i. We use the diagonal $x_{i,i}$ to define $y = 0.y_1y_2y_3\ldots$, where $y_i = 4$ whenever $x_{i,i} = 5$ and $y_i = 5$ otherwise. By construction y is neither ambiguous nor member of the list $(x_i)_{i\in\mathbb{N}}$ and thus $\mathbb{R}$ is not countable.

With the preceding example we actually had a jump start into set theory. As will be seen, set theory in several aspects is an attempt of dealing with the issues arising from different cardinalities of different sets. At the end of the nineteenth century, before Gödel's incompleteness proofs, when the hope of formalizing all of mathematics was still high, problems such as the above already provided evidence that sets (especially the infinite ones) would not be as easily captured as hoped for.

Example 2.3 (Russell's Paradox [Rus03, p. 101])**.** Russell's paradox could be renamed as the quest of defining a set of all sets, or the question of whether the collection $\mathscr{U}$ containing all sets is itself a set. If $\mathscr{U}$ is the *universe* of all sets (if X is a set then $X \in \mathscr{U}$), does it hold that $\mathscr{U} \in \mathscr{U}$? The somewhat surprising answer is: No. The universe of all sets can itself not be a set.

In a bit more detail assume $\mathscr{U} \in \mathscr{U}$. We further assume that for a given set X, variable symbol x and a predicate $\tau(x)$ with x as only free variable we have that also $\{x \in X \mid \tau(x)\}$ is a set.[1] It follows that also $\mathscr{X} = \{x \in \mathscr{U} \mid x \notin x\}$ is a set.

Now the question of whether or not $\mathscr{X} \in \mathscr{X}$ seems a natural follow up. If $\mathscr{X} \in \mathscr{X}$ by definition however we get $\mathscr{X} \notin \mathscr{X}$ as it should contain only those sets that do not contain themselves. If $\mathscr{X} \notin \mathscr{X}$ on the other hand again by definition we are required to let $\mathscr{X} \in \mathscr{X}$ as

[1]This principle will be called *restricted comprehension* after formal definition in Section 2.2 and is in general considered a desired property of sets.

it would not contain itself. Hence we get $\mathscr{X} \in \mathscr{X} \iff \mathscr{X} \notin \mathscr{X}$, a contradiction leading to $\mathscr{X}$ not being a set and thus also $\mathscr{U}$ should not be a set.

The preceding example illustrated why not every mathematical object can be a set, and hence modern set theory distinguishes between sets and proper classes, the latter describing mathematical collections of sets. As far as the paradox is concerned we could continue by asking for classes of classes, a class of all classes and so on. As far as this work is concerned however we settle for remarking that we will need a concise definition of what a set is. To this end in Section 2.2 we will introduce and further discuss Zermelo-Fraenkel set theory and its $8+1$ axioms [FBHL73]. The most important takeaway from set theory probably is, that most mathematical objects of relevance can be described as sets.

It was shown by Gödel [Göd31] that consistent mathematical theories covering natural number arithmetics (and hence set theory) always contain sentences that within the theory can neither be proven nor disproven. Consequently, the set theory of our choice (if consistent) will provide such statements. Out of nine axioms of set theory not all are undisputed, the most controversy however arises around the axiom of choice. It was disputed for some time whether incompleteness of mathematical theories actually concerned relevant sentences, as Gödel's construction of course was rather artificial. This dispute did lead to so called *independence proofs* [Kun83], i.e. research dedicated to showing that meaningful statements can be assumed both true and false with regards to the same starting theory.[2] The axiom of choice was one of the surprising representing statements and subsequently did arise in different forms in many areas. To illustrate intuition as well as controversy we follow up by an example attributed to Bertrand Russell.

Example 2.4 (Russell choosing socks and shoes [Rus93, pp. 125-127])**.** Assume you do have a collection of pairs of shoes, and another collection of pairs of socks. The task is to give a rule, selecting exactly one representative for each of these pairs. For the shoes one such rule would be to select all the left shoes. Due to their symmetric nature for the socks this rule does not work out. Consequently, to select exactly one sock from each pair of socks we do need further information on the overall setting.

For a more formal investigation regarding the problem touched in above example we refer to [HT06]. There the authors use this example as starting point for a surprising look at set theory without general choice. For another peek into the matter and intuitively different approach we further refer to Example 6.21 and Section 6.3 on variations of choice.

Choice does provide a very intuitive definition for a rather complicated issue. For practical purposes we often prefer to use an alternative formulation, that is we will use the equivalent Zorn's Lemma [Zor35]. While Axiom of Choice by naming is clearly marked as an assumption we might not be able to prove, Zorn's Lemma seems to suggest that there is a proof. In an

[2] Where under the assumption that the original theory did not contain contradictions also its extension with the statement or its negation is free from contradictions.

axiomatic sense however each one implies the other. We have been searching the literature and trying to come up with intuitive examples illustrating Zorn's Lemma, but in the end decided to follow up with an example closer to the further on discussed matters.

Example 2.5. Consider any subset of the power set of the natural numbers, i.e. $X \subseteq \wp(\mathbb{N})$. For any two members $x, y \in X$ exactly one of four cases holds, either $x = y$, or $x \subset y$, or $y \subset x$, or x and y are incomparable. For any chain of comparable members $x_1 \subseteq x_2 \cdots \subseteq x_n \in X$ clearly x_n is a maximum. Further even for infinite chains $x_1 \subseteq x_2 \cdots \in X$ we have that $\bigcup_i x_i$ is an upper bound. Assume that for each infinite chain $x_1 \subseteq x_2 \cdots \in X$ the upper bound is a member of X as well, i.e. $\bigcup_i x_i \in X$. Is it safe to conclude that X has a maximal element, i.e. some $x \in X$ such that there is no $y \in X$ with $x \subset y$?

This question should be inspiration and not lead to frustration, as Zorn's Lemma basically states that the answer is yes. Together with the knowledge of the equivalence between Zorn's Lemma and the Axiom of Choice however this means that in general there might not be a constructive way of selecting such maximal elements.

Regarding the incompleteness theorems one is probably best advised with [Smo77], a survey on the matter. At this point we also want to mention the halting problem [Tur37] as a practical issue dealing with incompleteness. As far as references regarding set theory in general and naive set theory in particular are concerned probably the first work to mention is [Hal60], a mathematician's introduction into the matter particularly highlighting common misconceptions and mistakes. From the perspective of a set theorist we recommend [Dev94] as an extended introduction, also [Hau14] and its translation [Hau62] as the first and still relevant handbook on set theory, as well as [Sup60] with a more up-to-date bibliography. Finally, if it comes to set theory the ultimate, most recent and seemingly all-encompassing book of course is Jech's Millenium edition [Jec06],

2.2 Axiomatic Set Theory

The mathematical approach to dealing with the previous examples (or as they are sometimes paraphrased: anomalies and paradoxes) is that we try to clear our mind on what we actually want to say and formalize the language in which we are doing so. Since Euclid's Elements, *a collection of definitions, postulates (axioms), propositions (theorems and constructions), and mathematical proofs of the propositions*, published circa 300 BC [Wik16a], this effort is commonly attempted by an axiomatic approach. Zermelo-Fraenkel set theory is nowadays widely considered as the base of mathematics and gives a formal framework to discuss infinity. The following definition introduces an important infinite set.

Definition 2.6 (Infinity)**.** A set x is called *inductive* if $\emptyset \in x$ and for each $y \in x$ also $y \cup \{y\} \in x$. The smallest inductive set is commonly referred to as ω.

The basic intuition behind axiomatic set theory is that it assumes that every mathematical object can in its essence be described as a set. For instance we might identify the number 0 with $\emptyset$, the empty set; and the number 1 with $\{\emptyset\}$, the set containing the empty set. Further the axioms supposedly reflect principles that are deemed desirable. We will carry forward by giving critical remarks on such desires and a discussion on their use. Observe that as variables (sets, sets of sets or elements of sets) we strictly use only lower letters here, mainly as to set theory everything is a set. Be aware that we further assert (the axioms of) predicate logic, the details of which we do not discuss but might be looked up for instance in [Smu95]. The language of set theory extends first order predicate calculus with *equality* ($=$) and *membership* ($\in$).

In the following definition we list the eight[3] axioms of interest as given on the first content page of [Jec06]. For a more formal discussion see [Jec06, Chapter 1] or [Wik16l]. Notably, as pointed out in Example 2.3 the collection of all sets is not a set itself. We will use the term *class* to refer to arbitrary collections of sets and *proper class* if the collection in question is not a set.

Definition 2.7 (Zermelo-Fraenkel Set Theory (ZF-Axioms) [Jec06, page 3])**.**

1. Axiom of Extensionality (EXT)
   If x and y have the same elements, then $x = y$.

2. Axiom of Pairing (PAIR)
   For any a and b there exists a set $\{a,b\}$ that contains exactly a and b.

3. Axiom schema of Restricted Comprehension (RCO)[4]
   If P is a property (a formula in the language of set theory), then for any x there exists a set $y = \{u \in x \mid P(u)\}$ that contains all those $u \in x$ that have property P.

4. Axiom of Union (UN)
   For any x there exists a set $y = \bigcup x$, the union of all elements of x.

5. Axiom of Power Set (POW)
   For any x there exists a set $y = \wp(x)$, the set of all subsets of x.

6. Axiom of Infinity (INF)
   There exists an inductive set.[5]

7. Axiom schema of Replacement (REP)
   If a class F is a function, then for any x there exists a set $y = F(x) = \{F(u) \mid u \in x\}$.

8. Axiom of Regularity (REG)
   Every non-empty set has an $\in$-minimal element.

[3]Existence of some set is sometimes listed as zeroth axiom, but can also be deduced.
[4]This is called *Axiom Schema of Separation* in [Jec06].
[5]By definition inductive sets have infinitely many members. Together with RCO thus ω is a proper set.

Remark 2.8. Observe that PAIR does not define pairs (or tuples) but rather sets with at least two given members. That is, for the resulting set z we have $\{x,y\} \subseteq z$, by RCO we then get $z' = \{x,y\}$. However even in z' we do not distinguish between $\{x,y\}$ and $\{y,x\}$. The established mathematical definition [Wik16e] of *ordered pair* or *two-tuple* (using REP and PAIR) would be $(x,y) = \{\{x\},\{x,y\}\}$, which also allows for a formal definition of n-tuples by induction.

Remark 2.9 (Restricting comprehension). Observe the subtle definition of RCO, where we allow comprehension only for sets whose existence we already have ensured otherwise. That way constructions such as $\{x \mid x \notin x\}$, cf. Example 2.3, are not possible.

Of course there is quite some historical development on ZF, for the interested reader Wikipedia [Wik16l] by now is a recommendable starting point. There are additions to above axioms that allow further insights into set theory (for instance Neumann-Bernays-Gödel or Kripke-Platek set theory), modifications that allow so-called urelements or exotic approaches that allow the universe of all sets to be a set (such as New Foundations). However ZF (and ZFC as discussed below) has gained the most acceptance, and again we refer the interested reader to a discussion about axiomatic set theory on Wikipedia [Wik16h].

ZF is aimed at reducing mathematics only to the most agreeable intuitions. This pedantic approach leads to pedantic formulas, which is the main reason we decided to omit formal and hence formulaic definitions of ZF axioms. Observe that in above definition this still leads to some cumbersome wordings, such as in PAIR we do not require $\{x,y\}$ to be a set itself, but rather to be a subset of some set z.[6]

By nature axiomatic set theories fall prey to incompleteness [Fra05, Ber11, Ber13] and hence the never-ending doubt of possible inconsistencies. This is one reason that set theorists sometimes give an extra thought on which axioms they actually use in their proofs, just in case one of them turns out to be malfunctioning. Another reason is independence and the spirit of competition in minimizing theories and thus maximizing impact.

If a given theory (such as ZF) is assumed to be consistent, one might be able to show that an augmented theory (such as ZF+choice as discussed below) is still consistent. If one is then able to show that the negatively augmented theory (ZF+¬choice) is also consistent, this is called independence. Independence hence is compatibility of a theory with some claim as well as with its counterclaim. One much disputed independence to ZF is the axiom of choice. Before we introduce the axiom of choice however we first discuss how mathematics can be developed under ZF.

2.3 Formal Mathematics

As mentioned earlier for arithmetics we can interpret the empty set as the number zero. Although intuitively arithmetics should also incorporate addition, the formal Peano axioms [Pea99] do not

[6]Also observe that in other definitions (e.g. UN) in this listing we did let go of cumbersome definitions for the benefit of readability.

require such sophistication. Rather arithmetics uses more basic definitions and derives further operations such as addition and multiplication from them. We proceed by illustrating how Peano arithmetics (and hence Gödel's incompleteness) is embedded in ZF.

Example 2.10 (Peano-Arithmetics in ZF)**.** Giuseppe Peano in 1899 formalized natural numbers and regarding operations in an axiomatic way [Wik16d]. We append a ZF-interpretation of natural numbers, the so-called von Neumann construction.

1. The number zero is identified with the empty set, i.e. $0 = \emptyset$. Existence of the empty set is for instance granted by INF (there is an infinite set x) and RCO (we may restrict x to none of its members, $\emptyset = \{y \in x \mid y \neq y\}$).

2. Given a natural number i, the successor function S is defined as $S(i) = i \cup \{i\}$. Since all natural numbers are defined as sets, the image under S is granted by *REP*. We hence recursively identify $1 = 0+1 = S(0) = \{\emptyset\} = \{0\}$, $2 = 1+1 = S(1) = \{\emptyset, \{\emptyset\}\} = \{0,1\}$, $3 = 2+1 = S(2) = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0,1,2\}, \ldots$

3. The Peano axioms further require that 0 is not a successor to any number and that numbers whose successor is the same already are the same themselves. Both statements are granted by our definition.

4. The collection of natural numbers $\mathbb{N}$ is a set by INF and RCO, i.e. $\mathbb{N} = \omega$, the smallest inductive set.

To highlight its importance we separate the above construction from the principle of induction. The Peano axiom of induction formally states that given some set of natural numbers x where $0 \in x$ and for each $y \in x$ also $S(y) \in x$ then already $x = \mathbb{N}$. Obviously this holds for the given interpretation. The use of induction now is that,

1. given a *base case*, i.e. a property φ that holds for 0,

2. and the *inductive step*, i.e. if φ holds for natural number i it also holds for $S(i)$,

3. we can conclude that φ holds for all natural numbers.

Remark 2.11 (Arithmetics and other number sets). The Peano axioms only require a successor function and no other arithmetical functions. The reason for this is that definition of a successor function suffices to derive further functions. For instance addition is recursively defined as $i+0 = i$ and $i+S(j) = S(i+j)$. For the given mathematical operations we can then further define the set of integers $\mathbb{Z}$, the set of rational numbers $\mathbb{Q}$, the set of real numbers $\mathbb{R}$ and so on.

While mathematical induction is a very useful tool, in the case of arbitrary infinities it does not help in proving statements for bigger sets. As highlighted in Example 2.2 the set of real numbers $\mathbb{R}$ is not countable, i.e. is distinctively bigger than $\mathbb{N}$. However there is a bijection between $\mathbb{R}$ and $\wp(\mathbb{N})$ (so-called Dedekind cuts). More generally the same example works for

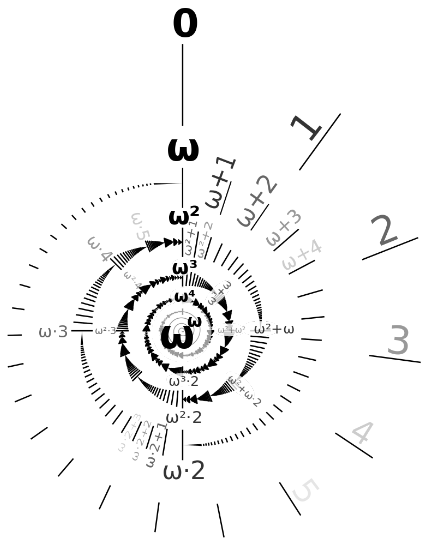


Figure 2.3: This spiral represents all ordinal numbers less than ω^ω, cf. Example 2.15. `https://commons.wikimedia.org/wiki/File:Omega-exp-omega-labeled.svg`.

arbitrary sets x and their by *POW* granted power set $\wp(x)$, i.e. there is no bijection between x and $\wp(x)$. This immediately means that not only $\mathbb{R}$ is bigger than $\mathbb{N}$, but also that $\wp(\mathbb{R})$ is bigger than $\mathbb{R}$ and so on. To deal with such infinities we introduce the notion of ordinals for being able to formally talk about the size of sets and subsequently extend induction to bigger sets. The following definitions closely follow [Wik16f].

Definition 2.12 (Well-Ordering)**.** A set together with a binary relation $>$ is a *well-ordering* if

- $>$ *is total:* For all members x, y, exactly one of the statements $x > y$, $x = y$ or $y > x$ is true.
- $>$ *is transitive:* For any elements x, y, z if $x > y$ and $y > z$ then $x > z$.
- Every non-empty subset has a *least* element, that is, it has an element x such that no element that is $< x$ is in the subset, where $y < x$ is another way of saying $x > y$.

Definition 2.13 (Ordinals)**.** A set x is an *ordinal* if and only if x is strictly well-ordered with respect to set membership ($\in$) and every element of x is also a subset of x. Two well-ordered sets have the same *order type* if and only if there is a bijection from one to the other that converts the relation in the first one to the relation in the second one.

Remark 2.14 (Order types). Given a well-ordered set, we define its *ordinal number* as the unique ordinal of the same order type. An ordinal number can only be used to describe the order type of a well-ordered set and not the order type of a well-ordered proper class. We distinguish between *successor ordinals* x where there is an ordinal y such that $x = y \cup \{y\}$, and *limit ordinals* x without predecessor where $x = \bigcup_{y<x} y$. By definition there are no other ordinals.

Example 2.15. Hence the set of natural numbers $\mathbb{N}$, which according to Example 2.10 is the same as the smallest inductive set ω, is an ordinal. Each natural number is an ordinal as well. Further we have that $\omega+1=\omega\cup\{\omega\}$ is an ordinal and so on, cf. Figure 2.3. Ordinal arithmetics subsequently defines $x+1=x\cup\{x\}$. It holds that each ordinal α contains only and all those ordinals smaller than α.

Apparently ordinal numbers are an extension of natural numbers. By incorporating a limit step into mathematical induction we can also extend this principle to achieve transfinite induction, i.e. statements over all ordinals or any well-ordered set.

Theorem 2.16 (Transfinite Induction)**.** *Consider some well-ordered set α and some ordinal β. If α is inductive and its order type is β, then already $\alpha=\beta$. In the same way there is only one inductive class, the class of all ordinals.*

Definition 2.17 (Transfinite Induction)**.** For cases where the principle of induction does not suffice we may use the following:

1. Given a *base case,* i.e. a property φ that holds for 0,
2. the *inductive step*, i.e. if φ holds for some ordinal α it also holds for $S(\alpha)$,
3. and the *limit step*, i.e. for any limit ordinal λ if φ holds for all $\alpha<\lambda$ then it holds for λ,
4. we can conclude that φ holds for all ordinal numbers.

Often one is not interested in properties of ordinals but rather constructions that exceed any limiting ordinal. Thus transfinite induction can be seen as unnecessary baggage and one might use *transfinite recursion* instead. For transfinite recursion *we construct a sequence of objects, one for each ordinal.* [Wik16k].

Observe that ordinals as such compare order types, not exactly the size of sets in terms of equal size for sets with bijective functions. For instance although ω and $\omega+1$ have different order type there are bijective mappings, such as $0\mapsto\omega$ and for $i>0$: $i\mapsto i+1$.

Definition 2.18 (Cardinals)**.** Consider some set x. If there is a smallest ordinal α such that there is a bijection between x and α then α is called the *cardinality* or *cardinal number* of x. In case x has a cardinality α, we write $|x|=\alpha$. For a cardinal α we call α^+ the successor cardinal of α which is the smallest cardinal bigger than α.

Example 2.19 (Hilbert's hotel revisited)**.** Recall Example 2.1. Intuitively Hilbert's hotel provides ω many rooms. We assume that every room is occupied. Apparently, making space for a finite number of additional guests does not pose a problem. As observed before we can even host ω many additional guests, and as can easily be seen this still applies for $\omega*\omega$ and ω^n for any $n\in\omega$ many arrivals. What happens if we consider ω^ω additional guests? We will advance on this question in Example 2.24.

We have briefly touched Zorn's Lemma before. In order to being able to give a formal definition we are still missing a few formal notations.

Definition 2.20 (Partial Order, Chain, Maxima)**.** Given a set x, a *partial order* is a relation $\sqsubseteq$ that is

- *reflexive:* for each $a \in x$ we have $a \sqsubseteq a$;
- *transitive:* for any $a, b, c \in x$ with $a \sqsubseteq b$ and $b \sqsubseteq c$ we also have $a \sqsubseteq c$;
- and *anti-symmetric:* for any $a, b \in x$ with $a \sqsubseteq b$ and $b \sqsubseteq a$ we already have $a = b$.

In case $a \sqsubseteq b$ and $a \neq b$ we may write $a \sqsubset b$. Observe that we do not require the relation to be the set-theoretic subset-relation in this definition, however for our purposes we will never be far.

Given a set x with a partial order $\sqsubseteq$, any sequence $(a_i)_{i<\alpha}$ for some ordinal α of members $a_i \in x$ is called a *chain* (if there is danger of confusion we may write $\sqsubseteq$-chain) if for each $i < j < \alpha$ we have $a_i \sqsubseteq a_j$. Finally, given some set $y \sqsubseteq x$, a member $a \in y$ is called a *maximum* of y if for each $b \in y$ we have $b \sqsubseteq a$, it is called *maximal* if there is no $b \in y$ with $a \sqsubseteq b$.

We now enter the question of whether each set has a cardinality. Without further ado we thus introduce three (with regards to ZF) equivalent statements. The remarkable aspect is that given ZF each of these statements implies the others [Jec73] and neither follows from ZF alone.

Definition 2.21 (Choice, Zorn and Well-Ordering, ZFC)**.** We will use ZFC to refer to ZF combined with AC, i.e. ZFC is Zermelo-Fraenkel set theory with choice.

- Axiom of Choice (AC)
  For each set x of non-empty sets there is a choice-function $f\colon x \to \bigcup x$ selecting one member of each $y \in x$: $f(y) \in y$.
- Zorn's Lemma (ZL)
  If for some partially ordered set x every chain has a maximum in x, then x contains at least one maximal element.
- Well-Ordering Principle (WO)
  Every set can be well-ordered.

Remark 2.22 (A very brief history of set theory). Before the 1870s (and Georg Cantor and Richard Dedekind) mathematicians did not elaborately discuss set theoretic issues. Only after discovering several paradoxes (such as Example 2.3) the need for an axiomatic approach to the basic objects called sets was acknowledged. Ernst Zermelo in 1908 proposed the first such axiomatic set theory which was slightly adjusted to better match expectations by Abraham Fraenkel and Thoralf Skolem in 1922 to further on be called Zermelo-Fraenkel set theory [Wik16l]. It was also Ernst Zermelo who first formulated AC in 1904 to prove the well-ordering theorem [Zer04]. Georg Cantor considered the well-ordering theorem as a “*grundlegendes Denkgesetz*” [Can83], a substantial principle of thought already in 1883. As with WO, also ZL was initially proven to follow from ZFC by Kazimierz Kuratowski in 1922 and independently by Max Zorn in 1935 [Moo12]. The equivalence followed later and in particular independence of AC and ZF was only shown by Kurt Gödel's constructible universe in 1938 [GB40] and the counterpart by Paul Cohen's Forcing method in 1963 [Coh63]. Nowadays by a quantitative measure probably most research areas assume ZFC to be the legit base of mathematics. It is still noteworthy that there are well-researched models of set theory which contradict some of the ZFC-axioms. For instance the Axiom of Determinacy (discussed in Section 6.4) is in contraposition to the Axiom of Choice.

Controversy and confusion around the equivalence of AC, WO and ZL are probably quite common. After all, as human beings every object we encounter is strictly finite and there is no proof of any existing material infinities. Hence intuition regarding infinite objects is naturally hard to come by. Hardly anyone ever summarized this observation better then Jerry Bona.

> “*The Axiom of Choice is obviously true, the well-ordering principle obviously false, and who can tell about Zorn's lemma?*”
> – Jerry Bona [Kra02]

In the line of our arguing about cardinals it becomes apparent that sets that have a cardinality immediately can also be well-ordered. It can be shown [Kun83] that AC, ZL and WO are independent from ZF and hence it is consistent with ZF to assume sets that do not have a cardinality. Under the assumption of ZFC every set on the other hand can be well-ordered and hence has a cardinality. As the following remark however highlights, cardinality even in the case of ZFC is not a straightforward concept.

Remark 2.23 (CH and GCH). We already know that for any set α its power set $\wp(\alpha)$ has bigger cardinality, i.e. $|\alpha| < |\wp(\alpha)|$. We also know that $|\mathbb{R}| = |\wp(\omega)|$. What we do not know however is whether $|\mathbb{R}|$ is the next cardinality after ω. The assumption that this is the case is known as *continuum hypothesis (CH)*, and the general assumption that the cardinality of the power set of any infinite cardinal is the smallest bigger cardinal ($\omega \leq \alpha \implies |\alpha|^+ = |\wp(\alpha)|$) is known as *general continuum hypothesis (GCH)*.

Georg Cantor assumed CH to be true and it was the first on David Hilbert's famous list of important questions from 1900 [Wik16c]. Kurt Gödel showed consistency of CH and GCH with ZF as well as with ZFC [GB40]. The technique of forcing on the other hand allows for construction of models of ZFC where GCH does not hold [Coh63]. Wacław Sierpiński showed that ZF together with GCH implies AC and hence AC and GCH are not independent.

Example 2.24 (Hilbert's hotel and infinite sets of guests)**.** Recall Example 2.1 and the question posed in Example 2.19. As we could observe in Example 2.2 there is no bijective mapping between $\mathbb{N}$ and $\mathbb{R}$. This observation is equivalent to the fact that, even if Hilbert's hotel is empty to begin with, if there is a set of guests arriving where each guest is uniquely identified by some $x \in \mathbb{R}$, the concierge has to deny most of the customers accommodation. Apparently, it is consistent with ZF that $\mathbb{R}$ does not have any well-ordering and thus $|\mathbb{R}|$ is not necessarily meaningful. In ZFC however $\mathbb{R}$ necessarily provides an ordering, but as highlighted by the controversy around CH and GCH, ZFC still does not guarantee unique well-orderings.

Around 1950 Nicolas Bourbaki [Bou49] and Ernst Witt [Wit50] presented a result regarding fixed points we will repeatedly make use of and hence present in the following. We will also append a short proof to illustrate working with set theoretic operators. The theorem strongly relies on a lemma by Friedrich Hartogs [Har15] from 1915, which states that for any set x there is a least ordinal α such that there is no injection from α into x. This basically means that regardless of the set considered there is always a bigger ordinal. Observe that neither the theorem nor the lemma require AC, but already work within ZF. Hence cardinality could also be defined without WO as equivalence classes of sets. This distinction however is not important for this thesis, but we will make use of the fact that the Bourbaki-Witt theorem does not require AC.

Definition 2.25 (Chain complete posets)**.** A *partially ordered set* (*poset*) is a set x together with a partial order $\sqsubseteq$ on x. It is called *chain complete* if every chain $y \sqsubseteq x$ in it (cf. Definition 2.20) has a least upper bound $sup(y) \in x$.

Theorem 2.26 (Bourbaki-Witt)**.** *If x is a non-empty chain complete poset and $\varphi \colon x \to x$ is such that $x \sqsubseteq \varphi(x)$ for all x, then φ has a fixed point.*

Proof. Pick some $y \in x$ and define a function f recursively on the ordinals as $f(0) = y$, $f(\alpha + 1) = \varphi(f(\alpha))$, and for limit ordinals β with $\{f(\alpha) \mid \alpha < \beta\}$ being a chain in x we get $f(\beta) = sup(f(\alpha) \colon \alpha < \beta\})$.

Due to Hartogs' Lemma the function f can not be strictly increasing since we eventually need to reach an ordinal that is strictly bigger than x. Hence f reaches a constant $f(\alpha) = f(\triangle)$ for some ordinal $\triangle$ and all $\alpha > \triangle$. Thus $\varphi(f(\triangle)) = f(\triangle)$ is a fixed point of φ. $\square$

Definition 2.27 (Inflationary functions)**.** A function φ as described in the Bourbaki-Witt theorem is called *inflationary* or *progressive* function.

Remark 2.28 (Zorn vs Bourbaki-Witt). Observe that the Bourbaki-Witt theorem bears resemblance to ZL. However for Bourbaki-Witt we require an explicit function to define a successor given some element, while for ZL we rather have an arbitrary poset. And indeed, when proving ZL this successor function is where AC comes into play.

As a closing remark to this section on set theory we present our premises on the use of set theoretic axioms. Namely, we will use ZFC unless otherwise stated. That is, we will be able to apply Zorn's lemma and can assume any given set to be accompanied by some well-ordering relation and hence possess a cardinality. We will still occasionally highlight the benefits of doing so, and in particular in Chapter 6 we will elaborate on the intrinsic differences for argumentation in a world without choice, or with choice, or with variations of choice.

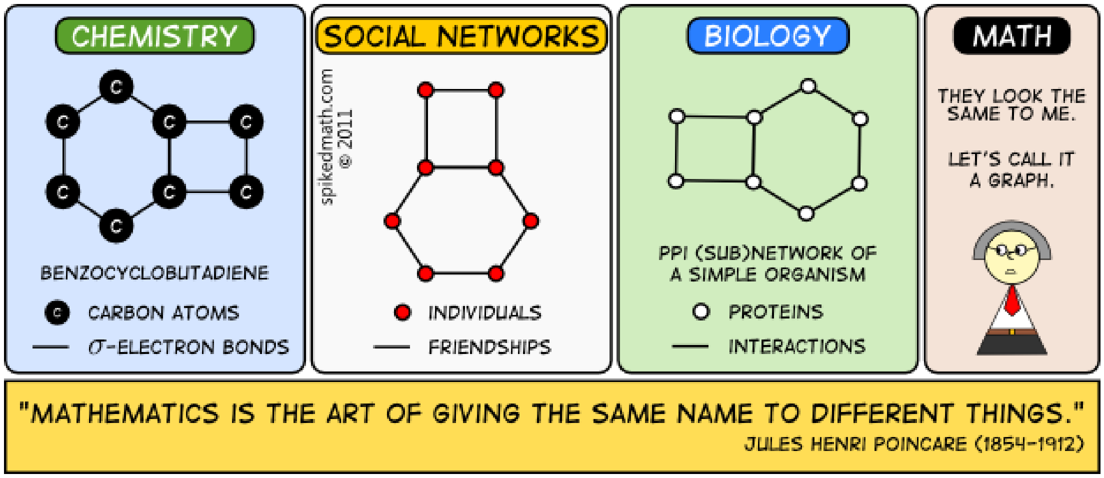


Figure 2.4: Comic illustration of the nature of graphs, cf. Remark 2.30. Comic CC-licensed at `http://spikedmath.com/382.html`

2.4 Graph Theory

Now that we have established what objects we have in mind when talking about sets, we continue by discussing a formal similarity between abstract argumentation and a mathematical field called graph theory (also see [W$^+$01, Bol98]). Graph theory tends to be well-known among computer scientists, which is why this section will not cover so much examples as mere plain definitions.

Definition 2.29 (Digraph and Graph)**.** A *digraph* is a pair $D = (V,E)$ where V is called its *vertex* set and $E \subseteq V \times V$ is called its *edge* set. Subsequently members of V are called *vertices* and members of E are called (directed) edges. If the direction of an edge is not of importance we might write $\{a,b\} \in E$ to denote $\{(a,b),(b,a)\} \cap E \neq \emptyset$.

In the case of '$(a,b) \in E$ implies $(b,a) \in E$' we call D a *symmetric* digraph. In the case of '$(a,a) \notin E$ for all $a \in V$' we call D a *loop-free* digraph. A loop-free symmetric digraph is also called a *graph*.

Remark 2.30 (The use of graphs)*.* The origin of graph theory is commonly attributed to Leonhard Euler and the seven bridges of Königsberg in 1736 [Wik16i]. Euler's approach was a generalization of the question whether there exists a tour crossing each bridge exactly once. He first introduced the concept of multigraph (different edges might connect the same vertices) and vertex degree (a measure of related edges). Since then graph theory has gained remarkable popularity in many different research areas, cf. Figure 2.4.

This widespread use is due to the simple basic definitions and flexible interpretation of what a vertex and an edge actually represent. However observe, that most commonly edges reflect a concept of connectedness. For instance in the bridges of Königsberg, we have land as nodes while bridges are (symmetric) edges. Apparently besides multigraphs there are further generalizations and modifications allowing for instance edges between more than two vertices. For the purpose of this thesis it suffices to consider digraphs and graphs as presented in Definition 2.29 only.

Remark 2.31 (Graphical representation of graphs). As illustrated by Figure 2.4 for graphical representations of graphs we draw vertices as circles, nodes or dots, and edges as lines. For digraphs we use arrows to denote the direction of an edge, where the origin is meant to be at the shaft, while the target is at the tip. In particular this means that for symmetric edges we occasionally might use lines without arrows.

While graphs allow for a wide field of applications the following notions have proven useful ever and again and will be referred to in this thesis as well.

Definition 2.32 ((Di)Graphs and Structure)**.** Given a digraph $D = (W,F)$, we refer to its set of vertices also by $V_D = W$, and its set of edges also by $E_D = F$. Two vertices $a,b \in V_D$ are said to be *adjacent* or *neighbors* if $\{a,b\} \in E_D$. The *cardinality* of a (di)graph G is given by the number of its vertices $|G| = |V_G|$.

Given a set of vertices $X \subseteq V_D$, we call $X_D^+ = \{b \in V_D \mid \exists a \in X, (a,b) \in E_D\}$ its *out-set* or *outward neighborhood* in D, $X_D^- = \{b \in V_D \mid \exists a \in X, (b,a) \in E_D\}$ its *in-set* or *inward neighborhood* in D, and $X_D^\pm = X_D^+ \cup X_D^-$ its *neighborhood*. We further call $|X_D^+|$ the *out-degree* of X, $|X_D^-|$ the *in-degree* of X, and $|X_D^\pm|$ the *(vertex-)degree* of X in D. The same notions will occasionally also be used for single vertices x analogously, e.g. $x_D^+ = \{b \in V_D \mid (x,b) \in E_D\}$. Further, if unmistakeable from context we may drop the subscript D.

For any digraph D and set X, the by X *induced subgraph* of D is defined as $D|_X = (X, E_D \cap (X \times X))$. The *inverse digraph* of D is given as $D^{-1} = (V_D, E_D^{-1})$ where $E_D^{-1} = \{(b,a) \mid (a,b) \in E_D\}$. The *complimentary digraph* of D is given as $D^c = (V_D, V_D \times V_D \setminus E_D)$.

Given two digraphs C and D, we further define their *union* component wise as $C \cup D = (V_C \cup V_D, E_C \cup E_D)$ and their *intersection* as $C \cap D = (V_C \cap V_D, E_C \cap E_D)$. Analogously for a family of digraphs $\mathrm{D} = (D_i)_{i \in I}$ we define $\bigcup \mathrm{D} = \bigcup_{i \in I} D_i$ and $\bigcap \mathrm{D} = \bigcap_{i \in I} D_i$ by transfinite recursion starting with $\bigcup\{D\} = \bigcap\{D\} = D$. We do not define these operations for empty index sets or empty sets of digraphs.

Two digraphs C and D are called *isomorphic*, written $C \equiv D$, if there is a bijection $f \colon V_C \to V_D$ such that $(x,y) \in E_C$ if and only if $(f(x), f(y)) \in E_D$. C is called a *subgraph* of D, written $C \subseteq D$, if $V_C \subseteq V_D$ and $E_C \subseteq E_D$, accordingly D is then called a *supergraph* of C. C is called *(the by V_C) induced subgraph* of D if $C \subseteq D$ and $E_C = (V_C \times V_C) \cap E_D$. C and D are called *disjoint* if $V_C \cap V_D = \emptyset$. Further on we may use $C \cap D = \emptyset$ as abbreviation for $C \cap D = (\emptyset, \emptyset)$.

A sequence of vertices $x_1, x_2, \ldots, x_n$ is called a *(directed) path* in D if $\{x_i, x_{i+1}\} \in E_D$ (in the directed case $(x_i, x_{i+1}) \in E_D$) for each $i \in \{1, 2, \ldots, n-1\}$. Formally we require paths to contain any vertex and make use of any edge at most once. A (directed) path $x_1, x_2, \ldots, x_n$ is called a *circle* (*cycle*) if also $\{x_n, x_1\} \in E_D$ (in the directed case $(x_n, x_1) \in E_D$). Any set of vertices X is called *connected* if for each couple $a,b \in X$, $a \neq b$ there is a path in between. A connected set of vertices is called *strongly connected* if for each such couple we have directed paths in both directions. We refer to the length of a path/circle/cycle $X = (x_1, x_2, \ldots, x_n)$ as $|X| = n$, if n is even/odd we call the path/circle/cycle an *even/odd-(length) path/circle/cycle*.

With this basic structural notations we proceed by recalling basic structural observations. In Definition 2.29 we already discussed symmetric and loop-free digraphs.

Definition 2.33 ((Di)Graph Properties)**.** A given digraph D is called

- *bipartite* if there is a partition $X \cup Y = V_D$, $X \cap Y = \emptyset$ such that for each edge $(a,b) \in E_D$ we have either $a \in X, b \in Y$ or $b \in X, a \in Y$;
- for a total of six different cases, *(odd/even-)circle/cycle-free* if there is no (odd/even) circle/cycle in D;[7]
- *complete* if for each $a, b \in V_D$, $a \neq b$ we have $(a,b) \in E_D$;
- *planar* if there is a graphical presentation on a plane (equivalently on the plane of a sphere) such that no two edges are crossing each other;
- *finite* if $|D| < \omega$ and *infinite* otherwise.

Remark 2.34 (Obvious (Di)Graph Properties). Obviously circle-freeness implies cycle-freeness but not the other way around. Similarly cycle-free (circle-free) digraphs are both, odd-cycle (odd-circle) and even-cycle (even-circle) free. A bit less plain but still sufficiently obvious: bipartite digraphs are odd-circle-free.

To round up our definitions regarding graphs and digraphs we introduce a few further concepts that make talking about digraph properties and functions a lot easier.

Definition 2.35 (Distance in Digraphs)**.** Given a digraph D and sets of vertices $X, Y \subseteq V_D$, we define the distance $dist(X,Y)_D$ between X and Y in D as the minimum length over all undirected paths from vertices $x \in X$ to vertices $y \in Y$. Similarly for $x, y \in V_D$ we define distances between x and Y, as well as between x and y by w.l.o.g. $X = \{x\}$ and $Y = \{y\}$.

If no ambiguity arises we might drop the subscript D. For $X = \emptyset$ and non-empty Y we define $dist(X,Y) = \omega$. For any $X \subseteq V_D$ we have $dist(X,X) = 0$, for vertices $x \neq y$ with $(x,y) \in E_D$ this gives $dist(x,y) = 1$. This notion of distance apparently is symmetric and undirected.

Definition 2.36 (k-neighborhood)**.** Given a digraph D and vertex (set) x, for any natural number k we define the k-neighborhood of x as

$$x_D^k = \{y \in V_D \mid dist(x,y) \leq k\}.$$

Observe that this definition of k-neighborhood allows for a very handy operator for $k = 0$, i.e. for vertex sets $x \subseteq V_D$ we get $x_D^0 = x$, for vertices $x \in V_D$ we get $x_D^0 = \{x\}$, which means that the 0-neighborhood will always be a set of vertices. To generalize this observation we will use the following notation.

[7] Circle-freeness is the main reason we need to restrict paths regarding the multiplicity of contained vertices and edges.

Definition 2.37 (0-neighborhood)**.** Given a vertex x, we define $x^0 = \{x\}$. Given a vertex set x, we define $x^0 = x$.

As sketched in Chapter 1 in abstract argumentation we are interested in sets of arguments. Inspired by the resemblance with graph theory we hence introduce the following graph notions as *graph-semantics* and briefly discuss their intended purpose and origins.

Definition 2.38 ((Di)Graph Semantics)**.** Given a digraph D, a set of vertices X is called

- an *independent* set if there are no $x, y \in X$ with $(x, y) \in E_D$ (and possibly $x = y$), i.e. $X \cap X^+ = X \cap X^- = \emptyset$;
- a *maximal independent* set if there is no independent set Y with $X \subset Y$;
- a *maximum independent* set if there is no independent set Y with $|X| < |Y|$;
- a *dominating* set if $X \cup X^+ = V_D$;
- a *minimal dominating* set if there is no dominating set Y with $Y \subset X$;
- a *minimum dominating* set if there is no dominating set Y with $|Y| < |X|$;
- a *kernel* if $X \cap X^- = \emptyset$ and $X \cup X^- = V_D$;
- a *semi-kernel* if $X \cap X^- = \emptyset$ and X has outward neighbors only to vertices from its inward neighborhood, i.e. $X^+ \subset X^-$;
- a *clique* if for each $x \neq y \in X$ we have $\{x, y\} \in E_D$.

We implicitly define maximal/minimal semi-kernels/cliques and combinations of above, e.g. *independent dominating* sets.

Remark 2.39 ((Di)Graph Semantics). The given notions are used in rather different areas of research. It is easy to see that independent dominating sets (see [GH13] for a recent overview on this combination) are the dual (under inverted direction of the edge relation) to kernels, but other than that there is no equivalence for the above.

The origins and intuitions of the above definitions are as follows:

- Independent sets are superficially close to argumentation in intuition as we ask for vertices that are not member of each other's neighborhood. One motivation for this notion is to distinguish sets where no vertex can influence another directly [Kor74]. Interestingly, the independent sets of any digraph are precisely the cliques in the digraph's complement. Finally the popular field of vertex coloring [Wik16b] requires a partition into pairwise independent sets.

- Domination is a very mathematical topic, in the sense that it appears to be of theoretical interest [HL90], while applications often are reduced to computational complexity of computing dominating sets or related parameters. A running example is that of an ice-cream truck company that attempts to serve a neighborhood in such a way that every street has access to at least one truck at a nearby crossing.
- Kernels and semi-kernels stem from game theory [vNM07]. In particular there is a strong relation between two player games and existence of such sets [GG07, GN84].

We introduced graph semantics mainly for reference purposes. As we will define comparable argumentation semantics we will use related results from the graph theory literature. Hence it turns out to be helpful to relate to the given definitions. Further investigations regarding the relation of graph semantics (via argumentation semantics) though are postponed to Chapter 5.

Chapter 3

Syntax of Abstract Argumentation

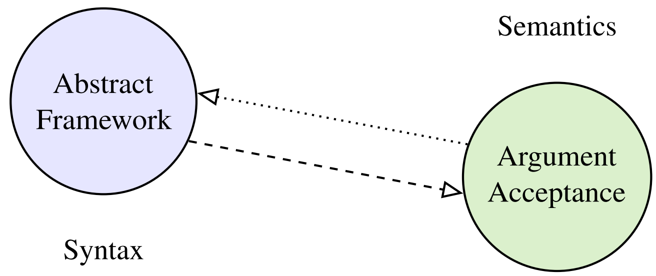


Figure 3.1: A schematic argumentation work flow reduced to the needs of this thesis.

As pointed out in Chapter 1 in this thesis we do not investigate the full argumentation process depicted in Figure 1.6 but rather mostly the interspace between Abstract Argumentation Frameworks and Abstract Argumentation Semantics as depicted in Figure 3.1. To allow for a precise language regarding this interspace we distinguish between syntax and semantics, respectively, as structural and justification-related properties. This chapter is dedicated to introducing and discussing the syntax of abstract argumentation. This approach allows us to tie the bonds to graph theory even stronger. However observe that "syntax" and "syntactic" are informational terms, the main value being of didactic nature. We will highlight cases where a strong distinction between syntax and semantics might be misleading. Without further ado we now present the core definition of this thesis, abstract argumentation frameworks.

Definition 3.1 (Standard AF Definition)**.** An (abstract) *argumentation framework* (AF) is an ordered pair $F = (A, R)$, where A is an arbitrary set of *arguments* and the two-valued relation $R \subseteq A \times A$ represents the *attacks* and is sometimes also called the *attack-relation* of F. For $(a, b) \in R$ we say that *a attacks b* (in F). For a given AF $F = (B, S)$ we sometimes use $A_F = B$ and $R_F = S$ to, respectively, denote its sets of arguments and attacks.

Observe that in this thesis we build argumentation with axiomatic set theory as foundation. This allows us to assume arbitrary sets of arguments instead of requiring existence of a countable or uncountable universe of arguments and thus a domain arguments are bound to.[1]

[1]By the way, this approach does not mean that we might not be able to distinguish arguments and sets of arguments, as one might think if considering that e.g. 1 is both a number and a set $1 = \{0\}$, cf. Example 2.10. If

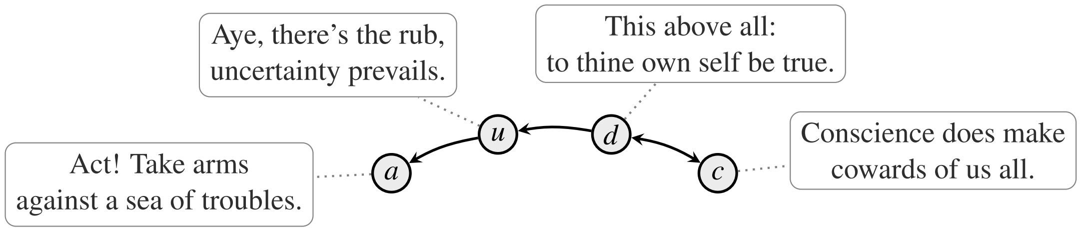


Figure 3.2: Natural Language Example, "Is it about time for action?", cf. Example 3.4.

Looking at Definition 3.1 it appears that it strongly resembles Definition 2.29 regarding mathematical objects called digraphs. And indeed we can and will identify AFs as digraphs. The previous definition mainly serves the purpose of giving a traditional introduction of abstract argumentation frameworks. For all practical purposes we will use the following, equivalent and extending, definition. Recall that by Definition 2.36 x^0 always gives a set of arguments regardless of whether x is an argument (then $x^0 = \{x\}$) or a set of arguments (then $x^0 = x$).

Definition 3.2 (AF Definition via Digraphs)**.** An (abstract) *argumentation framework* (AF) is a digraph F, where we interpret $A_F = V_F$ as its *arguments*and $R_F = E_F$ as its *attacks* . For arguments $x, y \in A_F$ (or argument sets $x, y \subseteq A_F$, or combinations) with $y^0 \cap x^+ \neq \emptyset$ we say that *x attacks y* in F. We further define the *range* of an argument (set) x as $x^* = x^0 \cup x^+$.

Remark 3.3 (Inherited Definitions from Graph Theory). By Definition 3.2 we simply define AFs as digraphs. The notions established for digraphs hence carry over to AFs, in particular we depict AFs such that nodes represent arguments and arrows represent attacks. To avoid confusion for most of this thesis we talk about AFs, symmetric AFs, arguments, attacks, and symmetric attacks instead of, respectively, digraphs, graphs, vertices, directed edges, and undirected edges. However we will make use of the concepts and observations from Section 2.4.

In particular for AFs F, argument (set) x we inherit the definitions for *cardinality* $|F|$, (*outward* and *inward*) *neighborhood* x^+, x^-, $x^\pm$, *k-neighborhood* x^k, *induced subframeworks* (induced sub-AFs) $F|_x$, *inverse AF* F^{-1}, *complementary AF* F^c. For two digraphs F, G we inherit the *union* $F \cup G$, *intersection* $F \cap G$, as well as union and intersection over families of AFs, the *subframework* (sub-AF) relation $F \subseteq G$, *isomorphism* $F \equiv G$, *disjoint* AFs. We further inherit *directed* and *undirected paths*, *circles* and *cycles*, *connected* and *strongly connected* components, as well as *k-neighborhood* and *distance*. Also like digraphs, AFs can be *symmetric*, *loop-free*, *bipartite*, *(even/odd) circle/cycle-free*, *complete*, *planar*, *finite* and *infinite*.

To illustrate these syntactic definitions we now continue by giving a first formal example of abstract argumentation. Although interpretation of arguments as some real world (or any world) entities is not strictly necessary for the work conducted in this thesis, we will occasionally give natural language examples and interpretations for a better understanding of the why and how.

needed we could for instance define arguments as sets of cardinality 2 where for some fixed ordinal α any argument is of the form $\{\alpha, \{\alpha, \beta\}\} = (\alpha, \beta)$. Then sets of arguments will never contain α as a member and can thus not be confused with arguments.

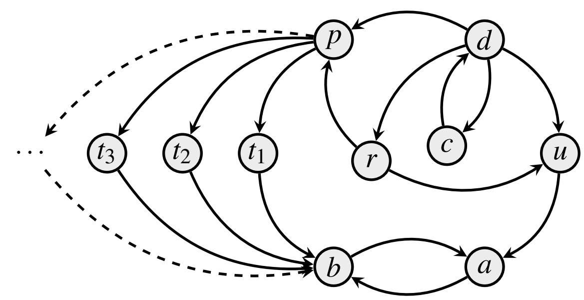


Figure 3.3: Argumentation Framework extracted from Shakespeare's Hamlet, cf. Examples 1.4 and 3.5.

Example 3.4. Consider the AF as depicted in Figure 3.2. We have argument c attacking and being attacked by argument d, which in turn attacks u, and u finally attacks a. Hence the AF F can also be defined as $A_F = \{c,d,u,a\}$ and $R_F = \{(c,d),(d,c),(d,u),(u,a)\}$. By Definition 3.2 we have that $\{d,u\}^* = \{d,u\} \cup \{c,u,a\} = A_F$ and thus by Definition 2.38 $\{d,u\}$ is a dominating set, it is not independent, yet minimal and even minimum dominating. The AF F is finite, bipartite, odd-circle-free, planar and connected. The (maximal) strongly connected components of F are $\{c,d\}$, $\{u\}$ and $\{a\}$.

The attentive reader who is familiar with the introduction might recognize some of the arguments presented in Example 3.4. They are for one taken from Example 1.3 as c: "*conscience*", d: "*determination*", u: "*uncertainty*", and a: "*to act, not to be*". For another the quotes in Figure 3.2 are very freely cited from William Shakespeare's play "Hamlet". This example will serve as a prototypical example of a finite AF throughout the thesis. We follow up by presenting a prototypical example of an infinite AF, the AF from Example 1.4, where we replace argument $\neg b$: "*not be*" with the equivalent from Example 3.4 a: "*act*". Observe that we are free to define an AF by explicitly giving its formulaic definition, but in particular for illustrating examples it suffices to give a graphical presentation.

Example 3.5. Consider the AF G from Figure 3.3. We have discussed its natural language interpretation in Section 1.1, we now give further observations regarding Definition 3.2 and Remark 3.3. The AF is infinite and planar, we have the AF F from Example 3.4 as proper sub-AF. The only cycles (and thus strongly connected components of more than one argument) are $\{a,b\}$ and $\{c,d\}$. Thus G is odd-cycle-free, but it is not odd-circle-free and thus also not bipartite since e.g. r,p,d is an odd circle of length 3. The longest directed path (of length 6) is c,d,r,p,t_1,b,a. We have $p^+ = \{t_1,t_2,\dots\} \subset \{a,t_1,t_2,\dots\} = b^-$. And for the set $S = \{a,c,r,t_1,t_2,\dots\}$ we have that $S^* = A_G$ and $S^+ \cap S = \emptyset$, hence S is an independent dominating set.

The remainder of this chapter is dedicated to introducing and discussing syntactic subtleties of abstract argumentation. To that end in Section 3.1 we present structural properties and first distinguish between local and global issues of AFs. In Section 3.2 we elaborate on the topic of modifications, relate to relevant literature and highlight specific examples.

3.1 Locality vs. Globality

In Chapter 4 we will use several syntactic notions (such as range) to properly define argumentation semantics. As mentioned before we see the informal term syntactic as a mode of describing the intuition of structural information. As exemplary such properties we have already introduced that of range and of cardinality. However there is a significant difference between range and cardinality in that intuitively range is local, while cardinality is global. For a better understanding of this intuitive differentiation we give the following definition.

Definition 3.6 (Local and Global Properties)**.** Given natural number k and argument (set) x, an AF-property φ is called *k-local (over x)* if it is decidable by looking only at the k-neighborhood of x, i.e. for any AF F we have $\varphi_x(F) = \varphi_x(F|_{x^k})$. Given an argument (set) x, an AF-property φ is called *local (over x)* if it is k-local for some k. An AF-property is called *global* if it is not local.

As already hinted at, given an argument (set) x, the range x^* is a local function and so are out-neighborhood x^+, in-neighborhood x^-, neighborhood $x^\pm$ and the plain property of attack. In these cases it suffices to consider the 1-neighborhood, indicating 1-local functions. Cardinality of an AF is a global property and hence so are being finite or infinite, as we need to consider all arguments available. Regarding Definition 2.38 we have that independence and semi-kernel are local, while all the other graph-semantics are of global nature. The case with semi-kernel might not be obvious, we will highlight this in Definition 3.9 when defining admissibility. Regarding the digraph-properties from Definition 2.33 apparently all of them are global. The benefit of local functions and properties is that they guarantee semi-efficient computability. This thesis does not focus on computational complexity, so we merely regard locality as a tool for distinguishing between less and more advanced properties/functions. We refer to [Spa13] for a nuanced discussion of local translations in the context of abstract argumentation semantics. The other use we make of locality/globality is that it allows us for a natural order of introducing properties. We proceed by presenting some local properties.

Definition 3.7 (Syntactic Conflict)**.** Given some AF F, argument sets (or arguments) $x, y \subseteq A_F$ (or $x, y \in A_F$, or some combination), we define as predicates

- *(syntactic) conflict*, written $[x,y]_F^{cnf}$, as an attack in between x and y, i.e. $x^\pm \cap y^0 \neq \emptyset$;
- *(syntactic) independence*, written $\{x,y\}_F^{ind}$, as digraph independence, i.e. $x^\pm \cap y^0 = \emptyset$;
- *(syntactic) attack*, written $(x,y)_F^{att}$, as an attack from x to y, i.e. $x^+ \cap y^0 \neq \emptyset$.

If no ambiguity arises we may drop the subscript F. Similarly we can define above notions on single arguments x (or argument sets x) by $[x]^{cnf} = [x,x]^{cnf}$, $(x)^{att} = (x,x)^{att}$ and $\{x\}^{ind} = \{x,x\}^{ind}$, where apparently $[x]^{cnf} = (x)^{att}$ and thus $\{x\}^{ind}$ exactly reflects digraph independence.

relation	$S = \{a,d\}$	$T = \{c,u\}$	$U = \{d,p\}$
$S = \{a,d\}$	$\{S\}^{ind}$	$(S,T)^{att}$	$(S,U)^{att}$
$T = \{c,u\}$	$(T,S)^{att}$	$\{T\}^{ind}$	$(T,U)^{att}$
$U = \{d,p\}$	$[U,S]^{cnf}$	$(U,T)^{att}$	$(U)^{att}$

Table 3.1: Table illustrating syntactic conflict relations between various sets of arguments for the AF from Figure 3.3, cf. Example 3.8.

Observe that by definition the syntactic attacks of any given AF are a subset of the syntactic conflicts. Attacks however are directed, while conflict (and independence) are symmetric relations. Hence attack can be seen as fine-tuning of conflict, and intuitively indeed for common argumentation procedures looking for conflicts precedes detailed investigation of attack relations. Independence on the other hand is the dual of conflict, i.e. for any AF F and any two sets of arguments $S,T \subseteq A_F$ exactly one of $[S,T]_F^{cnf}$ or $\{S,T\}_F^{ind}$ holds. To further illustrate these predicates we have another look at this chapter's running examples.

Example 3.8. Regarding AFs F from Example 3.4 and G from Example 3.5 with $F \subset G$, here for any argument sets $x,y \subseteq A_F$ we further have that $[x,y]_F^{cnf}$ iff $[x,y]_G^{cnf}$, $\{x,y\}_F^{ind}$ iff $\{x,y\}_G^{ind}$, and $(x,y)_F^{att}$ iff $(x,y)_G^{att}$.[2] For instance for the sets $S = \{a,d\}$, $T = \{c,u\}$ and $U = \{d,p\}$ we have the relations as crafted in Table 3.1. Observe that the given relations are not defined for AF F upon set U as p is not defined in F.

Conflict as a syntactic (and later on as a semantic) feature of AFs is of a very basic nature. Due to this peculiarity conflicts appear implicitly in most work on argumentation yet hardly ever are featured as research interest on their own. We dedicate and build most of Part III on the concept of syntactic and semantic conflicts and their interplay. Semantics (as discussed in Chapter 4) can be seen as predicates on sets of arguments, giving justification or acceptance states. The following definitions are in some work regarded as semantics on their own, due to their character we tend to think of them as syntactic as well and will henceforth discuss them in both chapters.

Definition 3.9 (Local evaluation)**.** Given AF F, we present

- *conflict-freeness*: given argument $x \in A_F$ (or argument set $x \subseteq A_F$), x is called *conflict-free* if there is no conflict among x, i.e. $\{x\}_F^{ind}$;[3]
- *defense*: given arguments $x,y,z \in A_F$ (or argument sets $x,y,z \subseteq A_F$, or combinations), we say that x *defends* z *against* y if x attacks all attackers of z among y, i.e. $(z^- \cap y^0) \subseteq x^+$. We say that x *defends* z (in F) if it attacks all attackers of z, i.e. $z^- \subseteq x^+$;

[2] This is a very substantial relation between F and G, we will refer to G as an embedding modification of F in Section 3.2. Often it is a general assumption of argumentation that monotonic extension of knowledge (i.e. adding information) results in AFs that keep established attack and independence relations intact.

[3] Yes, this is another (argumentation theoretic) name for digraph independence.

- *admissibility*: given argument $x \in A_F$ (or argument set $x \subseteq A_F$), x is called *admissible* if $\{x\}_F^{ind}$ and x defends itself, i.e. $x^- \subseteq x^+$.

Example 3.10. Regarding AFs F from Example 3.4 and G from Example 3.5 we state the following local observations:

- the set $\{c,u\}$ does not defend itself in G but does so in F;

- hence $\{c,u\}$ is admissible in F, but only conflict-free in G.

Naturally, when looking at argument harvesting procedures one might wonder whether a given procedure results in specific types of AFs. In particular structured argumentation (as the name might suggest) seems to press for this assumption. As final local property we discuss the property of superseding AFs as introduced in [CO14] with ASPIC-, a derivate of ASPIC+, in mind. As a heads-up, in this thesis in the first place we are not interested whether restrictions are justified or natural, but mainly whether restrictions also restrict expressiveness of AFs in the sense of [DS17, DDLW15]. We hence will not overly discuss intuition behind ASPIC-derivates, structured AFs or AF types.

Definition 3.11 (Superseding)**.** For a given AF F and arguments $x,y \in A_F$ we say that x *supersedes* y if $y^+ \subseteq x^+$ and $x^- \subseteq y^-$. Let AFs $F \subseteq G$ with $F = G|_{A_F}$, then F *supersedes* G if for each $y \in A_G$ there is some $x \in A_F$ that supersedes y in G.

Example 3.12. By definition any AF supersedes itself. For the AFs F from Example 3.4 and G from Example 3.5 we have that although $F \subset G$ no superseding happens. However for the finite AF $H = G|_{\{a,b,c,d,p,r,t_1,u\}}$ we have $H \subset G$ and $A_G \setminus A_H = \{t_2, t_3, \ldots\}$. By definition for any t_i for $i > 1$ there is always t_1 such that $t_i^+ = t_1^+$ and $t_i^- = t_1^+$. Thus H supersedes G, even in a strong sense of equality for the required relations. Observe that a further restriction $G|_{\{a,b,c,d,p,t_1,u\}}$ does not give a superseding AF anymore: while for r we have that $r_H^+ = r_G^+ \subset d_G^+ = d_H^+$ we also have $r_G^- = r_H^- = \{d\}$ but we do not have $d \in d^-$.

The reason for discussing superseding AFs is that certain properties might carry over from the smaller to the bigger AF. Especially in the case of the smaller AF being finite and the bigger AF being infinite this allows for an immediate computation of such properties for both. Speaking of finite and infinite AFs, we recall that the property of being finite or infinite is of global nature. We thus continue by giving further global syntactic definitions.

Definition 3.13 (Global issues: further AF classes)**.** Given some AF F, an argument (set) x is called *finitary* if $|x^-| < \omega$. If all arguments $x \in A_F$ are finitary, then also F is called *finitary* [Dun95]. F is called *finitely/finitarily superseded* [CO14] if there is a finite/finitary AF G that supersedes F. F is called *well-founded* [Dun95], if there is no infinite downwards path (a path without starting point), i.e. no sequence $x_1, x_2 \ldots \in A_F$ such that $(x_{i+1}, x_i)_F^{att}$ for each $i < \omega$.

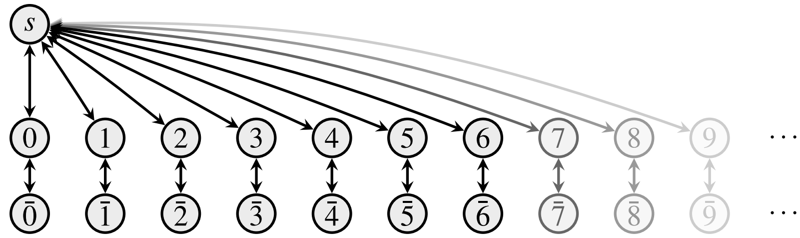


Figure 3.4: A planar, odd-circle-free, bipartite, symmetric and loop-free AF that is not finitarily superseded, cf. Example 3.14.

Example 3.14. Consider the AF F as depicted in Figure 3.4 with $A_F = \{s\} \cup \{i, \bar{i} \mid i \in \mathbb{N}\}$ and $R_F = \{(s,i),(i,s),(i,\bar{i}),(\bar{i},i) \mid i \in \mathbb{N}\}$. As visible in the illustration this AF is planar, by definition it further is symmetric loop-free and due its tree like structure odd-circle-free. With the partition $A = \{i \mid i \in \mathbb{N}\}$ and $B = \{s\} \cup \{\bar{i} \mid i \in \mathbb{N}\}$ it apparently is also bipartite.

We claim that the AF is not finitarily superseded. To this end observe that any argument x superseding some i in F needs $i_F^+ \subseteq x_F^+$, which means that we need $s, \bar{i} \in x_F^+$. Then the only argument $x \in A_F$ for which that is the case is i itself. Thus arguments i can be superseded only by themselves. Similarly the only argument superseding s in F (and thus attacking all i) is s itself. For $\bar{i}$ observe that since $(\bar{i})_F^- = \{i\}$ any argument x that supersedes $\bar{i}$ might attack at most argument i. Since this again is only the case for $\bar{i}$ itself the only AF superseding F is F itself. Since F is not finitary it is also not finitarily superseded.

Definition 3.15 (Global issues: strongly connected components, cf. Definition 2.32)**.** An argument set x is called a *strongly connected component* (SCC) (of F) [NSS94] if it is strongly connected and there is no strongly connected proper superset $y \supset x$. We collect the SCCs of F in $SCC(F)$, i.e. $C \in SCC(F)$ if and only if C is a strongly connected component of F. Finally we define the SCC-component function such that for each AF F we have $SCC_F : A_F \to SCC(F)$, and for each $x \in A_F$ we get $x \in SCC_F(x)$.[4]

Example 3.16. Regarding AFs F from Example 3.4 and G from Example 3.5. We have that F is finite and hence finitary but not well-founded as there is an infinite downwards path $c,d,c,d,c,d,c\ldots$. As F is a proper sub-AF of G, neither can G be well-founded. Finally as b has infinitely many attackers $\{t_1,t_2\ldots\}$, G is infinite and not finitary. Further consider the AF $H = G|_{\{a,b,c,d,p,r,t_1,u\}}$ from Example 3.12. As H is finite (and thus finitary) we have that H finitely (and finitarily) supersedes G.

Our interest in symmetric AFs, bipartite AFs, odd-cycle-free AFs is very similar in that such structures potentially reduce ambiguity of relations between arguments. The following definition (taken from [Dun95]) gives us some further fine-graining.

[4]Observe that for any argument of some AF there is always exactly one SCC that argument is member of, the SCC-component function is hence uniquely defined.

Definition 3.17 (Controversy)**.** Consider some AF F and arguments $x, y \in A_F$. We say that x *indirectly attacks/defends* y if there exists an odd/even number $n \in \mathbb{N}$ and directed path $x = z_0, z_1, \dots z_n = y$ with $(z_i, z_{i+1}) \in R_F$ for each $0 \leq i < n$. If x, both, indirectly defends and attacks y, then x is called *controversial* with respect to y.

An AF F is called

- *uncontroversial*, if there are no controversial arguments in F;
- *limited controversial*, if there exists no infinite downwards sequence $x_0, x_1, \dots$ such that x_{i+1} is controversial with respect to x_i.

Remark 3.18 (AF Properties). Expanding on Remark 2.34 we observe that

- bipartite AFs are always odd-circle-free;
- odd-circle-free AFs are always uncontroversial;
- uncontroversial AFs are always limited controversial;
- limited controversial AFs are always odd-cycle-free;
- well-founded AFs are always limited controversial;
- circle-free AFs are always odd-circle-free, cycle-free and limited controversial;
- cycle-free AFs are always odd-cycle-free;
- finite AFs are always finitary;
- finite/finitary AFs are always finitely/finitarily superseded and finitely superseded AFs are always finitarily superseded.

Further relations between these AF properties are possible but never granted. See Example 6.21 why odd-circle-free AFs being bipartite requires some variation of AC. Example 3.14 is witness for the superseded AFs of interest not being related to most other AF classes. Finding of further corresponding examples is left as an exercise for the attentive reader.

Also see Figure 3.5 for these results put into visual perspective.

3.2 Syntactic Modifications

Modifications in abstract argumentation have become a popular topic in recent years. There is work on intertranslatability between different semantics [DW11, Spa13, DS17], intertranslatability between different argumentation systems [BPW14, Pol16, Pol17], classification of modifications with semantics in mind [BB10, BB15, Bau14], and belief revision [CMKMM14, DHL$^+$15, FKIS09]. Naming and applicability for the various approaches still tend to differ. In this spirit we present an approach we call modification that allows us to incorporate several

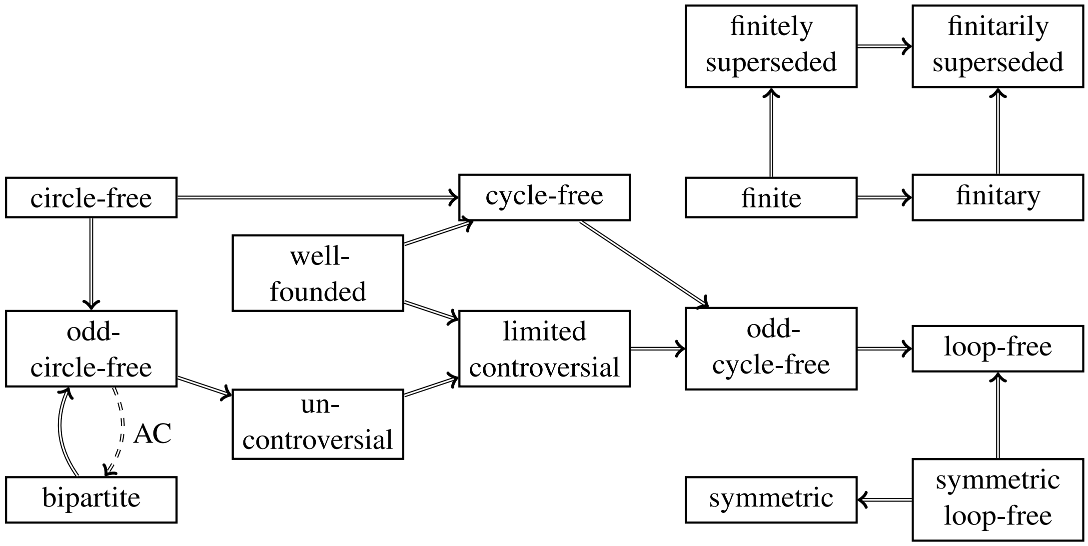


Figure 3.5: Syntactic AF classes put into relation. An arrow from x to y indicates that for each AF if x holds then so does y. No arrow indicates that x-AFs need not be y. The dashed arrow inscribed with AC indicates that this direction requires some variation of choice. See Remark 3.18 for a textual listing of these results.

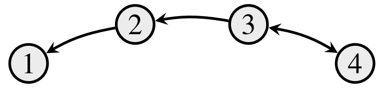
aspects in one notation highlighted under different names by different notations. However, for the purpose of this thesis in a stricter sense it suffices to consider translations between semantics and classification as referenced above.

Definition 3.19 (Modification)**.** A *modification* is a mapping from AFs to AFs, i.e. a modification φ, given any AF F the modification will produce another AF $G = \varphi(F)$.

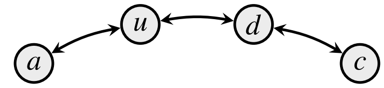
Without further insights or restrictions, modifications are kind of arbitrary tools. For instance, given the AFs F from Example 3.4 and G from Example 3.5, we can define modifications $\varphi(F) = G$ and $\psi(G) = F$. In this chapter we discuss syntax and syntactic notions and hence syntactic modifications. As syntactic modifications we understand mappings which for selected argument sets (or each single argument) are of local nature, i.e. do not bother about properties of an AF that are out of sight. As this statement might suggest in Chapter 5 we will further discuss semantic modifications and throughout the thesis also the interspace in between. Let us start with a few examples.

Example 3.20. Given some AF F, we define the following syntactic modifications:

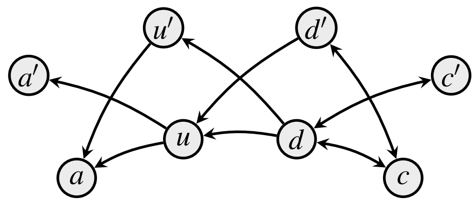
- any *renaming*, $rename(F) = G$ such that F and G are disjoint and isomorphic;
- *symmetrization*, $sym(F) = F \cup F^{-1}$;
- *loop-removal*, $looprm(F) = F|_{\{x \in A_F \mid \{x\}^{ind}_F\}}$;
- *argument shadow*, $shad_{arg}(F) = (A_F \cup A'_F, R_F)$, where $A'_F = \{x' \mid x \in A_F\}$(such that A_F and A'_F are disjoint);



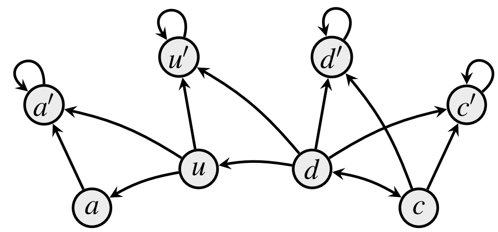
(a) $rename(F)$



(b) $sym(F)$



(c) $shad(F)$



(d) $shad^*(F)$

Figure 3.6: A selection of modifications from Example 3.20 as applied to AF F from Example 3.4.

- *loop shadow*, $G = shad_{loop}(F) \supseteq shad_{arg}(F)$ with $R_G = R_F \cup \{(x,x'),(x',x') \mid x \in A_F\}$;
- *out shadow*, $G = shad^+(F) \supseteq shad_{arg}(F)$ with $R_G = R_F \cup \{(x,y') \mid (x,y) \in R_F\}$;
- *in shadow*, $G = shad^-(F) \supseteq shad_{arg}(F)$ with $R_G = R_F \cup \{(x',y) \mid (x,y) \in R_F\}$;
- *range shadow*, $G = shad^*(F) = shad_{loop}(F) \cup shad^+(F)$;
- *shadow*, $G = shad(F) = shad^-(F) \cup shad^+(F)$;
- *full shadow*, $G = shad_{full}(F) = shad(F) \cup shad_{loop}(F)$;
- *strict shadow*, $G = shad_{strict}(F) = (A_{shad_{full}(F)}, R_{shad_{full}(F)} \setminus R_F)$.

Observe that restriction to some subsets $F|_S$, inverse F^{-1} and complement F^c are syntactic modifications as well. We will talk about the use of these and the above modifications in subsequent chapters, for now we mainly use them as illustration for the general concept of modification. To this end we also refer to Figure 3.6, where a selection of above modifications is illustrated for the AF F from Example 3.4.

Modifications might add or remove arguments and add or remove attacks, modifications might replace arguments as well as weaken or strengthen attacks regarding the interplay of specific arguments. For a categorization of modifications we will use the following definition.

Definition 3.21 (Syntactic Modification Types)**.** A modification φ is called

- *covering*, if for any AF F we have $F \subseteq \varphi(F)$;
- *embedding*, if it is covering and for any AF F we have $R_{\varphi(F)} \cap (A_F \times A_F) = R_F$;
- *monotone*, if for AFs $F \subseteq G$ we have $\varphi(F) \subseteq \varphi(G)$;
- *modular*, if for any AF F we have $\varphi(F) = \bigcup_{G \subseteq F} \varphi(G)$;

- *(k-)local over x* for argument (set) x, if there is a natural number k such that for any AF F we have $\varphi(F) = F \cup \varphi(F|_{x_F^k})$;
- *(k-)local*, if there is a natural number k such that for any AF F we have $\varphi(F) = \bigcup_{x \in A_F} \varphi(F|_{x_F^k})$;
- *global*, if it is not local;
- *outbound*, if it is embedding and for each AF F and all attacks $(x,y) \in (R_{\varphi(F)} \setminus R_F)$ we have $x \in A_F$ and $y \in (A_{\varphi(F)} \setminus A_F)$;
- *inbound*, if it is embedding and for each AF F and all attacks $(x,y) \in (R_{\varphi(F)} \setminus R_F)$ we have $y \in A_F$ and $x \in (A_{\varphi(F)} \setminus A_F)$;
- *secluded*, if for any AF F we have $F \subseteq \varphi(F)$ and $A_F = A_{\varphi(F)}$;
- a *deletion*, if for any AF F we have $\varphi(F) \subseteq F$.

Translations as in [DW11, Spa13, DS17] do have a strong semantic bias in that their intention is to transform AFs for matching semantic evaluation of different semantics. This is why this kind of modification in this thesis we would describe as a semantic modification. However, a lot of the introduced translations (sometimes called transformations if the semantic component is less strong) are of syntactic nature in that they are purely structural modifications. In particular (putting aside the semantic requirements) our definitions of covering, embedding, monotone, and modular modifications match exactly the definitions from [DW11] of covering, embedding, monotone, and modular translations, respectively. Our definitions of locality in modifications closely resemble the definitions of locality in transformations from [Spa13] yet do show subtle differences, where a local modification is the same as a finite diameter local transformation. Local transformations are aimed at allowing for some fine-tuning in between monotonicity and modularity, local modifications on the other hand target structural manipulation of AFs with minimal impact on the overall structure.

Expansions and deletions as defined in [BB10, BB15, Bau14] were introduced to characterize equivalence notions between AFs given some semantics. However, although the motivation again is of semantic nature, the definitions of expansion and deletion are of syntactic nature to begin with. Semantics aside, we have that normal expansions are the equivalent of embedding modifications and thus embedding transformations, i.e. they can be seen as a central notion in both lines of research. Further our definitions of outbound, inbound, and secluded modification match exactly the definitions from [BB10] of weak, strong, and local expansions, respectively. It should be pointed out that local expansions and local modifications are very different in nature.

We continue by categorizing the syntactic modifications introduced so far.

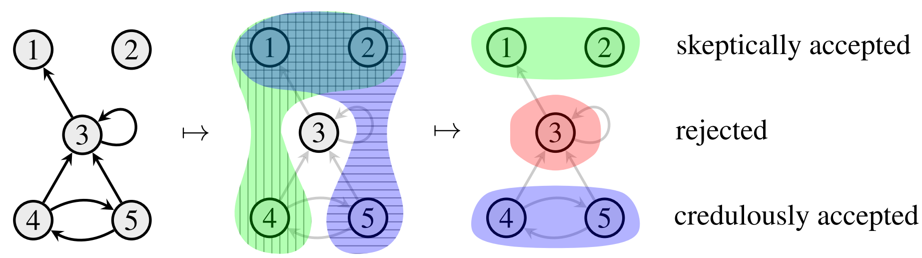
Example 3.22. We have that

- *identity*, $id(F) = F$ for any AF F is a covering, embedding, monotone, modular, 1-local, outbound, inbound, secluded modification and a deletion as well;

- restriction to subsets and loop-removal are deletions;
- loop-removal is a 1-local modification (we need to keep non-loop attacks);
- inverse, complement and symmetrization are secluded, 1-local modifications;
- renaming is a 1-local modification;
- all shadows but the strict shadow are embedding 1-local modifications;
- argument shadow is an outbound as well as an inbound modification;
- and finally, out shadow, loop shadow and range shadow are outbound modifications.

Chapter 4

Semantics of Abstract Argumentation



(a) Origin AF. (b) Semantics applied. (c) Justification states.

Figure 4.1: Scheme of applying semantics to AFs, cf. Example 4.1.

In this chapter we formally introduce argumentation semantics and semantic features of argumentation and AFs, i.e. we follow the transitions depicted in Figure 4.1: from argumentation structures in Figure 4.1a to semantic evaluation in Figure 4.1b to justification states in Figure 4.1c. We start out by giving an overview and discussing basic definitions. In Section 4.1 we introduce fixed point operators to be used later on. Such use can be seen as tools for the transition between Figures 4.1a and 4.1b. In Section 4.2 we present the core semantic definitions and elaborate on relevant properties and relations. There we discuss different ways of approaching Figure 4.1b. Section 4.3 is dedicated to reasoning with AFs and argumentation semantics, i.e. we discuss the entity appearing in Figure 4.1c as well as the transition leading there from Figure 4.1b.

Example 4.1 (Computational scheme of argumentation semantics)**.** Abstract argumentation frameworks (AFs) as discussed in Chapter 3 in a mathematical sense simply are directed graphs, where vertices represent arguments and edges represent attacks. Directed graphs and hence AFs are commonly visualized similar to Figure 4.1a. We depict arguments as circles (possibly with argument names inside) and attacks as arrows from the attacking to the attacked argument. Hence the given AF F is mathematically described by argument set $A_F = \{1,2,3,4,5\}$ and attack set $R_F = \{(3,1),(3,3),(4,3),(5,3),(4,5),(5,4)\}$.

In this chapter we discuss argumentation semantics, which formally are mappings from AFs to sets of sets of arguments (the transition from Figure 4.1a to Figure 4.1b). In Figure 4.1b we have marked two distinct sets of arguments in a Venn Diagram, the sets are $S_1 = \{1,2,4\}$

and $S_2 = \{1,2,5\}$. Hence for the semantics σ of choice we have $\sigma(F) = \{S_1, S_2\}$. For this introductory example it is not important how we arrive at these sets, nor which semantics we consider in Figure 4.1. We will repeatedly return to this example throughout the chapter. Semantics are intended to provide a means of justification in that the resulting sets represent justifiable selections of arguments. We might say that S_1 as well as S_2 are justified.

In Figure 4.1c finally we have sketched acceptance states of the arguments from F. Intuitively we might prefer arguments that occur in each single of the sets suggested by a semantics, hence regard the arguments $1,2 \in S_1 \cap S_2$ as skeptically justified. Arguments that do not occur in any of these extensions might as well be considered as rejected. The remaining arguments are justified, yet not by every extension set and are hence considered as credulously accepted. Credulous acceptance as a weak state of justification of course also covers arguments $1,2$.

Dung in [Dun95] introduced abstract argumentation together with the still most state of the art argumentation semantics as core notion, which are conflict-free, naive, admissible, complete, grounded, preferred, and stable semantics. The argumentation community extended, adapted and invented [BG09, BCG11] several further semantics, of which we will investigate the from our point of view most popular, which are stage [Ver96], semi-stable [Cam06, CV10, CCD12], cf2 and stage2 [BGG05, GW13, DG16], ideal and eager [Cam07] semantics. We will also introduce and briefly discuss reasoning modes [PRW03, RA06, EW06, DW11], equivalence notions [BB15, DS12, DDLW15], and AF classes with particular features [CDM05, Wey11, BDL$^+$14, DSLW16]. Without further ado we now give a formal definition of argumentation semantics.

Definition 4.2. An *(argumentation) semantics* σ is a mapping from AFs to sets of sets of arguments such that for any given AF F we have $\sigma(F) \subseteq \wp(A_F)$. Given some semantics σ and some AF F, we refer to sets $S \in \sigma(F)$ as *σ-extensions* of F. An (arbitrary) set of sets of arguments is subsequently often called an *extension set*.

Remark 4.3 (ZFC and argumentation semantics). For an implicit definition of semantics via predicate formulas for extension sets observe that that regarding fragments of ZFC first: extensions are definable if the formula φ is definable; and second: extension sets, i.e. the application of some semantics σ to some AF F, is a set by POW and RCO, that is

$$\sigma(F) = \{x \in \wp(A_F) \mid \varphi(x,F)\}.$$

With this very encompassing definition of argumentation semantics, combined with Definition 3.2 of AFs as digraphs, naturally we receive a wide range of argumentation semantics. We might even expect to occasionally stumble upon one or another. And indeed, as emphasized in the following example, up to this point this thesis already introduced quite some.

Example 4.4. Recall Definition 2.38, we have that the properties of (maximal/maximum) independent sets, (minimal/minimum) dominating sets, kernels, and (maximal/maximum) semi-kernels also define argumentation semantics. Observe that for any given AF F the independent dominating sets of F are exactly the kernels of F^{-1}, and similarly the maximal independent sets of F are exactly the independent dominating sets of $sym(looprm(F))$, cf. Example 3.20.

Recall Definition 3.9 of local syntactic properties, we have that conflict-freeness and admissibility are argumentation semantics. Further conflict-freeness matches independence, and, given some AF F and argument set $S \subseteq A_F$, we have that S is admissible in F if and only if S is a semi-kernel of F^{-1}. Similarly S is conflict-free in F if and only if it is admissible (and conflict-free) in $sym(F)$.

One might wonder if so far introduced modifications give further relations between semantics or allow for desirable new semantics. In this thesis we focus on elaborating on less tempting ideas. Martin Caminada in one of his talks at COMMA 2012 vehemently emphasized that instead of bending semantics for use cases it would be very advisable to focus on specifications for actually being able to compare and generalize use cases. And if one really needed to change or create a semantics it deems us reasonable to do so by following well-thought through guidelines such as the axiomatic outlines from [CA05]. Semantics are meant to give justification. In most cases where instantiated argumentation frameworks and their computed extensions do not satisfy the expectations, one should consider adapting one's expectations or instantiation procedures instead of inventing new semantics. We firmly believe that Dung argumentation with the semantics provided in this chapter still provide sufficiently many open, interesting and important questions for decades of research to come. The comparison of semi-kernel and admissibility above mainly serves the purpose of interconnecting related results of argumentation theory with graph theory. Further for each semantics discussed we will give a motivation to highlight their intended purpose.

4.1 Fixed point operators

This section is an intermezzo, a prelude to the actual content of this chapter. In abstract argumentation we sometimes make use of fixed point operators. At least for the motivation of argumentation semantics such recursive functions play an important role. While it is not strictly necessary to introduce and separately discuss the operators for the definition of most semantics, we still do so, mainly since we will make use of them again in Part II.

The operators discussed in this section formally are local syntactic modifications related to an initial argument set, cf. Section 3.2. The driving force behind the modifications in this section is that specific properties of argument sets allow for exploitation, e.g. implicitly lead to the definition of desired semantics. We start with the neutrality function as introduced in [Pol87] and emphasized for abstract argumentation in [Gro12, GM15].

Definition 4.5. Given some AF F, the *neutrality function* $\mathcal{N}_F$ is defined on sets of arguments $S \subseteq A_F$ as the sets of arguments not attacked by S in F, that is

$$\mathcal{N}_F(S) = A_F \setminus S^+_F.$$

For a fixed point of the neutrality function we observe that for any AF F and argument set $S \subseteq A_F$, the identity $\mathcal{N}_F(S) = S$ implies that $S^+ \cap S = \emptyset$ and that $S \cup S^+ = A_F$. For the AF F from Example 4.1 for instance $S_1 = \{1,2,4\}$ is a fixed point of the neutrality function. We continue with the characteristic function from [Dun95], sometimes also called defense function.

Definition 4.6. Given some AF F, the *characteristic function* (or *defense function*) $\mathcal{D}_F$ is defined on sets of arguments $S \subseteq A_F$ as the sets of arguments defended by S in F, that is

$$\mathcal{D}_F(S) = \{x \in A_F \mid x^- \subseteq S^+\}.$$

By definition fixed points of the neutrality function need to be conflict-free, the same does not hold for the characteristic function. For instance for the AF $(\{a\},\{(a,a)\})$ the argument set $\{a\}$ is a fixed point of the characteristic function as it defends itself against the attack from itself. Now consider the following well-known relations.

Fact 4.7. *For any AF F and argument sets $S,T \subseteq A_F$ we have*

$$\begin{aligned} S \subseteq T &\Rightarrow \mathcal{N}_F(S) \subseteq \mathcal{N}_F(T), \\ S \subseteq T &\Rightarrow \mathcal{D}_F(S) \subseteq \mathcal{D}_F(T), \\ \mathcal{D}_F(S) &= \mathcal{N}_F(\mathcal{N}_F(S)). \end{aligned}$$

For the characteristic function observe that fixed points are sets which defend themselves but no other arguments. For the AF F from Example 4.1 such fixed points for instance are the set $\{2\}$ or the set A_F (which is not conflict-free!). By Fact 4.7 also $S_1 = \{1,2,4\}$ is a fixed point of the characteristic function in F, as it already is a fixed point of the neutrality function. The set $\{2,4\}$ however is not a fixed point as 4 also defends 1. For the next fixed point operator we regard AF modifications rather than manipulation of argument sets.

Recall strongly connected components (SCCs) from Definition 3.15. Further observe that attacks between SCCs might be considered stronger than other attacks. If for some AF F we have $(x,y) \in R_F$ and $SCC_F(x) = SCC_F(y)$ then y could still defend itself (or some attacker of x) against x. However if $SCC_F(x) \neq SCC_F(y)$ then this is not the case and even no other argument $z \in SCC_F(y)$ can defend y against x. This stronger notion of attack is motivation for the following fixed point operator. Be aware that the literature [GW13, DG16] does use different but equivalent definitions.

Definition 4.8. Given AF F and argument set S, the *SCC-reduct* $\mathcal{R}_S(F)$ of F by S is defined as

$$\begin{aligned} A_{\mathcal{R}_S(F)} &= \{a \in A_F \mid a \notin (S \setminus SCC_F(a))^+_F\}, \\ R_{\mathcal{R}_S(F)} &= \{(x,y) \in A_{\mathcal{R}_S(F)} \times A_{\mathcal{R}_S(F)} \mid SCC_F(x) = SCC_F(y)\}. \end{aligned}$$

In words the SCC-reduct removes arguments from an AF F that are strongly attacked by an argument set S (in the sense that the attack ranges between different SCCs) and further separates SCCs of F. Be aware that removal of arguments can result in changes of the SCC-structure for $\mathcal{R}_S(F)$ if compared to F. The given SCC-reduct is an equivalent notion of the separation used in Theorem 3.11 of [GW13]. We investigate this function for an exemplary AF.

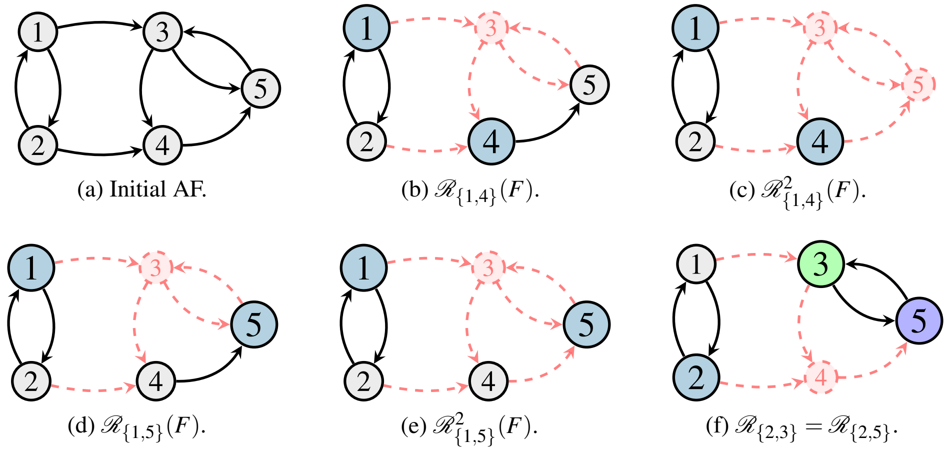


Figure 4.2: Illustration of the SCC-reduct operator as discussed in Example 4.9.

Example 4.9. Consider the AF F from Figure 4.2a with argument set $A_F = \{1,2,3,4,5\}$ and attack set $R_F = \{(1,2),(2,1),(1,3),(2,4),(3,4),(3,5),(4,5),(5,3)\}$, consisting of two SCCs where the first ($\{1,2\}$) is a two-cycle and the second ($\{3,4,5\}$) is a three-cycle. We further investigate the SCC-reduct of the (maximal conflict-free) sets $S_1 = \{1,4\}$, $S_2 = \{1,5\}$, $S_3 = \{2,3\}$, and $S_4 = \{2,5\}$. For $S_1 = \{1,4\}$ we first remove arguments that are attacked by S_1 where the attack is between different SCCs, which in this case means we remove argument 3. Further we only consider attacks among formerly strongly connected arguments, which means we also let go of the attack $(2,4)$. As there are still distinct but connected SCCs (namely sets $\{4\}$ and $\{5\}$) in $F' = \mathscr{R}_{\{1,4\}}(F)$ (see Figure 4.2b) the resulting AF F' is not yet a fixed point. Hence we can apply the SCC-reduct again on F', which now results in removal of argument 5 as depicted in Figure 4.2c. For $F'' = \mathscr{R}_{\{1,4\}}(F') = \mathscr{R}^2_{\{1,4\}}(F)$ however the initial AF is fully separated into non-connected SCCs, which means that we now have a fixed point.

Similarly for $S_2 = \{1,5\}$ we have two applications of the SCC-reduct as depicted in Figures 4.2d and 4.2e. Observe that in this case one of the resulting SCCs (which is $\{4\}$) does not contain any arguments from the generating argument set S_2.

For $S_3 = \{2,3\}$ and $S_4 = \{2,5\}$ it appears that already after the first application we arrive at a fixed point, as $\{3,5\}$ are strongly connected even without argument 4. Hence the SCC-reduct can break SCCs, as it occurred for S_1 and S_2, but for other constellations SCCs can still remain intact although we did remove arguments from them.

We thus have three different fixed point operators where two of them (neutrality and defense function) have as input and as output sets of arguments, and the third has as input and output AFs. The important aspect here is that input and output are of the same type and recursive application of the operator is thus possible.

Definition 4.10 (Fixed points)**.** Given a fixed point operator φ, a fixed point formally is the incidence of inputs that output themselves, i.e. inputs x such that $\varphi(x) = x$. For ordinals α we may use $\varphi^\alpha(x)$ to refer to the α^{th} application of φ to x, e.g. $\varphi^3(x) = \varphi(\varphi(\varphi(x)))$. In case φ

applied to x arrives at a fixed point y there is a minimal ordinal α such that $\varphi^{\alpha}(x) = y$, we will use $\triangle$ as superscript to denote such fixed points, i.e. $\varphi^{\triangle}(x)$ is a fixed point and $\triangle$ can then be used to refer to this minimal ordinal.

For the SCC-reduct it appears that at each application we may remove arguments or attacks, but never add any. Hence the arguments we remove from F can be represented as an inflationary function, cf. Definition 2.27. Thus, as soon as $\mathscr{R}_S^i(F) = \mathscr{R}_S^j(F)$ for $i < j$ (which is bound to happen due to Bourbaki-Witt, cf. Theorem 2.26) we can conclude $\mathscr{R}_S^{\triangle}(F) = \mathscr{R}_S^i(F)$, i.e. a fixed point. For neutrality and defense function this is not the case as the following example illustrates.

Example 4.11. Consider the AF $F = (\{x,y,z\},\{(x,y),(y,z),(z,x)\})$, a directed cycle of three arguments. We have $\mathscr{N}_F(\{x\}) = \{x,z\}$, $\mathscr{N}_F(\{x,z\}) = \{z\}$, $\mathscr{N}_F(\{z\}) = \{y,z\}$, and so on. Hence $\mathscr{D}_F(\{x\}) = \{z\}$, $\mathscr{D}_F(\{z\}) = \{y\}$, and $\mathscr{D}_F(\{y\}) = \{x\}$. Thus for some AFs there are input sets that result in endless repetition of the same patterns without fixed points. However $\mathscr{N}_F(\emptyset) = A_F$ and $\mathscr{N}_F(A_F) = \emptyset$ and thus both, $\emptyset$ and A_F are fixed points of the defense function.

4.2 Formal argumentation semantics

In this section we present and discuss the abstract argumentation semantics of interest. We will do so by giving a definition for a collection of semantics, talking about motivation and context and then illustrating the modes of operation on minimal examples. We first start with the semantics originally proposed by Phan Minh Dung in [Dun95].

Definition 4.12 (Original Dung semantics)**.** Given some AF F,

- an argument set $S \subseteq A_F$ is called a *conflict-free* extension (or set), written $S \in cf(F)$, if S is conflict-free in F, S it is called an *admissible* extension (or set), written $S \in ad(F)$, if S is admissible in F, also see Definition 3.9.
- An admissible set $S \in ad(F)$ is called a *complete* extension (or set), written $S \in co(F)$, if it contains all by S defended arguments, i.e. for $x \in A_F$ we have $\{S,x\}_F^{ind}$ and $x_F^- \subseteq S_F^+$ if and only if $x \in S$. A complete set $S \in co(F)$ is called a *grounded* extension, written $S \in gr(F)$, if it is minimal complete, i.e. for any $T \in co(F)$ with $T \subseteq S$ already $T = S$.
- A conflict-free set $S \in cf(F)$ is called a *naive* extension, written $S \in na(F)$, if it is maximal conflict-free, i.e. for any $T \in cf(F)$ with $T \supseteq S$ already $T = S$. An admissible set $S \in ad(F)$ is called a *preferred* extension, written $S \in pr(F)$, if it is maximal admissible, i.e. for any $T \in ad(F)$ with $T \supseteq S$ already $T = S$.
- Finally, a conflict-free set $S \in cf(F)$ is called a *stable* extension, written $S \in st(F)$, if it has full range, i.e. $S^* = A_F$.

Among above semantics conflict-free, admissible, and naive are not always considered proper semantics (of Dung's formalism). The reason for this is that *cf* and *ad* mainly serve as conditions that lead to definitions of semantics, and *cf* as well as *na* lack the feature of actually considering attacks but merely treat every attack as symmetric conflict. We include these three here for illustration purposes, and for compatibility with the literature.

If one is to accept that conflict-freeness is a necessity of argumentation systems in that we do not want justified sets of arguments to contradict themselves, then admissibility must be seen as core notion of Dung's abstract argumentation semantics. Indeed, the definitions of complete, grounded, preferred and stable semantics can be seen as straightforward consequences of observations regarding admissibility. Here the characteristic/defense function comes into play. Complete semantics results as any conflict-free fixed point of the characteristic function. Preferred semantics can also be defined as maximal conflict-free fixed points, grounded semantics as minimal fixed points. To be a bit more precise, grounded semantics is "the" least fixed point of this operator (indicating that there is only one), we will talk a bit more about unique status semantics in and around Definition 4.31. Stable semantics now takes the concept of admissibility to the top in that instead of only attacking attackers it requires attacking of every outside argument. We summarize the relation between above semantics and the fixed point operators of neutrality and defense function in the following lemma.

Lemma 4.13 (Fixed point operators and Dung's semantics)**.** *Given AF F and argument set S, we have that*

- $S \in cf(F)$ *if and only if* $S \subseteq \mathcal{N}_F(S)$*;*
- $S \in st(F)$ *if and only if* $\mathcal{N}_F^{\triangle}(S) = S$*;*
- $S \in ad(F)$ *if and only if* $S \subseteq \mathcal{D}_F(S)$ *and* $S \subseteq \mathcal{N}_F(S)$*;*
- $S \in co(F)$ *if and only if* $\mathcal{D}_F^{\triangle}(S) = S$ *and* $S \subseteq \mathcal{N}_F(S)$*;*
- $S \in gr(F)$ *if and only if* $\mathcal{D}_F^{\triangle}(\emptyset) = S$*.*

The relation between stable extensions and admissibility can also be seen in the corresponding terms from graph theory, where kernels are the equivalent of stable extensions and semi-kernels are the equivalent of admissible sets. There the motivation is that for any AF F and admissible set $S \in ad(F)$ for the restriction $F|_{S^1}$ the set S becomes a stable extension, i.e. $S \in st(F|_{S^1})$. The following lemma illustrates well-known relations.

Lemma 4.14. *Given an AF F and argument set $S \subseteq A_F$, the following hold:*

- $S \in st(F)$ *iff S is a kernel of F^{-1} iff S is minimal dominating F;*
- $S \in cf(F)$ *iff* $\{S\}_F^{ind}$*;*
- $S \in na(F)$ *iff S is a maximal independent set of F iff S is a maximal clique of F^c;*
- $S \in ad(F)$ *iff S is a semi-kernel of F^{-1}.*

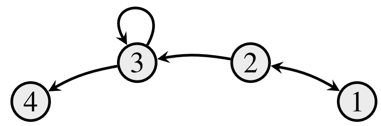


Figure 4.3: AF illustrating differences of Dung's semantics, cf. Example 4.15.

As it turns out among Dung's semantics stable stands out in that it might deliver an empty extension set.[1] For instance for the AF $F = (\{x\}, \{(x,x)\})$ for $\sigma \in \{cf, ad, na, co, pr, gr\}$ we have $\sigma(F) = \{\emptyset\}$ while $st(F) = \emptyset$. This difference might appear to be of subtle nature yet it leads to a situation we will call collapse and subsequently investigate in Part II. Collapse in brief terms is the situation where acceptance for independent components of some AF (i.e. not connected by undirected paths) can not be evaluated separately. For instance for the AF $G = (\{x,y\}, \{(x,x)\})$, where x and y represent disjoint connected components, we still have $st(G) = \emptyset$ while $\{y\} \in \sigma(G)$ and $\{y\} \in \sigma(G|_{\{y\}})$ as well as $\{y\} \in st(G|_{\{y\}})$. We now proceed by giving a minimal example where all Dung semantics result in different extension sets.

Example 4.15 (Dung semantics compared)**.** Consider the AF F as depicted in Figure 4.3 with $A_F = \{1,2,3,4\}$ and $R_F = \{(1,2),(2,1),(2,3),(3,3),(3,4)\}$. We have

$$
\begin{array}{rrcllllll}
 & cf(F) & = & \{\ \emptyset, & \{1\}, & \{2\}, & \{4\}, & \{1,4\}, & \{2,4\}\ \}, \\
 & ad(F) & = & \{\ \emptyset, & \{1\}, & \{2\}, & & & \{2,4\}\ \}, \\
\text{and} & co(F) & = & \{\ \emptyset, & \{1\}, & & & & \{2,4\}\ \}. \\
\text{Subsequently we get} & gr(F) & = & \{\ \emptyset, & & & & & \ \}, \\
 & na(F) & = & \{ & & & & \{1,4\}, & \{2,4\}\ \}, \\
 & pr(F) & = & \{ & \{1\}, & & & & \{2,4\}\ \}, \\
\text{and} & st(F) & = & \{ & & & & & \{2,4\}\ \}.
\end{array}
$$

Remark 4.16 (Subset relations for Dung semantics)*.* Observe that the implicit subset relations from above example can be shown to hold for arbitrary AFs, that means the following hold for any AF F:

$$st(F) \subseteq na(F) \subseteq cf(F); \qquad st(F) \subseteq pr(F) \subseteq co(F) \subseteq ad(F) \subseteq cf(F).$$

By Example 4.15 all of these inequalities are proper. A further observation is that each conflict-free set is subset of some naive extension. Similarly each admissible set is subset of some complete extension which is subset of some preferred extension. For the AF $F = (\emptyset, \emptyset)$ however we have $\sigma(F) = \{\emptyset\}$ for all introduced semantics, i.e. it is possible for them to coincide. A final observation is that for each preferred (and thus for each complete, admissible, or grounded) extension S, there is a naive extension T containing it, i.e. $S \subseteq T$.

Stable semantics is often regarded as the most desirable semantics. In cases where we only deal with AFs that provide stable extensions we prefer to only consider stable semantics. This preference is a bit more easily captured when considering that for given AF F and stable extension $S \in st(F)$ any argument $x \in A_F$ is either accepted ($x \in S$) or attacked ($x \in S^+_F$).

[1] We will discuss this coarse statement in more detail and in relation to axiomatic set theory in Chapter 6.

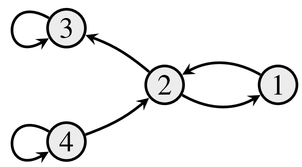


Figure 4.4: AF illustrating differences of range based semantics, cf. Example 4.19.

Definition 4.17 (Labelling Semantics)**.** Consider as given some AF F, semantics σ and extension $S \in \sigma(F)$. Any argument $x \in A_F$ is said to *labelled in* if $x \in S$, it is labelled *out* if $x \in S_F^+$, and it is labelled *undecided* if $x \in A_F \setminus S_F^*$.

For any AF, argument and extension set there is always exactly one label we can assign. By definition stable semantics provides only two-valued (i.e. only in and out labels for any extension) interpretations for any given AF structure. However, for this reason any given AF augmented with a single disjoint and self-attacking argument does not have any stable extensions. As a workaround for this issue several approaches were introduced, we proceed by presenting range-based semantics [Ver96, CV10].

Definition 4.18 (Range based semantics)**.** Given a semantics σ and some AF F, an extension set $S \in \sigma(F)$ is called *range-maximal* for σ in F if there is no $T \in \sigma(F)$ such that $S_F^* \subset T_F^*$. Intuitively for the introduced semantics with possibly $S \subset T$ for extensions $S, T \in \sigma(F)$ we only need to distinguish between *cf* and *ad*. Given some AF F, an argument set $S \subseteq A_F$ is called a

- *stage extension*, written $S \in sg(F)$, if it is conflict-free and range-maximal;
- *semi-stable extension*, written $S \in sm(F)$, if it is admissible and range-maximal.

Observe that the range of any stable extension is the full set of arguments of the framework. Hence all stable extensions provide the same range. If there is no stable extension however we might consider different range-maximal extensions. This is, in a nutshell, the motivation for range based semantics. The only question is whether we start from conflict-free or from admissible sets. The definition of stable semantics works with either. Thus, stage and semi-stable semantics coincide with stable semantics for AFs F with $st(F) \neq \emptyset$, yet might show different behaviour for other AFs. For illustration purposes we hence use an AF without stable extensions.

Example 4.19 (Range based semantics compared)**.** Consider the AF F as depicted in Figure 4.4 with $A_F = \{1,2,3,4\}$ and $R_F = \{(1,2),(2,1),(2,3),(4,2),(3,3),(4,4)\}$. We have $sg(F) = \{\{2\}\}$ and $sm(F) = \{\{1\}\}$.

Remark 4.20 (Subset relations for range based semantics). Observe that in Example 4.19 stage and semi-stable semantics differ in the greatest possible way, i.e. they can not accept a single argument simultaneously. As range maximality implies subset maximality we can still extend the notions from Remark 4.16 in the following way:

$$st(F) \subseteq sg(F) \subseteq na(F) \subseteq cf(F); \quad st(F) \subseteq sm(F) \subseteq pr(F) \subseteq co(F) \subseteq ad(F) \subseteq cf(F).$$

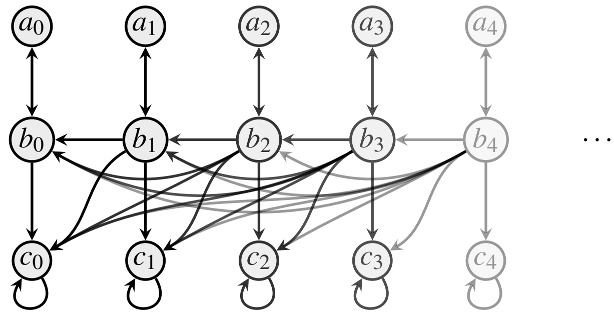


Figure 4.5: An illustration of the AF from Example 4.21.

Although stage is derived from naive semantics and semi-stable is derived from preferred semantics, the relation of each preferred extension being contained in some naive extension does not carry over. However this downside comes with an upside. As range based semantics focus on range, we still get that for any AF F and every semi-stable extension $S \in sm(F)$ there is some stage extension $T \in sg(F)$ such that $S_F^+ \subseteq T_F^+$.

In summary stage and semi-stable semantics serve the purpose of falling back to stable semantics whenever possible and otherwise providing default extensions. Naturally, the question occurs whether stage and semi-stable semantics (and others) always provide extensions. We will elaborate on this question extensively in Part II. For now we settle for giving one more example (from [Ver96]) illustrating that for infinite AFs we might run into troubles.

Example 4.21. Consider the AF F depicted in Figure 4.5 with $A_F = \{a_i, b_i, c_i \mid i \in \mathbb{N}\}$ and $R_F = \{(a_i, b_i), (b_i, a_i), (b_i, c_i), (c_i, c_i) \mid i \in \mathbb{N}\} \cup \{(b_j, b_i), (b_j, c_i) \mid i < j \in \mathbb{N}\}$. Here conflict-free and admissible sets and hence naive and preferred extensions coincide, i.e. for $A = (a_i)_{i \in \mathbb{N}}$ and $S_n = (A \setminus \{a_n\}) \cup \{b_n\}$ we have $na(F) = pr(F) = \{A\} \cup \{S_n \mid n \in \mathbb{N}\}$. However none of these extensions is maximal in range since $A^+ \subset S_n^+ \subset S_{n+1}^+$ for any $n \in \mathbb{N}$. Thus we get $sg(F) = sm(F) = st(F) = \emptyset$.

We will discuss classes of AFs that guarantee existence of stage/semi-stable extensions in Part II. For finite AFs F also $\wp(A_F) < \omega$ holds and thus incidents like the one from Example 4.21 can not occur. Formal argumentation traditionally puts focus mostly on finite AFs. Targeting finite AFs the following scheme was introduced to cover one particular aspect of preferred semantics. Very roughly spoken observe that for any AF F we can compute all preferred extensions by partitioning F into SCCs (Definition 3.15), ordering the SCCs by directed path reachability and computing the preferred extensions component by component. This observation leads to the introduction of SCC-recursiveness and SCC-based semantics.

Definition 4.22 (SCC-recursiveness and semantics)**.** Given semantics σ, we define the SCC-derived semantics σ_{SCC} recursively for AFs F and argument sets $S \subseteq A_F$ as

- case $SCC(F) = 1$, then $S \in \sigma(F)$;
- otherwise for each $C \in SCC(F)$ we have $(C \cap S) \in \sigma_{SCC}(F|_{C \setminus (S \setminus C)_F^+})$.

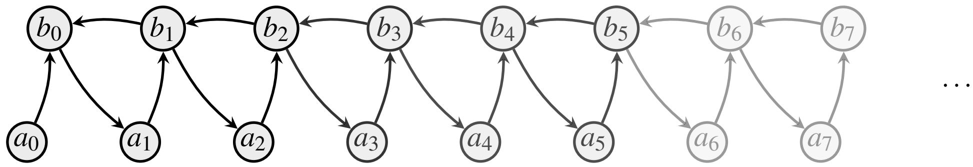


Figure 4.6: AF where the original schema of SCC-recursiveness is not applicable as discussed in Example 4.23.

Observe that this definition slightly differs from the original definition [BGG05] as rephrased in [GW13]. Mainly here we do not define component-defeated arguments, as for the purpose of this thesis we will not make use of this concept. Another issue we need to address at this point is that of well-definedness. SCC-recursiveness was introduced for finite AFs, but we also consider non-finite AFs. Hence the following observation becomes of importance.

Example 4.23 (Original SCC-recursiveness not well-defined, cf. Remark 4.37)**.** Consider the AF F as depicted in Figure 4.6 with $A_F = \{a_i, b_i \mid i \in \mathbb{N}\}$ and $R_F = \{(a_i, b_i), (b_i, a_{i+1}), (b_{i+1}, b_i) \mid i \in \mathbb{N}\}$. Intuitively SCC-recursiveness looks first at the first SCC, then at the second and so on. Hence we would assume $S = \{a_i \mid i \in \mathbb{N}\}$ to be a na_{SCC}-extension. However by definition we never arrive at the case $|SCC(F)| = 1$

Initially there are only two SCCs, $\{a_0\}$ and the rest which we call $C = A_F \setminus \{a_0\}$. For C we now need $C_0 \cap S \in na_{SCC}(F|_{\{a_i,b_i|i>0\}})$. However this restriction is isomorphic to F (i.e. $F \equiv F|_{\{a_i,b_i|i>0\}}$) and thus decision on whether or not S is an extension depends recursively on its own result. Definition 4.22 is thus not well-defined.

In [GW13] we find a characterization of na_{SCC}, introduced for computational reasons. For defined cases the characterizations are equivalent (as shown for the finite case in [GW13]) Because of our need for applicability also in the infinite case we will thus make use of the following definition. We will talk about definedness in Section 4.3.

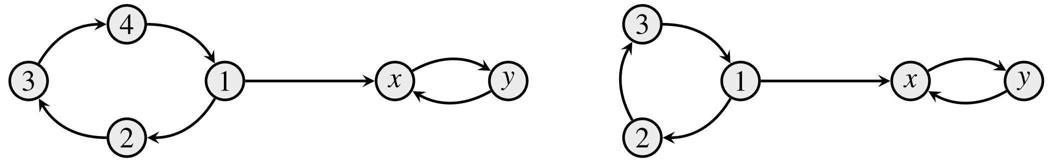
Definition 4.24. Given semantics σ, we define the SCC-derived semantics σ_{SCC}. For any AF F and $S \in A_F$ we have

$$S \in \sigma_{SCC}(F) \qquad \iff \qquad S \in \sigma(\mathscr{R}_S^{\triangle}(F)).$$

One of the main motivations for introducing SCC-recursiveness into abstract argumentation was different handling of odd- and even-length cycles. We follow up with an example from [BG03] illustrating this aspect.

Example 4.25. Consider the AFs F from Figure 4.7a and G from Figure 4.7b. We have

$$\begin{aligned} na(F) &= \{\{1,3,y\},\{2,4,x\},\{2,4,y\}\}, & na(G) &= \{\{1,y\},\{2,x\},\{2,y\},\{3,x\},\{3,y\}\},\\ pr(F) &= \{\{1,3,y\},\{2,4,x\},\{2,4,y\}\}, & pr(G) &= \{\{\quad y\}, \qquad\qquad\qquad\qquad\},\\ na_{SCC}(F) &= \{\{1,3,y\},\{2,4,x\},\{2,4,y\}\}, & na_{SCC}(G) &= \{\{1,y\},\{2,x\},\{2,y\},\{3,x\},\{3,y\}\}. \end{aligned}$$



(a) Four-cycle and two-cycle.

(b) Three-cycle and two-cycle.

Figure 4.7: Different behaviour of semantics on even- and odd-length cycles, cf. Example 4.25.

Observe that even- and odd-length cycles are treated very differently for preferred (and any admissibility-based semantics) although they present the same principle. For instance we might have as arguments $1,2,3,4$ (or $1,2,3$) the witnesses of a crime scene where each of them questions the reliability of the next one. By implication we get that for preferred semantics in F both, x and y are acceptable, while in G only y is acceptable. This example illustrates that for instance na_{SCC} intuitively handles such instances in a more fair way.

SCC-recursiveness is aimed at fair treatment of odd-length cycles in AFs, consequently the only representatives discussed in the literature are not admissibility-based. It can easily be seen that $cf_{SCC}(F) = cf(F)$ for any AF F, hence we are left with na_{SCC} and sg_{SCC}, as discussed with the following briefer names in [GW13, DG16].

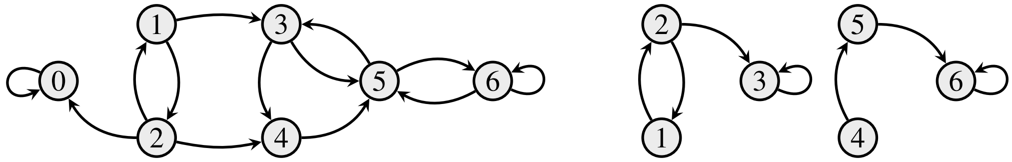
Definition 4.26 (Cf2 and stage2 semantics)**.** The *cf2* and *stage2* semantics are respectively defined as

$$c2 = na_{SCC} \qquad\qquad \textit{and} \qquad\qquad s2 = sg_{SCC}.$$

Apparently by definition any *c2* or *s2* extension will be maximal conflict-free and hence a *na* extension. Sets that are not conflict-free but lose their conflict by application of the SCC-reduct necessarily also lose arguments. Sets that are not maximal conflict-free can not be maximal conflict-free in the SCC-reduct. We will present a few examples illustrating SCC-based semantics in the following. We start with an already known example.

Example 4.27. Consider the AF F from Example 4.9 and Figure 4.2a with argument set $A_F = \{1,2,3,4,5\}$ and attack set $R_F = \{(1,2),(2,1),(1,3),(2,4),(3,4),(3,5),(4,5),(5,3)\}$. We have $S_1 = \{1,4\}$, $S_2 = \{1,5\}$, $S_3 = \{2,3\}$, $S_4 = \{2,5\}$ as maximal conflict-free sets and thus candidates for cf2 and stage2 extensions. The application of the SCC-reduct for these sets is illustrated in Figure 4.2. For S_1 we receive $\mathscr{R}^{\triangle}_{S_1}(F) = (\{1,2,4\},\{(1,2),(2,1)\})$ for which S_1 is a stable (and hence naive and stage) extension. For S_2 we receive $\mathscr{R}^{\triangle}_{S_2}(F) = (\{1,2,4,5\},\{(1,2),(2,1)\})$ for which S_2 is not maximal conflict-free. For S_3 and S_4 we receive $\mathscr{R}^{\triangle}_{S_i}(F) = (\{1,2,3,5\},\{(1,2),(2,1),(3,5),(5,3)\}$ for which both, S_3 and S_4 are stable extensions. Thus $c2(F) = s2(F) = \{S_1,S_3,S_4\}$. Observe that in this case also $st(F) = sg(F) = sm(F) = pr(F) = \{S_1,S_3,S_4\}$.

For the next example we flesh out further differences between Dung semantics and SCC-based semantics but also between cf2 and stage2 semantics.



(a) Modification of Figure 4.2a. (b) *c2* and *s2* in opposition to *sg*.

Figure 4.8: Cf2 and stage2 semantics, an illustration of SCC-recursiveness, cf. Example 4.28.

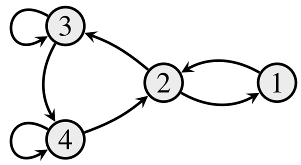


Figure 4.9: AF illustrating differences between $s2$ and sg, pr, cf. Remark 4.29.

Example 4.28. Consider the AF F from Figure 4.8a, a modification of the AF from Example 4.9 in that we have added arguments 0 and 6, and the AF G from Figure 4.8b. We have

$$\begin{array}{rlrl}
na(F) = \{\{1,4\},\{1,5\},\{2,3\},\{2,5\}\}, & & na(G) = \{\{1,4\},\{1,5\},\{2,4\},\{2,5\}\}, & \\
c2(F) = \{\{1,4\}, \qquad \{2,3\},\{2,5\}\}, & & c2(G) = \{\{1,4\}, \qquad \{2,4\} \qquad \}, & \\
s2(F) = \{\{1,4\}, \qquad\qquad \{2,5\}\}, & & s2(G) = \{\{1,4\}, \qquad \{2,4\} \qquad \}, & \\
sg(F) = \{ \qquad\qquad\qquad \{2,5\}\}, & & sg(G) = \{ \qquad\qquad \{2,4\},\{2,5\}\}, & \\
st(F) = \{ \qquad\qquad\qquad \{2,5\}\}, & & st(G) = \{ \qquad\qquad\qquad\qquad \}. &
\end{array}$$

Remark 4.29 (Subset relations for SCC-recursive semantics). Taken from [DG16], for any (finite) AF F we have

$$st(F) \subseteq s2(F) \subseteq c2(F) \subseteq na(F).$$

By illustration of Example 4.28 these relations might be proper. This example further illustrates that for some AFs (such as G) stage is incomparable with both cf2 and stage2.

To see that preferred/semi-stable semantics and stage2 semantics might disagree to maximal extent consider a modification F of the AF from Example 4.19 and Figure 4.4, where $A_F = \{(1,2,3,4\}$ and $R_F = \{(1,2),(2,1),(2,3),(3,3),(3,4),(4,4),(4,2)\}$ as depicted in Figure 4.9. We have $pr(F) = sm(F) = \{\{1\}\}$ while $sg(F) = s2(F) = \{\{2\}\}$.

It is easy to see (for an intuition see Definition 5.14) that for finite AF F and each $S \in pr(F)$ there is some $T \in c2(F)$ with $S \subseteq T$. Stage semantics can provide extensions that do not accept arguments from any initial SCC, which makes the following lemma all the more remarkable.

Lemma 4.30. *For finite AF F we have $sg(F) \cap s2(F) \neq \emptyset$.*

Proof. We first define the accumulation of initial components of F as $B = \bigcup\{C \in SCC(F) \mid C_F^- \subseteq C\}$, apparently also $B_F^- \subseteq B$ holds. Now consider some stage extension of the restriction to this accumulation $U \in sg(F|_B)$, and some range extending stage extension $T \in sg(F)$ with $U^*_F \subseteq T^*_F$. Since $B^-_F \subseteq B$ we conclude that also $T \cap B \in sg(F|_B)$. Consequently neither $\mathbb{S}_{s2} = \{S \in s2(F) \mid T \cap B \subseteq S\}$ nor $\mathbb{S}_{sg} = \{S \in sg(F) \mid T \cap B \subseteq S\}$ are empty.

We want to further consider only extensions from $\mathbb{S}_{s2}$ and $\mathbb{S}_{sg}$. To this end we eliminate arguments being in conflict with $T \cap B$ and construct $G = F|_{A_F \setminus (T \cap B)^{\pm}_F}$. Observe that $sg(G) = \mathbb{S}_{sg}$ and $s2(G) = \mathbb{S}_{s2}$ hold. In case $F = G$ this leaves us with an AF where the only attacks are self-attacks and thus $\mathbb{S}_{sg} = \mathbb{S}_{s2} = \{S_\omega\}$ with $S_\omega = \{x \in A_G \mid (x,x) \in R_G\}\}$. Since F is finite and construction of G only removes arguments we can thus apply recursion and after finitely many steps deliver $S_\omega \in sg(F) \cap s2(F)$. $\square$

As emphasized in the discussion after Definition 4.12 grounded semantics stands out as unique status semantics. For a generalization one might observe that for any AF F with $S \in gr(F)$ the argument set S is the biggest admissible set contained in all complete extensions of F. This observation leads to the so called ideal family of semantics, whose fabulous members are for a reason referred to as unique status semantics [DMT07, Cam07].

Definition 4.31 (Unique-status semantics)**.** Given semantics σ and AF F, an argument set $S \in A_F$ is called a σ-*ideal* of F if it is a maximal admissible subset of all σ-extensions, i.e. if $S \subseteq \bigcap \sigma(F)$, $S \in ad(F)$ and for each $T \in ad(F)$ with $T \subseteq \bigcap \sigma(F)$ and $S \subseteq T$ already $S = T$. Observe that here $S \subseteq \bigcap \sigma(F)$ is an abbreviation for $\forall S' \in \sigma(F) : S \subseteq S'$.We identify the following by name:

- *grounded* semantics (*gr*) as *co*-ideal;
- *ideal* semantics (*id*) as *pr*-ideal;
- *eager* semantics (*eg*) as *sm*-ideal.

As unique-status semantics never possess more than one extension (in the finite case) we then also speak of "the" extension and for AF F and $\sigma(F) = \{S\}$ sometimes also write $\sigma(F) = S$. We refer to Chapter 6 for a discussion of the infinite case.

As *sm* and *st* either coincide or *st* does not provide any extensions, *st*-ideal does not provide enough insight for a separate treatment. The other candidates for ideal semantics are based on conflict-freeness rather than admissibility, the given three ideal semantics are hence the only ones we can justify making use of. We continue by giving a minimal example of maximal difference.

Example 4.32 (Unique-status semantics compared)**.** Consider the AF F as depicted in Figure 4.10 with $A_F = \{1,2,3,4,5,6\}$ and $R_F = \{(2,3),(3,2),(3,3),(4,5),(5,4),(5,6),(6,6)\}$.

$$
\begin{array}{rrcl}
\text{We have} & co(F) & = & \{\{1\},\{1,2\},\{1,2,4\},\{1,2,5\}\}, \\
 & pr(F) & = & \{\qquad\qquad\quad \{1,2,4\},\{1,2,5\}\}, \\
\text{and} & sm(F) & = & \{\qquad\qquad\qquad\qquad\quad\ \{1,2,5\}\}. \\
\text{Subsequently we get} & gr(F) & = & \{\{1\}\qquad\qquad\qquad\qquad\qquad\quad\}, \\
 & id(F) & = & \{\quad\ \{1,2\}\qquad\qquad\qquad\qquad\}, \\
\text{and} & eg(F) & = & \{\qquad\qquad\qquad\qquad\quad\ \{1,2,5\}\}.
\end{array}
$$

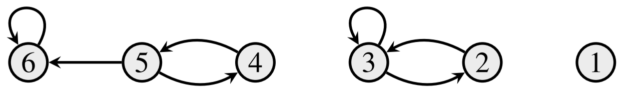


Figure 4.10: AF illustrating differences of unique status semantics, cf. Example 4.32.

When looking at above example one might assume that as for any AF F we have $\bigcap co(F) = gr(F)$ and thus $\bigcap co(F) \subseteq id(F)$ a similar relation might hold for the intersection of all preferred extensions and eager semantics. We contradict this assumption with the following example.

Example 4.33. Consider the AF F from Example 4.1 depicted in Figure 4.1a with $A_F = \{1,2,3,4,5\}$ and $R_F = \{(3,1),(3,3),(4,3),(5,3),(4,5),(5,4)\}$. We have

$$pr(F) = sm(F) = \{\{1,2,4\},\{1,2,5\}\}, \qquad \bigcap pr(F) = \bigcap sm(F) = \{1,2\},$$
$$co(F) = \{\{2\},\{1,2,4\},\{1,2,5\}\}, \quad \bigcap co(F) = gr(F) = id(F) = eg(F) = \{\{2\}\}.$$

Remark 4.34 (Subset relations for unique status semantics). For any AF F in general we have $\bigcap co(F) = gr(F) \subseteq id(F) \subseteq eg(F)$. Although $id(F) \subseteq \bigcap pr(F)$ and $eg(F) \subseteq \bigcap sm(F)$ in general hold as well, there is no necessary subset relation between $eg(F)$ and $\bigcap pr(F)$. Example 4.32 serves as witness for $\bigcap pr(F) \subset eg(F)$. Example 4.33 serves as witness for $eg(F) \subset \bigcap pr(F)$. Finally observe that *gr*, *id* and *eg* always result in a *co* extension. [Cam07, DDW13]

As final example of this section we return to the finite running examples.

Example 4.35. Consider the AF F from Example 3.4, Figure 3.2, and the AF G from Example 4.1, Figure 4.1a. For $\sigma \in \{pr, sm, sg, st, c2, s2\}$ and $\tau \in \{gr, id, eg\}$ we have

$$co(F) = \{\emptyset, \{c,u\}, \{d,a\}\}, \qquad co(G) = \{\{2\},\{1,2,4\},\{1,2,5\}\},$$
$$\sigma(F) = \{\quad \{c,u\}, \{d,a\}\}, \qquad \sigma(G) = \{\qquad \{1,2,4\},\{1,2,5\}\},$$
$$\tau(F) = \{\emptyset \qquad\qquad\qquad \}, \qquad \tau(G) = \{\{2\} \qquad\qquad\qquad \}.$$

Remark 4.36 (Locality and Globality of Argumentation Semantics). As final remark to this section we have a look at local/global nature of the introduced semantics. According to Definitions 3.6 and 4.2 the property of some argument set being a σ-extension can as well be local or global decidable. This allows us to distinguish between *local/syntactic* (*cf*, *ad*) and *global* (*gr*, *id*, *eg*, *pr*, *na*, *st*, *sm*, *sg*, *c2*, *s2*) semantics.

4.3 Reasoning with abstract argumentation semantics

In this section we discuss approaches of reasoning with argumentation semantics, i.e. regarding Example 4.1 we are now at Figure 4.1c. Before talking about acceptance and justification we first make sure that the semantics under consideration are defined for every AF.

Remark 4.37 (Well-Definedness). Formally, according to Definition 4.2 any semantics is well-defined as a mapping from AFs to extension sets. However, as illustrated in Example 4.23, implicitly defining some semantics via extension properties can lead to cases that are not well-defined, i.e. cases that formally are not semantics. To be clear, for any semantics σ, any AF F and any set $S \subseteq A_F$ *well-definedness* requires that exactly one of $S \in \sigma(F)$ and $S \notin \sigma(F)$ holds.

By Example 4.23 we know that Definition 4.22 does not give a (well-defined) semantics. For this reason in this thesis we use Definition 4.24 as fall-back for semantics based on SCC-recursiveness. The following lemma resolves the issue of well-definedness for all semantics under consideration.

Lemma 4.38. *All semantics $\sigma \in \{cf, ad, na, pr, st, sm, sg, c2, s2, co, gr, id, eg\}$ are well-defined. For arbitrary AF F and argument set $S \subseteq A_F$ exactly one of $S \in \sigma(F)$ or $S \notin \sigma(F)$ holds.*

Proof. We make use of axiomatic set theory, cf. Section 2.2. By PAIR, UN, POW and RCO we have that $S \times S \subseteq \wp(\wp(S))$ is a set. Then again by RCO we get that $\alpha = \{(x,y) \in S \times S \mid (x,y) \in R_F\}$ is a proper set. If $\alpha = \emptyset$ then $S \in cf(F)$, the precondition for all other semantics holds. Otherwise $S \notin \sigma(F)$.

Neutrality and defense function are well-defined. Hence ad, co, st, gr are well-defined as well. For sets $S, T \subseteq A_F$ the predicates $S \subseteq T$ and $S_F^+ \subseteq T_F^+$ are well-defined and hence also pr, na, sg, sm, id, eg are well-defined.

Finally the SCC-reduct $\mathscr{R}_S(F)$ always yields a fixed point since with each application we can only remove arguments/attacks and never add any. In more detail, for the set of sub-AFs $\mathbb{F} = \{G \subseteq F\}$ we can define the partial order $G \sqsubseteq H \iff G \supseteq H$ (Definition 2.20). Then for any non-empty chain $\{G_i = (A_i, R_i) \mid i \in \alpha\}$, $G_i \sqsubseteq G_j$ for $i < j$ we have that the intersection AF $G_\alpha = (\bigcap_{i\in\alpha} A_i, \bigcap_{i\in\alpha} R_i)$ is a least upper bound. Now observe that $(\mathbb{F}, \sqsubseteq)$ is a chain complete poset (Definition 2.25) and $\mathscr{R}_S$ serves as inflationary function (Definition 2.27). We can thus apply Bourbaki-Witt (Theorem 2.26) and conclude existence of $\mathscr{R}_S^\triangle(F)$. For $S \in \sigma_{SCC}(F)$ it then remains to verify that $S \in \sigma(\mathscr{R}_S^\triangle(F))$, hence also *c2* and *s2* are well-defined. $\square$

Observe that computation of fixed points from an input for *c2* and *s2* is a requirement for well-definedness, while for the other semantics that can be defined via fixed point the computation of this fixed point already results in an extension set. If for some AF (and fragment of ZFC) the empty set does not lead to a fixed point of the characteristic function, then the grounded semantics would still be defined but result in the empty extension set. The question of whether for a given fragment of ZFC some semantics produces extension sets is thus also of interest and will be discussed extensively in Chapter 6.

Definition 4.39 (Existence and Collapse)**.** A semantics σ fulfills *existence* for a given AF F if $\sigma(F) \neq \emptyset$, oppositely we say that σ *collapses* for F if $\sigma(F) = \emptyset$. Further by *non-empty existence* we refer to the occurrence of some $S \in \sigma(F)$ with $S \neq \emptyset$.

Example 4.40. Consider stable semantics and the AFs $F = (\emptyset, \emptyset)$, $G = (\{x\}, \emptyset)$, and $H = (\{x\}, \{(x,x)\})$. We have existence for F as $st(F) = \{\emptyset\}$, non-empty existence for G as $st(G) = \{\{x\}\}$, and collapse for H as $st(H) = \emptyset$.

As already hinted to in Example 4.1, Figure 4.1c we further distinguish between acceptance states. Due to the motivation (see Chapter 1) of abstract argumentation structures and semantics, given some AF and semantics, we might prefer arguments that appear in every extension over arguments that do not. With the following definition we formalise this intuition.

Definition 4.41 (Acceptance states)**.** Given semantics σ, AF F and argument x, we distinguish the following *acceptance states*, sometimes also called *justification states*.

- *rejection*: $x \in A_F \setminus \bigcup \sigma(F)$;
- *credulous acceptance*: $x \in \bigcup \sigma(F)$;
- *skeptical acceptance*: $x \in \bigcap \sigma(F)$.

Example 4.42. Consider the AF F from Example 4.1, Figure 4.1a and semantics σ with $\sigma(F) = \{\{1,2,4\},\{1,2,5\}\}$. As depicted in Figure 4.1c we have rejection of argument 3, credulous acceptance of arguments $1,2,4,5$ and skeptical acceptance of arguments $1,2$. Observe that skeptical acceptance always implies credulous acceptance. It might be argued that as argument 2 is not attacked at all while argument 1 is, there should be some further fine graining regarding their justification. This however is not focus of this thesis and in this case anyway reflected by the grounded extension. Justification states always implicitly give further fine graining by considering various different argumentation semantics.

Above acceptance states, existence and non-empty existence are well-established methods for investigating AFs [BCG11]. In particular these are questions that are naturally asked when researching computational complexity of argumentation semantics [DW09]. On an atomic level acceptance is a semantic evaluation of arguments. It deems us natural to also evaluate attacks semantically. While a publication is in the making a concise such evaluation finds its premier in this thesis. A very similar preceding investigation of conflict relations between arguments can be found in [BDL$^+$16].

Definition 4.43 (Semantic Conflict, cf. Definition 3.7)**.** Given some extension set $\mathbb{S}$ (or AF F and semantics σ with $\sigma(F) = \mathbb{S}$), argument sets x^0, y^0, we define[2]

- *(semantic) independence*, written $\{x,y\}^{ind}_{\mathbb{S}}$, if for each $a \in x^0$, $b \in y^0$ there is some $S \in \mathbb{S}$ with $a,b \in S$; x and y are then called *(semantically) independent* in $\mathbb{S}$;
- *(semantic) conflict*, written $[x,y]^{cnf}_{\mathbb{S}}$, if there is $a \in x^0$, $b \in y^0$ such that for all $S \in \mathbb{S}$ at most one of $a \in S$ or $b \in S$ holds; x and y are then called *(semantically) conflicting* in $\mathbb{S}$.

If no ambiguity arises we might drop the subscript $\mathbb{S}$. Similarly we can define above notions on single argument (sets) x by $\{x\}^{ind} = \{x,x\}^{ind}$ and $[x]^{cnf} = [x,x]^{cnf}$. Observe that semantic

[2] Observe that this definition talks about arbitrary argument sets, rather than sets of acceptable arguments. The reason for this is of notational nature. Whether or not to call any rejected argument semantically conflicting is debatable, but does not affect the results provided in this thesis.

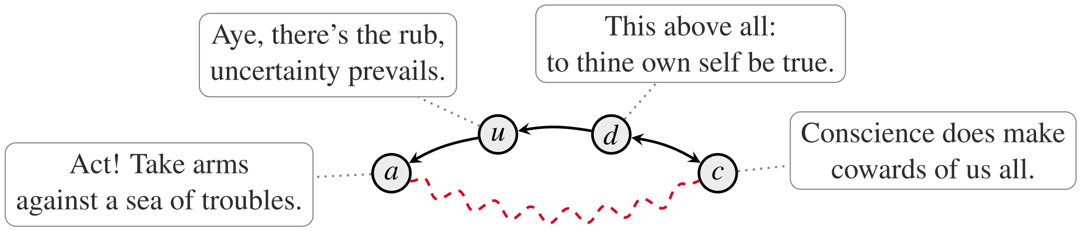


Figure 4.11: Semantic conflict and independence explained for Example 3.4, cf. Example 4.44.

attacks and conflicts due to the symmetric nature of extension sets are indistinguishable, we hence define $(x,y)^{att}_{\mathbb{S}} = [x,y]^{cnf}_{\mathbb{S}}$.

Observe that $\{x,y\}^{ind}$ and $\{x \cup y\}^{ind}$ are not the same as the first might not even imply $\{x\}^{ind}$, regardless of whether we consider syntactic or semantic conflicts. This can for instance be seen for a simple AF $F = \{\{x,y\},\{(x,x),(y,y)\}\}$. We will come back to the interplay between semantic and syntactic conflicts in Part III.

Example 4.44. Consider the AF F from Example 3.4, as depicted in Figure 4.11. As highlighted in Example 4.35 for $\sigma \in \{pr, sm, sg, st, c2, s2\}$ we have $\sigma(F) = \{\{c,u\},\{d,a\}\}$. This leads to semantic independences $\{c,u\}^{ind}_{\sigma(F)}$ and $\{d,a\}^{ind}_{\sigma(F)}$, and semantic conflicts $[c,d]^{cnf}_{\sigma(F)}$, $[c,a]^{cnf}_{\sigma(F)}$, $[d,u]^{cnf}_{\sigma(F)}$ and $[u,a]^{cnf}_{\sigma(F)}$. Further observe that we have syntactic conflicts only for $[c,d]^{cnf}_{F}$, $[d,u]^{cnf}_{F}$ and $[u,a]^{cnf}_{F}$, while aside from syntactic independences $\{c,u\}^{ind}_{F}$ and $\{d,a\}^{ind}_{F}$ there is also $\{c,a\}^{ind}_{F}$. Hence $[c,a]^{cnf}$ appears as semantic but not syntactic conflict, as depicted by a red dashed snake line in Figure 4.11.

In the context of Hamlet's reasoning we thus arrive at an interesting observation. While "*c: conscience*" and "*a: act*" do as such not contradict each other, for the given (fragment of an) AF they still are not jointly acceptable and hence present a semantic conflict. For the matter of manipulation thus semantic conflict introduces an option for introducing incompatibilities between arguments, where syntactically they seem compatible. And indeed, as hinted to in Example 1.4, we can interpret the introduction of argument r as conflict enforcement masterminded by William Shakespeare.

Chapter 5

Preliminary Properties of Abstract Argumentation Semantics

In the previous two chapters we separately introduced syntax and semantics of abstract argumentation. Syntax as formal structures also called argumentation frameworks (AFs), and semantics as evaluation methods of such AFs to establish a meaning of justification. In this chapter we combine syntax and semantics and present preliminary results and observations regarding their relations.

As a starter we recall that syntactic modifications (see Section 3.2) are of local nature (see Definition 3.21) in that for a given AF F and argument (set) x a modification $\varphi_x(F)$ might result in no changes to F beyond a fixed distance to x. Semantics on the other hand are global functions on AFs. For instance for stable semantics it does not suffice to consider even connected components when one is interested in credulous acceptance. With the following notion (cf. Definition 3.6) we introduce a property of predicates that reflects this distinguishing observation.

Definition 5.1 (Local decidability)**.** Given some argument (set) x, a predicate φ is called *(k-)local decidable over x* if there is a natural number k such that for any AF F we can decide φ on $F|_{x^k}$, i.e. $\varphi_x(F) = \varphi_x(F|_{x^k})$.

Example 5.2. Recall the definitions from Section 4.3 of rejection, credulous and skeptical acceptance, semantic conflict and semantic independence. For naive semantics all of these properties are local decidable. Rejected arguments need to be self-attacking, which means 0-local decidability for rejection and credulous acceptance. Skeptical acceptance is solely possible for arguments whose only neighbors are self-attacking, similarly semantic and syntactic conflict/independence coincide, which means 1-local decidability for skeptical acceptance, semantic conflict and semantic independence.

For any other semantics $\sigma \in \{co, pr, st, sg, sm, c2, s2, gr, id, eg\}$ we do not have local decidability. This can for instance be seen for any natural number n with the well-founded AF F_n with $A_{F_n} = (n+1) \cup \{a, b\}$ and $R_{F_n} = \{(i, i+1) \mid i < n\} \cup \{(n, a), (n, b)\}$, cf. Figure 5.1. For even n we have $\sigma(F) = \{0, 2, \ldots n\}$, while for odd n we have $\sigma(F) = \{0, 2, \ldots n-1, a, b\}$. This means that a and b are jointly accepted/rejected for odd/even n. Hence for any natural number k and

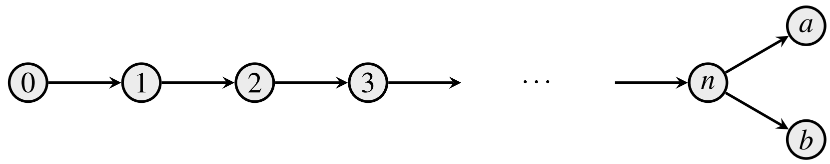


Figure 5.1: AF used in Example 5.2 to illustrate that reasoning questions from Section 4.3 are not locally decidable for complex semantics (see Definition 5.3).

k-locality around a (and b) we could at most consider an AF isomorphic to F_{k-1} and thus we get that rejection, credulous/skeptical acceptance, semantic conflict/independence are not k-local decidable for σ.

This example once more illustrates substantial differences between conflict-free and naive semantics as opposed to the remaining semantics introduced in Chapter 4. For future reference and clarification we use this prejudice for a distinguishing definition.

Definition 5.3 (Common vs. complex semantics)**.** We call semantics $\sigma \in \{ad, cf, na\}$ *common semantics*, and $\tau \in \{co, pr, st, sg, sm, c2, s2, gr, id, eg\}$ *complex semantics*.

The remainder of this chapter is organised as follows: In Section 5.1 we introduce and discuss semantic criteria that appear to be desired properties for argumentation semantics of interest. These criteria range from necessities to extras. In Section 5.2 we collect and elaborate on various notions from the literature we subsume as equivalence criteria. Such notions will be used frequently throughout this thesis. Finally, in Section 5.3 we collect acquired knowledge in a compendium of AF classes of interest.

5.1 Extension evaluation criteria

With Remark 4.37 (well-definedness) we introduced a semantic property we essentially require for semantics σ under consideration, i.e. that for any AF F and argument set $S \subseteq A_F$ we can decide whether or not S is a σ-extension of F. In this section we will first (Subsection 5.1.1) introduce further such properties that we require for any reasonable abstract argumentation semantics, and then (Subsection 5.1.2) properties that seem reasonable but for one reason or another are not fulfilled by all semantics under consideration.

5.1.1 Fair argumentation semantics

Definition 5.4 (Basic criterion)**.** A semantics σ is called *basic* if there is an AF F such that there is some σ-extension $S \in \sigma(F)$ and some argument $x \in S$:

$$\exists\, AF\ F\ \exists S \in \sigma(F) : S \neq \emptyset.$$

The basic criterion essentially requires semantics to sometimes accept some arguments at least credulously. Semantics that do not fulfill this property can have only empty extension sets and empty extensions, i.e. for a semantics σ that is not basic and any AF F either $\sigma(F) = \emptyset$ or $\sigma(F) = \{\emptyset\}$ holds. Clearly, such semantics do not provide much insight into argumentation processes and hence are not considered in this thesis. This property is very fundamental, we introduced it in [Spa16b] for the proof of Theorem 5.17. Next we present a property that might not hold for natural language argumentation but is a precondition for abstract argumentation. For the definition of this condition we will use a broader modification type derived from the renaming modification from Example 3.20.

Definition 5.5 (Renaming AFs)**.** By *renaming* we refer to an arbitrary modification ρ, as an injective function from argument sets to argument sets. We define accordingly for any AF F the ρ-renaming as $\rho(F) = (\rho(A_F), \rho(R_F))$ where $\rho(A_F) = \{\rho(x) \mid x \in A_F\}$ (and hence $|\rho(A_F)| = |A_F|$) and $\rho(R_F) = \{(\rho(x), \rho(y)) \in \rho(A_F) \times \rho(A_F) \mid (x, y) \in R_F\}$. Further for sets $\mathbb{S}$ of sets of arguments (extension sets) we implicitly define $\rho(\mathbb{S}) = \{\rho(S) = \{\rho(x) \mid x \in S\} \mid S \in \mathbb{S}\}$.

Hence a renaming of an AF F results in an isomorphic AF G with possibly distinct arguments $A_F \cap A_G = \emptyset$. The following property could be defined in various ways, we choose this approach for readability. Observe that this property is most often called language independence, which from our point of view however appears to be especially confusing and misleading as we are not actually talking about different languages but rather only different names.

Definition 5.6 (Name independence)**.** A semantics σ is called *name independent* if for each renaming $\rho(F)$ the semantic evaluations of F and $\rho(F)$ are isomorphic as well:

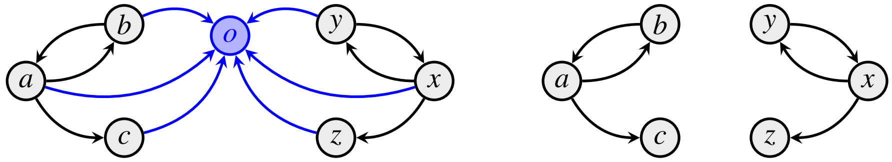
$$\forall \textit{ AF } F \; \forall \textit{ renaming } \rho : \sigma(\rho(F)) = \rho(\sigma(F)).$$

Name independence hence can be seen as a formalization of the "*abstract*" in "*abstract argumentation*". Name independence can also be seen as a fairness condition in that we want to evaluate arguments solely based on the attack relations, not on their names. Semantics that are not name independent might for instance treat apples different from oranges without giving an argument for the unequal treatment. Similarly we might observe that distinct connected components of AFs should allow us to evaluate them separately to combine results afterwards.

Definition 5.7 (Component independence)**.** A semantics σ is called *component independent* if for each AF F the semantic evaluation of separate connected components can be processed separately:

$$\forall \textit{ AFs } F, G \textit{ with } F \cap G = \emptyset\colon\ \sigma(F \cup G) = \{S \cup T \mid S \in \sigma(F), T \in \sigma(G)\}$$

The essential meaning of component independence is as follows: Consider a distinct partition of some AF $H = F \cup G$ with $F \cap G = \emptyset$. The σ-extensions of H then are expected to treat F and G essentially the same way and can thus be computed by listing all possible combinations of σ-extensions of F and G separately.



(a) One connected component, AF F.   (b) Two connected components, AF G.

Figure 5.2: AFs from Example 5.9 for the illustration of an artificial semantics that is not component independent.

Example 5.8. Component independence does not mean that a semantics might not collapse. For instance for stable semantics and the AF $F = (\{a,b\},\{(a,a)\})$ with two distinct connected components $\{a\}$ and $\{b\}$, we have $st(F|_{a^0}) = \emptyset$ and $st(F|_{b^0}) = \{\{b\}\}$. By component independence we expect $st(F) = \{S \cup T \mid S \in st(F|_{a^0}), T \in st(F|_{b^0})\} = \{S \cup T \mid S \in \emptyset, T = \{b\}\} = \emptyset$, and indeed F does not have a stable extension.

Component independence is called well-definedness in [Spa16b]. The intention there was to point out how fundamental this property is. Since we have the need for a more traditional use of well-definedness, and to highlight the similarity of the concept with name independence we changed the name for this thesis. All semantics under consideration clearly are component independent. The above definition might not seem intuitive at first sight, which is why we follow up with a (very artificial) example of a semantics that violates component independence.

Example 5.9. Consider a semantics σ such that for connected AFs F we have $\sigma(F) = pr(F)$. For syntactically disjoint AFs σ first computes the sub-extensions for each connected sub-AF and then collects all sub-extensions of same cardinality into one unifying extension.

For instance for the AF F depicted in Figure 5.2a and the AF G depicted in Figure 5.2b we have that $G = F|_{\{a,b,c,x,y,z\}}$, F consists of only one connected component while G consists of two disjoint connected components. Observe that in F each argument among a,b,c,x,y,z attacks o, which due to admissibility results in o being rejected. Further observe that $F|_{\{a,b,c\}} = G|_{\{a,b,c\}}$ and $F|_{\{x,y,z\}} = G|_{\{x,y,z\}}$ are isomorphic AFs. We get

$$\begin{aligned}
\sigma(F) &= \{\{a,x\},\{a,y,z\},\{b,c,x\},\{b,c,y,z\}\},\\
\sigma(F|_{\{a,b,c\}}) = \sigma(G|_{\{a,b,c\}}) &= \{\{a \quad\}, \qquad\qquad\qquad \{b,c \qquad\}\},\\
\sigma(F|_{\{x,y,z\}}) = \sigma(G|_{\{x,y,z\}}) &= \{\{\quad x\}, \qquad\qquad\qquad \{\qquad y,z\}\},\\
\sigma(G) &= \{\{a,x\}, \qquad\qquad\qquad \{b,c,y,z\}\}.
\end{aligned}$$

For a semantics σ to violate component independence it is required to treat components differently. While such a different handling could be achieved by simply also violating name independence, it might as well involve just counting components or other structural differentiations as illustrated in Example 5.9. Such structural manipulations in turn can emulate language dependence. For this line of thought think of encoded argument structures that serve

	cf	*ad*	*co*	*na*	*pr*	*st*	*sg*	*sm*	*c2*	*s2*	*gr*	*id*	*eg*
well-defined	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
basic	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
name independent	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
component independent	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
fair	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

Table 5.1: Satisfaction summary of necessary semantic properties with regards to argumentation semantics made use of in this thesis, cf. Remark 5.11.

as markers. Hence we would expect reasonable semantics to respect component independence. The following definition pays respect to these expectations.

Definition 5.10 (Fair semantics)**.** An argumentation semantics σ is called *fair* if it is basic, name independent and component independent.

As stated before with this we conclude semantic properties we see as necessities and shortly proceed to more contested criteria.

Remark 5.11. We refer to Table 5.1 for an almost superfluous summary of prior semantic properties in the light of abstract argumentation semantics of interest. In this table a checkmark in line φ and column σ means that semantics σ fulfills the φ-criterion.

5.1.2 Evaluation properties

In this subsection we are back to more conventional ways. The following three criteria (I-maximality, directionality and crash-resistance) are well researched in the finite case. In this thesis we extend focus to the infinite case and with Theorem 5.17 highlight that the complicated notion of crash-resistance (and with it contaminating AFs and non-interference) are redundant to an AF never collapsing. This insight seems to have been used frequently intuitively (e.g. [Wey11]), yet to the best of our knowledge was not published anywhere before [Spa16b]. The step from crash to collapse requires the formalization of the criteria of basic and component independence which also are new in [Spa16b]. We start with the less disputed criteria.

Definition 5.12 (I-maximality)**.** An extension set $\mathbb{S}$ is called *I-maximal* if there are no $S,T \in \mathbb{S}$ with $S \subset T$. A semantics σ is called *I-maximal* if for any AF F already $\sigma(F)$ is I-maximal.

Example 5.13. Consider the AF $F = (\{a,b\},\{(a,b),(b,a)\})$. We have $cf(F) = ad(F) = co(F) = \{\emptyset,\{a\},\{b\}\}$. Hence cf, ad and co semantics are not I-maximal.

We have talked about the motivation for SCC-recursiveness in Chapter 4. Here comes a formal definition of directionality. Observe that there are similarities with component independence. The latter requires unconnected components to be treated independently. Directionality on the other hand says that strongly connected components can be treated sequentially.

Definition 5.14 (Directionality)**.** A semantics σ is called *directional* if for each AF and initial component of the SCC-tree, i.e. argument set C such that C does not have any incoming attacks in F, the σ-semantic evaluation of the C-restriction $F|_C$ computes as C-restriction of the σ-semantic evaluation of F:

$$\forall\, AF\; F \forall C \subseteq A_F \textit{ with } C_F^- \subseteq C \mid \sigma(F|_C) = \{S \cap C \mid S \in \sigma(F)\}.$$

Example 5.15. Consider the AF $F = (\{a,b\},\{(a,b)\})$. We have $SCC(F) = \{\{a\},\{b\}\} = na(F)$. As $na(F|_{\{a\}}) = \{\{a\}\} \neq \{\{a\},\emptyset\} = \{\{a\}\cap\{a\},\{b\}\cap\{a\}\}$, naive semantics violates directionality. Now consider $G = (\{a,b,c\},\{(a,b),(b,a),(b,c),(c,c)\})$. We have $st(G) = \{\{b\}\}$, while restriction to the first component delivers $st(G|_{\{a,b\}}) = \{\{a\},\{b\}\}$. Thus also stable, stage and semi-stable semantics violate directionality.

The following definition should be handled with care, as we will immediately prepend a theorem to dispense with the elaborate plethora. Also note that since we will not make use of the following definition in this thesis we keep the involved notational peculiarities at a minimum. We refer to [BG07, BG09, BCG11] for a detailed discussion. Finally, we do not make use of instantiated argumentation (such as [MP14], [KAK$^+$11] or [BCDG13]) or labelling approaches in this thesis, hence we did not investigate whether our Theorem 5.17 carries over in a meaningful way to such systems. For abstract argumentation focussing on extension based semantics however the following can be considered redundant background noise.

Definition 5.16 (Non-interference, contamination and crash-resistance)**.** A semantics σ fulfills the *non-interference* property if for any AF F and separate connected component C ($C_F^{\pm} \subseteq C$) we have $\sigma(F|_C) = \{S \cap C \mid S \in \sigma(F)\}$. An AF F_0 such that for all disjoint AFs F_1 we have that $\sigma(F_0 \cup F_1) = \sigma(F_0)$ is called *contaminating* AF. A semantics σ is called *crash-resistant* if it does not provide contaminating AFs.

Subsequently we will call the occurrence of a semantics that is not non-interfering as *interference* of the semantics, the existence of a contaminating AF a *contamination*, and the event of a disjoint union with a contaminating AF a *crash*.

Observe that obviously component independence implies non-interference. However as Example 5.9 shows there might be non-interfering semantics without contaminating AFs and hence crash-resistance that are not component independent. We now follow up with the promised theorem, enabling us to let go of the complicated notions of interference, contamination and crash. Recall that some semantics σ is said to collapse if there is some AF F with $\sigma(F) = \emptyset$.

Theorem 5.17 (The simplicity of collapse)**.** *For fair argumentation semantics the notions of contamination, interference, crash and collapse are equivalent.*

Proof. Assume that for some AF F we have $\sigma(F) = \emptyset$, i.e. a collapse. By component independence for any disjoint AF G (with $A_F \cap A_G = \emptyset$) we get $\sigma(F \cup G) = \emptyset$ (as illustrated in

	cf	*ad*	*co*	*na*	*pr*	*st*	*sg*	*sm*	*c2*	*s2*	*gr*	*id*	*eg*
I-maximal	-	-	-	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
directional	✓	✓	✓	-	✓	-	-	-	✓	✓	✓	✓	-
crash-resistant	✓	✓	✓	✓	✓	-	✓	✓	✓	✓	✓	✓	✓

Table 5.2: Table indicating evaluation criteria for semantics of interest for finite AFs, cf. Remark 5.19.

Example 5.8), i.e. σ crashes on F and also violates the non-interference property with F being a contaminating AF.

Now assume σ does not collapse for any AF and consider some arbitrary syntactically disjoint AFs F and G ($A_F \cap A_G = \emptyset$). Since σ does not collapse on F or G we have $\mathbb{S} = \sigma(F)$ and $\mathbb{T} = \sigma(G)$ for some non-empty sets of sets of arguments $\mathbb{S} \neq \emptyset$ and $\mathbb{T} \neq \emptyset$. Observe that we could still have only empty extensions, $\mathbb{S} = \mathbb{T} = \{\emptyset\}$, in which case $\sigma(F \cup G) = \{\emptyset\}$ which is fine with non-interference. Thus consider that there is some $S \in \mathbb{S}$ with $S \neq \emptyset$, existence of such F is granted by σ being basic. By definition of semantics (extensions are subsets of the arguments) we get $\bigcup \mathbb{S} \cap \bigcup \mathbb{T} = \emptyset$. Finally by component independence we get $\sigma(F \cup G) = \{S \cup T \mid S \in \sigma(F), T \in \sigma(G)\}$, which satisfies the non-interference property. And in case there is $T \in \mathbb{T}$ with $T \neq \emptyset$ (such G exists by σ being basic and name independent) there will be $S \cup T \in \sigma(F \cup G)$ with $S \cup T \notin \mathbb{S} \cup \mathbb{T}$, i.e. σ is crash-resistant and there is no interfering AF. □

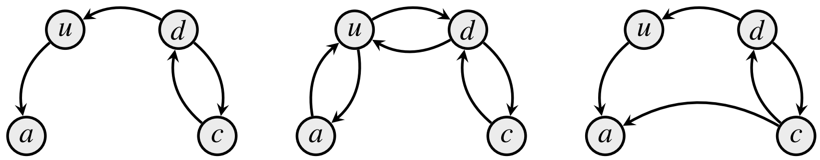
To comply with the literature we can hence give a renewed definition of crash-resistance avoiding the notion of crash, interference and contamination altogether.

Definition 5.18 (Crash resistance)**.** A semantics σ is called *crash resistant* if it never collapses, i.e. if there is no AF F with $\sigma(F) = \emptyset$.

Remark 5.19. The presented semantic evaluation criteria are a subset of the established criteria, see [BCG11]. At this point we only present a summary of semantics meeting criteria in the finite case, as taken from the literature [BCG11, DG16]. We will expand this investigation to the infinite case in Part II. For Table 5.2 observe that a checkmark in line φ and column σ means that semantics σ fulfills criterion φ for finite AFs, while a dash means that it does not.

5.2 Semantic Equivalence and Modifications

Semantic equivalence of AFs has gained interest in the last couple of years, see for instance [OW11, ABV14, GG14, CSAD15, BB15]. This covers notions where AFs are considered equivalent for a given semantics provided they deliver the same extension set, or where AFs are considered strongly equivalent if they still provide the same extension set under monotone modifications (see Definition 3.21), or where arguments in AFs are considered equivalent if they allow the same conclusions. Equivalence in a broader sense, ranging not only over single semantics but for the purpose of comparing AFs, can also be seen as a feature of investigations relating



(a) AF F (b) AF $G = sym(F)$ (c) AF H

Figure 5.3: Illustrational examples of semantic equivalence, cf. Example 5.21.

to intertranslatability [DW11, DS17, GM16] or signatures [DSLW16, BDL$^+$16, DDLW15]. In this section we will introduce and briefly discuss the most common notions. We will refer to these notions repeatedly throughout this thesis.

Definition 5.20 (Semantic equivalence I)**.** Given some semantics σ, AFs F and G are called *σ-equivalent* (or *semantically equivalent*) if $\sigma(F) = \sigma(G)$.

Example 5.21. Consider the variants of the AF from Example 3.4 depicted in Figure 5.3. We have AF F depicted in Figure 5.3a, AF G depicted in Figure 5.3b and AF H depicted in Figure 5.3c, with $A_F = \{c,d,u,a\}$, $R_F = \{(c,d),(d,c),(d,u),(u,a)\}$, and G,H being secluded modifications of F where $G = sym(F)$ and $R_H = R_F \cup \{(c,a)\}$.

For $\sigma \in \{pr, st, sg, sm, c2, s2\}$ we have

$$
\begin{aligned}
cf(F) = cf(G) = ad(G) &= \{\emptyset, \{c\}, \{d\}, \{u\}, \{a\}, \{c,u\}, \{d,a\}, \{c,a\}\} \\
ad(F) = ad(H) = cf(H) &= \{\emptyset, \{c\}, \{d\}, \qquad\qquad \{c,u\}, \{d,a\} \qquad\quad \} \\
na(F) = na(G) = \sigma(G) &= \{ \qquad\qquad\qquad\qquad\quad\;\; \{c,u\}, \{d,a\}, \{c,a\}\} \\
\sigma(F) = \sigma(H) = na(H) &= \{ \qquad\qquad\qquad\qquad\quad\;\; \{c,u\}, \{d,a\} \qquad\quad \}
\end{aligned}
$$

Hence for conflict-free and naive semantics AFs F and G are semantically equivalent while for admissible semantics and σ AFs F and H are semantically equivalent.

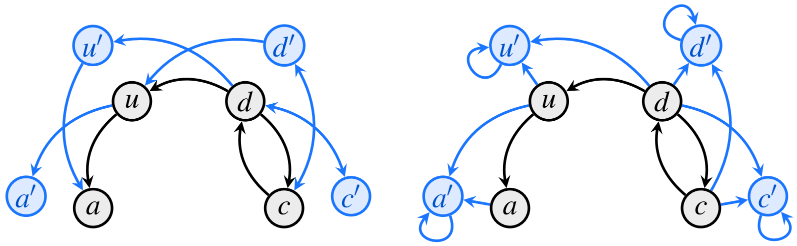
We observe that equivalence between AFs can immediately be extended to a notion of equivalence between AFs coupled with possibly different semantics.

Definition 5.22 (Semantic equivalence II)**.** Given AFs F,G and semantics σ,τ, we say that the pairs (F,σ) and (G,τ) are *(semantically) equivalent* if $\sigma(F) = \tau(G)$.

Obviously σ-equivalent AFs F,G are another way of stating that (F,σ) and (G,σ) are equivalent. For a slightly more elaborate equivalence we present the following example.

Example 5.23. Again consider the AFs F,G,H from Example 5.21 and the established semantic evaluation. We have that (F,cf), (G,cf), and (G,ad) are semantically equivalent. Similarly (F,σ), (H,σ) and (H,na) are semantically equivalent.

Intertranslatability refers to modifications that lead to semantically equivalent (or equivalent restricted to the initial argument set) modifications. Initially such modifications were sought for



(a) $shad(F)$ (b) $shad^*(F)$

Figure 5.4: Translations as applied to Example 3.4, cf. Example 5.25.

the benefit of using solver engines for multiple semantics. But intertranslatability has since then also been used to establish a sense of expressiveness.

Definition 5.24 (Intertranslatability)**.** A modification φ is called a *faithful translation* for $\sigma \to \tau$ if $\sigma(F) = \{S \cap A_F \mid S \in \tau(\varphi(F))\}$ and $|\sigma(F)| = |\tau(\varphi(F))|$, and *weakly faithful* with *remainder set* $\mathbb{T}$ if $\sigma(F) = \{S \cap A_F \mid S \in \tau(\varphi(F)) \setminus \mathbb{T}\}$ and $|\sigma(F)| = |\tau(\varphi(F)) \setminus \mathbb{T}|$. It is called an *exact translation* for $\sigma \to \tau$ if $\sigma(F) = \tau(\varphi(F))$ and *weakly exact* with *remainder set* $\mathbb{T}$ if $\sigma(F) = \tau(\varphi(F)) \setminus \mathbb{T}$. Remainder sets are required to be finite sets and are intended to contain simple sets such as $\emptyset$.

Example 5.25. Symmetrization is an exact translation for $na \to pr$ as is illustrated by AFs F and G from Example 5.21 with Figure 5.3a as origin AF and Figure 5.3b as translated AF.

The shadow modification *shad* as defined in Example 3.20 is a faithful translation for $co \to pr$ and $co \to st$. The loop shadow $shad_{loop}$ is an exact translation for $na \to sg$ and as well for $pr \to sm$. Proofs can be found in [Spa13, DS17] and [DW11]. We illustrate these results with application to the AF F from Example 5.21 in Figure 5.4. For instance for $co \to pr$ we have

$$\begin{array}{rclcccl} pr(shad(F)) & = & \{ & \{c',d',u',a'\}, & \{c',c,u',u\}, & \{d',d,a',a\} & \}, \\ co(F) & = & \{ & \emptyset, & \{c,u\}, & \{d,a\} & \}. \end{array}$$

Before we have first generalized semantic equivalence from a singular semantics over different AFs to different combinations of AFs and semantics. Translations as defined above are modifications matching the more general notion of semantic equivalence. Some observations call for a stricter form of translation.

Definition 5.26 (Semantic modification)**.** Given some semantics σ, a modification φ is called *(σ-)semantic* for AF F if $\sigma(F) = \sigma(\varphi(F))$. It is called *($\sigma$-)semantic* if it is σ-semantic for all AFs F.

Example 5.27. Consider AFs F, G, H from Example 5.21. The modification $R_H = R_F \cup \{(c,a)\}$ is pr-semantic for F. The modification *sym* is na-semantic for all AFs witnessed by $G = sym(F)$.

The idea of semantic modifications is, given AF F, to fix its extension set and alter the AF without changing this extension set. Research in this direction concerns structural modifications, i.e. syntactic modifications that do not have immediate semantic impact. If we take this idea a bit further we let go of AFs altogether and focus on extension sets.

Definition 5.28 (Realizability)**.** Given semantics σ, an extension set $\mathbb{S}$ is called σ*-realizable* if there is an AF F such that $\sigma(F) = \mathbb{S}$. Given a tuple of semantics $(\sigma_1, \sigma_2, \ldots \sigma_n)$, a tuple of extension sets $(\mathbb{S}_1, \mathbb{S}_2, \ldots \mathbb{S}_n)$ is called $(\sigma_1, \sigma_2, \ldots \sigma_n)$*-realizable* if there is an AF F such that $\sigma_1(F) = \mathbb{S}_1$, $\sigma_2(F) = \mathbb{S}_2$, $\ldots \sigma_n(F) = \mathbb{S}_n$.

Example 5.29. For conflict-free semantics any extension set $\mathbb{S}$ such that for all $S \subseteq \bigcup \mathbb{S}$ with $\{S\}^{ind}_{\mathbb{S}}$ we have $S \in \mathbb{S}$ is *cf*-realizable. This immediately leads to a realizability characterization of naive semantics as well. See [DDLW15] for a more detailed investigation.

The extension set $\mathbb{S} = \{\{1,2\},\{2,3\},\{1,3\}\}$ is not σ-realizable for any semantics $\sigma \in \{cf, ad, co, pr, st, sm, sg, c2, s2, gr, id, eg\}$. It is however realizable for minimal dominating sets as is witnessed by the digraph $D = (\{1,2,3\},\{(1,2),(2,3),(3,1)\})$.

Some argumentation semantics apparently allow for appealing characterizations regarding realizability. In [DDLW15] we find explicit notions for semantics $\sigma \in \{cf, ad, pr, st, sm, sg, na\}$. The following definition is almost borrowed from there (and [DSLW16]) with the main difference being that above consider finite AFs only.

Definition 5.30 ((Multi-dimensional) signatures)**.** The class Σ_σ of all possible extension sets for a given semantics σ (over a fixed argument set or universe of arguments $\mathbb{A}$) is called σ*-signature (over* $\mathbb{A}$*)*. An extension set $\mathbb{S}$ then *belongs* to the σ-signature (over $\mathbb{A}$) if $\mathbb{S} \in \Sigma_\sigma$. Analogue notions can be derived for multi-dimensional signatures over tuples of semantics. If $\mathbb{A}$ is important for an observation we might write $\Sigma_\sigma(\mathbb{A})$.

Example 5.31. For the empty universe of arguments $\mathbb{A} = \emptyset$ we have the empty AF $F = (\emptyset, \emptyset)$ as only possible realization and hence $\Sigma_\sigma(\emptyset) = \{\{\emptyset\}\}$ for all semantics under consideration.

For a universe consisting of one argument, e.g. $\mathbb{A} = \{x\}$ we do have three different AFs $(\emptyset, \emptyset)$, $(\{x\}, \emptyset)$ and $(\{x\}, \{(x,x)\})$. Hence $\Sigma_{st}(\{x\}) = \{\{\emptyset\}, \emptyset, \{\{x\}\}\}$.

Remark 5.32. Signature could also be defined as equivalence classes of AFs for which a given semantics provides the same extension sets. The substantial difference between our definition and the one given in [DDLW15] though is that we allow arbitrary argument sets for our argumentation structures. This difference immediately transfers back to the realizability notion as realizability over a fixed set of (additional) arguments as follows.

Definition 5.33 ($\mathbb{A}$-realizability)**.** Given a semantics σ and a set of arguments $\mathbb{A}$, an extension set $\mathbb{S}$ is called *(σ-)realizable in* $\mathbb{A}$ if there is an AF F such that $A_F \subseteq \mathbb{A}$ and $\sigma(F) = \mathbb{S}$. It is called *(σ-)$\mathbb{A}$-realizable* if it is realizable in $\mathbb{A} \cup \bigcup \mathbb{S}$. A realizing AF F is then called a *(σ-)realization of* $\mathbb{S}$ *in* $\mathbb{A}$ or, respectively, a *(σ-)$\mathbb{A}$-realization of* $\mathbb{S}$.

Example 5.34. We continue Example 5.31 with the observation that $\{\emptyset\}$ is $\mathbb{A}$-realizable for any universe $\mathbb{A}$ and any semantics σ under consideration. The extension set $S = \emptyset$ is *st*-$\mathbb{A}$-realizable

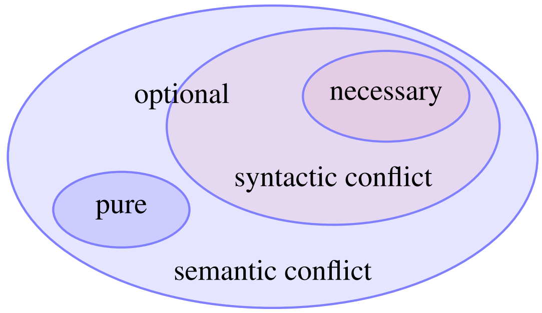


Figure 5.5: A Venn-diagram illustrating different levels of conflict, cf. Remark 5.37.

for any non-empty universe $\mathbb{A}$. For $\sigma \in \{sm, sg\}$ on the other hand S is σ-$\mathbb{A}$-realizable if and only if $\mathbb{A}$ is at least countable.

Above notions of intertranslatability, signature, semantic modifications and realizability from a certain point of view are variants or fine graining of semantic equivalence. In the following we will use these notions to characterise the possible interplay of semantic and syntactic conflicts

Definition 5.35. Consider semantics σ and extension set $\mathbb{S}$ (or AF F with $\sigma(F) = \mathbb{S}$). We will optionally restrict ourselves to a universe $\mathbb{A}$ of arguments in the following. A semantic conflict $[x,y]_{\mathbb{S}}^{cnf}$ or a semantic attack $(x,y)_{\mathbb{S}}^{att}$ is called

- *($\mathbb{A}$-)necessary (syntactic)* if for each ($\mathbb{A}$-)realization F it is also syntactic, i.e. $[x,y]_F^{cnf}$;
- *($\mathbb{A}$-)pure (semantic)* if for each ($\mathbb{A}$-)realization F we have syntactic independence $\{x,y\}_F^{ind}$;
- *($\mathbb{A}$-)optional* otherwise.

Example 5.36. Recall the AF F from Example 5.21, Figure 5.3a. As both $\{c\}$ and $\{d\}$ are admissible extensions but $\{c,d\}$ is not, we have that $[c,d]_{ad(F)}^{cnf}$ is necessary and due to admissibility and symmetry even $(c,d)_{ad(F)}^{att}$ and $(d,c)_{ad(F)}^{att}$ are necessary. Further $[c,d]_{cf(F)}^{cnf}$ is necessary due to the nature of conflict-free semantics, but neither $(c,d)_{cf(F)}^{att}$ nor $(d,c)_{cf(F)}^{att}$ are necessary since either one would suffice. In [BDL$^+$16] and in this thesis in Chapter 10 we find AFs with $\emptyset$-pure conflicts.

Remark 5.37. In Figure 5.5 we have an illustration of the possible conflict levels. Observe that for this illustration we assume some AF F and semantics σ as given. Only then we can talk about semantic and syntactic conflicts, conflicts that are necessarily syntactic or purely semantic. Finally this illustration as a Venn-diagram talks about sets. The sets considered are semantic conflicts to begin with. Hence members of these sets can be arbitrary pairs of sets of arguments.

5.3 Argumentation Framework Classes

Recall that we have already defined syntactic AF classes, i.e. symmetric, loop-free, bipartite, (even/odd) circle/cycle-free, complete, planar, finite and infinite AFs (Remark 3.3); finitary, finitely/finitarily superseded and well-founded AFs (Definition 3.13). In this section we will introduce AF classes based on semantic evaluation. We start with a class we will repeatedly (though implicitly) visit in Part II.

Definition 5.38 ((Non-)collapsing AFs)**.** Given semantics σ, an AF F is said to be *collapsing* if we have $\sigma(F) = \emptyset$, otherwise it is *non-collapsing*.

Apparently the topic of collapsing and not collapsing was already very important in Section 5.1. And apparently we would prefer to avoid the collapse of AFs whenever possible. This leads to a further generalization of this concept. Kernel-perfection is a widely discussed graph property [Ber85, GN84].

Definition 5.39 (Perfection)**.** Given semantics σ, an AF F is called σ*-perfect* if for each induced sub-AF ($G = F|_B$ for some argument set $B \subseteq A_F$) it does not collapse:

$$\forall B \subseteq A_F : \sigma(F|_B) \neq \emptyset.$$

With the following we present an often considered AF class. It was introduced in [Dun95] and its characterizing feature is that most complex semantics coincide, which allows for very general investigations, see [DB02]. We extend this notion to cover all multi-status I-maximal complex semantics.

Definition 5.40 (Coherent AFs)**.** An AF F is said to be *coherent* if $st(F) = pr(F)$. Further F is called *super-coherent* if $st(F) = pr(F) = sg(F) = c2(F)$.[1]

Example 5.41. Consider the AF $F = (\{x,y,z\}, \{(x,y),(y,x),(x,z),(y,z)\})$. We have $ad(F) = co(F) = \{\emptyset, \{x\}, \{y\}\}$, $cf(F) = ad(F) \cup \{\{z\}\}$, $na(F) = \{\{x\},\{y\},\{z\}\}$ while for any $\sigma \in \{pr, sg, sm, st, c2, s2\}$ we get $\sigma(F) = \{\{x\},\{y\}\}$, and thus $gr(F) = id(F) = eg(F) = \{\emptyset\}$. That is, for super-coherent AF F and $\tau \in \{cf, na, ad, co, gr, id, eg\}$ we can have $\sigma(F) \neq \tau(F)$.

Example 5.42. Consider the AF F from Example 5.2. For all complex semantics σ hence we have $\sigma(F) \neq \emptyset$, i.e. F is super-coherent and σ does not collapse on F. Further any induced sub-AF can easily be seen to provide unique extensions as well (this in fact is a property of well-founded AFs). Hence this AF is also σ-perfect.

Lemma 5.43. *Well-founded AFs F are super-coherent and we even get*

$$gr(F) = id(F) = eg(F) = co(F) = pr(F) = st(F) = sm(F) = sg(F) = c2(F) = s2(F) \neq \emptyset.$$

Proof. This was shown for coherence and Dung semantics in [Dun95]. Given an AF F with $st(F) \neq \emptyset$, we immediately get $sm(F) = sg(F) = st(F)$, ideal and eager semantics follow by definition. Since well-founded AFs provide initial SCCs for any induced sub-AF we can use the characteristic function not only to compute the grounded extension but also to compute the thus unique cf2 and stage2 extension. Due to Theorem 2.26 this already works for ZF. □

[1] See Remark 7.33 for the case of $st(F) = pr(F) = c2(F) = \emptyset$ while $sg(F) \neq \emptyset$ in models of set theory without AC. Proposition 6.49 together with Theorem 6.15 illustrates that in ZFC already AFs F with $st(F) = pr(F) = c2(F)$ are super-coherent.

Observation 5.44 (Running examples). Recall Examples 3.4, 3.5, and 4.1. All of these AFs are super-coherent by construction.

In [BDL$^+$14] focus of investigation was put on a semantic AF class that provides the substantial feature of only having accepted arguments. We will occasionally make use of this class and repeatedly provide examples from this class to illustrate that rejected (or self-attacking) arguments do not play a major role for the investigations of this thesis.

Definition 5.45 (Compact AFs)**.** Given semantics σ, an AF is called *compact* if $A_F = \bigcup \sigma(F)$.

Observe that compact AFs thus are $\emptyset$-realizable (compare Definition 5.33) and we could characterize this AF class alternatively by $\mathbb{A}$-realizability.

Example 5.46. Consider the AFs F from Example 3.4 and G from Example 3.5. For I-maximal complex semantics σ we have as extension sets $\sigma(F) = \{\{c,u\},\{d,a\}\}$ and $\sigma(G) = \{\{c,r,a,t_1,t_2\ldots\},\{d,a,t_1,t_2\ldots\}\}$. As $A_F = \{a,c,d,u\} = \bigcup \sigma(F)$ it turns out that F is σ-compact. On the other hand for G we have $A_G = \{a,b,c,d,p,r,u,t_1,t_2\ldots\}$ which contains arguments b,p,u which do not occur in any σ-extension of G. Hence G is not σ-compact.

The investigation [BDL$^+$14] of analytic AFs in regards of signatures did lead to a conjecture that involved the assumption that for stable semantics analytic AFs would show only semantic conflicts that are also syntactic. Put in other words this can also be called an assumption that stable semantics does not provide $\emptyset$-pure conflicts. We provided a counter example for this conjecture in [BDL$^+$16] (also see Chapter 10), which inspired investigation of AFs where semantic and syntactic conflicts correspond in a strong sense (see next definition) and also inspired the research this thesis lists in Part III.

Definition 5.47 (Analytic AFs)**.** Given semantics σ, an AF F is called *analytic* if for each pair of arguments $x,y \in A_F$ such that there is no $S \in \sigma(F)$ with $x,y \in S$ already $[x,y]_F^{cnf}$ holds.

Example 5.48. Recall Example 5.36 where we established that for naive and conflict-free semantics all conflicts are necessary and hence any compact AF F is analytic for naive and conflict-free semantics. The AF F from Example 3.4 though is not σ-analytic for any complex semantics. As highlighted in Example 4.44 we have $[a,c]_{\sigma(F)}^{cnf}$ while $\{a,c\}_F^{ind}$.

Part II

Infinite Argumentation Structures

Chapter 6

Existence of Extensions and Set Theoretic Principles

> The laws of determinacy and indeterminacy must be well understood. When the nature and origin of the experience cannot be explained in terms of the normal language of logic, we then ask,
> "Who was the experiencer?"
> I say, no one.
>
> Swami Satyananda, Bombay, March 1978 [Sat80]

In this chapter (built on insights from [Spa14]) we investigate the most basic set theoretic principles in regards of abstract argumentation semantics. That is, we take a look at set theoretic principles that grant or possibly hinder existence of extensions. As opposed to intuition built on the finite case this means that we will run into odd and possibly unexpected behaviour, for instance for models of ZF without AC. Beware that for this chapter in particular we try to capture as many set theoretic models of abstract argumentation as possible and hence often explicitly state which axioms we make use of. We will particularly highlight results that rely on or contradict the common assumption of AC, ZL, WO, see Definition 2.21.

Since existence is of main interest for this chapter, we are basically asking the question whether for a given collection of axioms, for some semantics σ and AF F we have $\sigma(F) \neq \emptyset$. By Lemma 4.38 for all semantics of interest the formula $S \in \sigma(F)$ is well-defined in ZF. Remark 4.3 highlights that thus the formula $\sigma(F) \neq \emptyset$ is well-defined in ZF as well.

As a first introspection we ask the most essential question. Are there always conflict-free and/or admissible sets? To build at least some tension regarding this question we highlight the following example, the most essential semantic bomb. See Chapter 7 for further examples of and an investigation into semantics bombs.

Example 6.1 (Mini-bomb)**.** Consider the AF $F = (\{x\}, \{(x,x)\})$, that is the AF consisting of a single self-attacking argument. For $\sigma \in \{cf, ad, na, pr, sm, sg, c2, s2, co, gr, id, eg\}$ we have

$$st(F) = \emptyset, \qquad \sigma(F) = \{\emptyset\}.$$

Thus F collapses for stable semantics and produces only the empty argument set as extension for all other semantics under consideration. In particular this means that no semantics provides non-empty extension sets here. But only for stable semantics we have that for any AF G with $F \cap G = \emptyset$ we can automatically conclude $st(F \cup G) = st(F)$. This is the reason we might refer to F as a st-bomb, cf. Definition 7.8.

The preceding example might have come as less of a shock to most readers, but essentially already provides all the ingredients for subsequent more 'shocking' revelations. We do not ask for conditions of non-empty existence but rather for plain existence. The following result is our starting point.

Theorem 6.2 (Existence of Conflict-free/Admissible Sets)**.** *In ZF every AF F provides conflict-free and admissible sets,* $cf(F) \neq \emptyset$*,* $ad(F) \neq \emptyset$*.*

Proof. Consider the set $S = \emptyset$. The empty set can not be attacked and is always defended. By definition we have $S \cap S^+ = \emptyset$ and hence $S \in cf(F)$, the empty set is not attacking any set of arguments, in particular not itself. Further, also by definition for any AF F we get $S_F^+ = \emptyset$ and $S_F^- = \emptyset$. Thus, since the empty set is neither attacking nor attacked by any arguments we also get $S_F^- \subseteq S_F^+$, i.e. $S \in ad(F)$. □

The previous results might seem trivial. Be prepared for a steep increment of non trivial matter for the following sections. In Section 6.1 we focus on results that are independent from the use of AC and equivalent formulations. In Section 6.2 we highlight results depending on AC and even the inherent correlation between existence of certain extensions and AC. In Section 6.3 we discuss variations of AC. In Section 6.4 we investigate alternate models of set theory and a specific axiom that contradicts AC yet might be necessary for other existence issues of abstract argumentation: the axiom of determinacy. In Section 6.5 we elaborate on relations between semantics in ZF and ZFC. In Section 6.6 we conclude this chapter and highlight our achievements.

6.1 Bourbaki-Witt and Hartogs' Lemma

For this section we focus on results that are independent from whether or not we believe in AC. That is, in general we are not able to facilitate transfinite induction (Theorem 2.16), and the only remaining friend in terms of induction thus is Theorem 2.26, Bourbaki-Witt making use of Hartogs' Lemma. That is, if we can define the induction steps without making use of a choice function, and we can provide a monotonically growing set, and there is a natural limit to the size of this set, then transfinite methods might not need AC. Observe that the following results are similarly already given in [Dun95]. Our main objective is to highlight which set theoretic axioms we actually make use of for each step.

From a more technical point of view this section foremost provides tools that are independent from variations of AC. Particularly one might consider cf, ad, co semantics as preliminary or secondary semantics and the defense function as helper function in the sense that their main purpose is to allow definition of more elaborate semantics. Then the only real result of this section is regarding existence of grounded extensions. For definition of grounded extensions via transfinite induction we require a base case (delivered by Theorem 6.2), and successor as well as limit steps. We proceed accordingly by putting the defense function (as candidate for a successor step) into perspective. As a function the well-definedness of $\mathscr{D}_F$ is given for models of set theory with REP.

Lemma 6.3 (Monotonicity of the defense function)**.** *Assume AF F and argument sets $S,T \subseteq A_F$ with $S \subseteq T$. It then follows that also $\mathscr{D}_F(S) \subseteq \mathscr{D}_F(T)$.*

Proof. To see this observe that by definition we get $S^+_F \subseteq T^+_F$ and thus subsequently for $x^-_F \subseteq S^+_F$ also $x^- \subseteq T^+_F$ holds. □

This covers inherited comparability of the defense function, we now turn to monotonicity. Observe that monotonicity of repeated iteration of $\mathscr{D}_F$ in general is not given. For instance for a directed three cycle F with $A_F = \{1,2,3\}$ and $R_F = \{(1,2),(2,3),(3,1)\}$ we have $\mathscr{D}_F(\{1\}) = \{3\}$, $\mathscr{D}_F(\{3\}) = \{2\}$, $\mathscr{D}_F(\{2\}) = \{1\}$ and so on. However for admissible sets there is an advantage.

Lemma 6.4 (Defense Function $\mathscr{D}_F$ on Admissible Sets)**.** *Given AF F and admissible set $S \in ad(F)$, we have $S \subseteq \mathscr{D}_F(S)$, and $\mathscr{D}_F(S) \in ad(F)$.*

Proof. To see this we need only consider the definitions. Admissible sets are defined as conflict-free sets (that is $S \cap S^+_F = \emptyset$) that defend themselves (that is $S^-_F \subseteq S^+_F$). The defense operator $\mathscr{D}_F$, given some argument set as input, returns all those arguments that are defended by that set (that is $(\mathscr{D}_F(S))^-_F \subseteq S^+_F$). Since for $S \in ad(F)$ we have self defense, naturally we get $S \subseteq \mathscr{D}_F(S)$. Since $\mathscr{D}_F(S)$ is already defended by S and S is a subset of $\mathscr{D}_F(S)$ we also get self defense of $\mathscr{D}_F(S)$: $(\mathscr{D}_F(S))^-_F \subseteq S^+_F \subseteq (\mathscr{D}_F(S))^+_F$.

Concerning conflict-freeness of $\mathscr{D}_F(S)$ assume for a contradiction a conflict $[\mathscr{D}_F(S)]^{cnf}_F$. Then by minimality of conflicts (Lemma 9.2) there are arguments $x,y \in \mathscr{D}_F(S)$ such that $(x,y)^{att}_F$. Regarding membership in $S' = \mathscr{D}_F(S) \setminus S$ or S we distinguish four cases.

1. $x,y \in S$, then already $[S]^{cnf}_F$;
2. $x \in S$, $y \in S'$, then by conflict-freeness of S we have that S can not defend y against x;
3. $y \in S$, $x \in S'$, then from $x^-_F \subseteq S^+_F$ it follows that $(S,y)^{att}_F$ and thus case (2);
4. $x,y \in S'$, then from $y^-_F \subseteq S^+_F$ it follows that $(S,x)^{att}_F$ and thus again case (2). □

The defense function on admissible sets thus provides a monotonic operator as required by Bourbaki-Witt. This means that for the transfinite induction defining grounded extensions we can use Lemma 6.4 as successor step. Next we take a look at limit steps, i.e. the question whether chains of *cf*/*ad* sets define in turn *cf*/*ad* sets, the union of which is defined granted UN.

Lemma 6.5 (Conflict-free and Admissible Chains)**.** *Consider some AF F, semantics* $\sigma \in \{cf, ad\}$ *and chain of* σ*-extensions* $(S_i)_{i\in\alpha}$ *(that is a well-ordered set* α*,* $S_i \in \sigma(F)$ *and for* $i < j$ *we have* $S_i \subseteq S_j$*). Then* $\bigcup_{i\in\alpha} S_i \in \sigma(F)$*.*

Proof. Define $S = \bigcup_{i\in\alpha} S_i$. For successor ordinals $\alpha = \beta + 1$ we naturally receive $S = S_\beta$ and thus $S \in \sigma(F)$. For $\sigma = cf$ and limit ordinals α assume for a contradiction $[S]_F^{cnf}$, a syntactic conflict in S. Then we may use minimality of conflicts, Lemma 9.2, i.e. there are arguments $x, y \in S$ with $[x, y]_F^{cnf}$. Now by definition of $S = \bigcup_{i\in\alpha} S_i$ there are $i, j \in \alpha$ with $x \in S_i$ and $y \in S_j$. Since α is well-ordered w.l.o.g. $i \leq j$ holds and thus $x \in S_j$ as well. Then immediately also $[S_j]_F^{cnf}$, a contradiction. Thus necessarily $\{S\}_F^{ind}$ and $S \in cf(F)$ hold.

Similarly for $\sigma = ad$, where we already know $S \in cf(F)$, assume for a contradiction $S \notin ad(F)$. We then have $S_F^- \setminus S_F^+ \neq \emptyset$ and hence there is $x \in S_F^-$ with $x \notin S_F^+$. Since $S_i \in ad(F)$ for all $i \in \alpha$ such x does not exist for any i, that is $x \notin S_i$ and thus $x^+ \cap S_i = \emptyset$ for all $i \in \alpha$. But then we also get $x^+ \cap S = \emptyset$ and thus $x \notin S_F^-$. □

The preceding results would already allow for existence results of grounded semantics. However, for the sake of completeness, we interlude with a result on complete semantics. Observe that uniqueness of minimal admissible fixed points of the defense function is a non-trivial result.

Lemma 6.6 (Uniqueness of Minimal Complete Sets)**.** *In ZF, for any AF F, given some admissible set* $S \in ad(F)$*, there is a unique minimal complete extension* $S' \in co(F)$ *with* $S \subseteq S'$*.*

Proof. Consider some admissible set $S \in ad(F)$ as given. The defense function (Definition 4.6) serves as increasing function as required by Theorem 2.26, see Lemma 6.4. We use as non-empty chain complete poset $ad(F)$ together with the usual subset relation. The partial order property is given by definition. For chain completeness consider Lemma 6.5.

Thus by Bourbaki-Witt (Theorem 2.26) $\mathscr{D}_F(S)$ eventually arrives at some fixed point $\mathscr{D}_F^\triangle(S)$ for the restriction of $ad(F)$ to all sets generated by $\mathscr{D}_F(S)$. This fixed point is again admissible and due to the nature of this fixed point a complete extension, $\mathscr{D}_F^\triangle(S) = \mathscr{D}_F(\mathscr{D}_F^\triangle(S))$. Further there is a unique minimal such fixed point. This is inherent to the proof of Bourbaki-Witt, which for a contradiction constructs an injective function from the class of all ordinals into the set A_F. Such injective functions then contradict Hartogs' Lemma. □

The beauty of Lemma 6.6 is that the defense function delivers unique minimal fixed points starting from any admissible set. We make use of this insight in the final result of this section for grounded and complete extensions.

Theorem 6.7 (Existence of Complete/Grounded Extensions)**.** *In ZF every AF F provides complete extensions,* $|co(F)| > 0$*. Moreover there always is exactly one grounded extension,* $|gr(F)| = 1$*.*

Proof. With Theorem 6.2 we have that $ad(F) \neq \emptyset$ which combined with Lemma 6.6 results in $co(F) \neq \emptyset$. Now since $\emptyset \in ad(F)$ and monotonicity of the defense operator (Lemma 6.3) We have that first $\mathscr{D}_F^{\triangle}(\emptyset) \in co(F)$ and further $\mathscr{D}_F^{\triangle}(\emptyset) \subseteq S$ for any $S \in co(F)$ and hence $\mathscr{D}_F^{\triangle}(\emptyset) \in gr(F)$ and $|gr(F)| = 1$. □

That is as far as we can get without the use of AC in the general sense. Recall Lemma 5.43 which highlights that already in ZF well-founded AFs are super-coherent. Hence, if for some AF every path has a starting point then existence of extensions for all considered semantics is granted already in ZF. Further, since AC is not relevant in the finite case, existence with ZF is also given for finite AFs and naive, preferred, stage, semi-stable, cf2, stage2 semantics. One might ask similar questions for other classes of AFs. We elaborate on such in Chapter 8.

6.2 Axiom of Choice

In this section we show further existence results with the use of AC and, most notably, we show that these results are not possible without a choice function. Results from this section have been featured already in [Spa14, BS15] but are genuine work of this thesis. To highlight the targets we first give a formal definition of existence.

Definition 6.8 (Existence of Naive/Preferred Extensions)**.** Define the following statements:

- $\exists NA$: existence of naive extensions, for any AF F we have $na(F) \neq \emptyset$;
- $\exists PR$: existence of preferred extensions, for any AF F we have $pr(F) \neq \emptyset$.

We proceed by showing that given AC (that is, in ZFC), both $\exists NA$ and $\exists PR$ are theorems and even *cf*/*ad* sets can always be extended to *na*/*pr* extensions.

Lemma 6.9 (Extending *cf* and *co* sets)**.** *In ZFC, given arbitrary AF F,*

1. *for any $S \in cf(F)$ there exists $S' \in na(F)$ s.t. $S \subseteq S'$, and*
2. *for any $S \in ad(F)$ there exists $S' \in pr(F)$ s.t. $S \subseteq S'$.*

Proof. For this proof we make use of Zorn's Lemma (Definition 2.21, an equivalent formulation of AC). Consider $(\sigma, \tau) \in \{(cf, na), (ad, pr)\}$, then by definition any τ-extension is a σ-extension as well. It remains to show that there are I-maximal σ-extensions (Definition 5.12). To this end observe that $\sigma(F)$[1] is a partially ordered set and for any chain $(S_i \in \sigma(F))_{i \in \alpha}$ we have that $S_\alpha = \bigcup_{i \in \alpha} S_i$ is a σ-extension as well by Lemma 6.5. We can hence apply Zorn's Lemma and thus receive a maximal σ-extension, that is a proper τ-extension. □

This lemma together with existence of conflict-free and admissible sets (or the empty set always being such a set, Theorem 6.2) gives the following proposition.

[1] Recall that we primarily define properties extensions have to fulfill, the extension set is thus defined as $\sigma(F) = \{x \in \wp(A_F) \mid x \in \sigma(F)\}$. That is, for its definition we make use of POW and RCO.

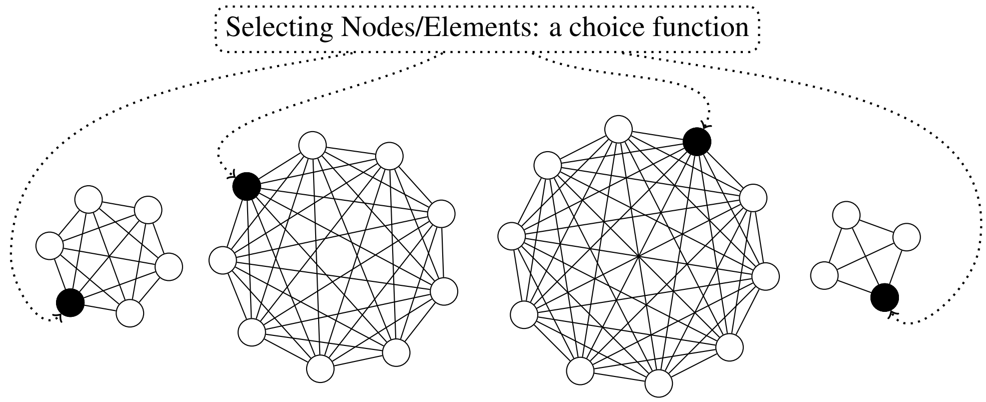


Figure 6.1: Illustration of Example 6.11 and Definition 6.12.

Proposition 6.10 (AC$\Rightarrow\exists NA$,$\exists PR$)**.** *In ZFC every AF F provides naive and preferred extensions,* $na(F) \neq \emptyset$, $pr(F) \neq \emptyset$*, that is* $\exists NA$ *as well as* $\exists PR$ *hold.*

In ZFC thus $\exists NA$ and $\exists PR$ are theorems. That however was the easy part of this section. For the remainder of this section we work on the other direction of an equivalence between choice and existence of naive/preferred extensions, or, in other words, impossibility of such existence proofs without the use of AC. To this end first consider the following example.

Example 6.11 ($\exists NA$,$\exists PR$ and ZF)**.** Consider as given a set of sets $\mathbb{S} = \{S_1, S_2, S_3, S_4\}$ where $|S_1| = 6$, $|S_2| = 7$, $|S_3| = 10$ and $|S_4| = 4$. For $i \neq j$ we further know $S_i \cap S_j = \emptyset$ but other than that we do not know any details about members of S_i. Nonetheless we can construct an AF F with $A_F = S_1 \cup S_2 \cup S_3 \cup S_4$ and symmetric attacks between different members of any given S_i. See Figure 6.1 for a visualization. Now assume $\exists NA$ or $\exists PR$. As illustrated in Figure 6.1 this yields an extension S where exactly one member of each set S_i is necessarily featured. We can thus use $\exists NA$ or $\exists PR$ to facilitate a choice function.

The previous example is the intuition behind the following definition. We have as input a set of sets, where we ask to choose one member of each. We thus need as output an AF where each extension represents a valid choice. We use as arguments the members of $\bigcup_{i\in\alpha} A_i$ indexed with i to indicate their original set membership, and for the attack structure we make use of complete sub-AFs consisting of different arguments with the same index. These complete sub-AFs then ensure that any *pr*/*na* extension defines a choice for the given input. Regarding ZF, this construction makes use of UN, POW and RCO.

Definition 6.12 (AC-framework)**.** Given a set of sets $(A_i)_{i\in\alpha}$, we abbreviate $x_i = (i,x)$ and define the symmetric and loop-free AF F with

$$A_F = \{x_i \mid i \in \alpha, x \in A_i\}; \qquad R_F = \{(x_i, y_i) \mid i \in \alpha, x \neq y \in A_i\}.$$

In above definition we use an index set α which might be an ordinal for illustrational purposes, but it might just as well be an arbitrary set. In research regarding set theoretic principles often attempts are made to reduce the amount of axioms used. For instance in [AL78] the authors investigate consistent subtheories consisting of any four axioms from EXT, UN, POW, REP, AC. Further nowadays many authors derive RCO from REP. For our purposes however it suffices to remark whether we are using ZF or ZFC. Now, given this definition, we use it to show that indeed AC follows from $\exists NA$ as well as from $\exists PR$.

Proposition 6.13 ($\exists NA$,$\exists PR \Rightarrow$AC)**.** *Considering ZF, if every AF provides a naive/preferred extension, then also AC holds.*

Proof. We assume a set of sets $(A_i)_{i\in\alpha}$ as given and use the AF F from Definition 6.12. Since F consists of exactly one symmetric loop-free clique for each set A_i any σ-extension (for $\sigma \in \{na, pr, sg, sm, st, c2, s2\}$) of F will contain exactly one member of each component to be identified with exactly one member of each set A_i. Existence of a σ-extension for this AF thus implies existence of a choice function for the initial set of sets. That is, given $S \in \sigma(F)$, we define $f(A_i) = \{x \in A_i \mid x_i \in S\}$. □

This result already is very powerful. To round up the subsequent equivalence theorem with the following lemma we further elaborate on the class of AFs implicitly used for Definition 6.12.

Lemma 6.14. *In ZF any symmetric loop-free AF F (aka. graph) is super-coherent and we even have* $na(F) = pr(F) = st(F) = sg(F) = sm(F) = c2(F) = s2(F)$.

Proof. First observe that in symmetric AFs conflict-freeness and admissibility coincide as each argument is self-defending. Also connected components and strongly connected components coincide since directed and undirected paths are the same for symmetric AFs. We thus have $na(F) = pr(F) = c2(F)$ and $sm(F) = sg(F) = s2(F)$. Naturally any *st*-extension is also a *pr*-extension. Now for any argument $x \in A_F$ we have $\{x\} \in ad(F)$ and hence for each $S \in pr(F)$ necessarily $x \in S^*_F$ (either $x \in S$ or S attacks x) holds, i.e. $pr(F) \supseteq st(F)$ and thus $pr(F) = st(F)$. Any *sm*-extension is a *pr*-extension by definition. Thus either *sm* collapses and with it all other semantics, or $st(F) \neq \emptyset$ and again the claim follows. □

Now finally we turn to one of the core results of this chapter, a collection of insights regarding so far established equivalence of statements on existence of extensions with AC.

Theorem 6.15. *In ZF the following are equivalent:*

- *AC, ZL and WO, Definition 2.21;*
- *$\exists NA$: every AF provides a naive extension;*
- *$\exists PR$: every AF provides a preferred extension;*
- *every symmetric loop-free AF provides a σ-extension for $\sigma \in \{st, sm, sg, c2, s2\}$.*

Proof. Essentially the summary of the previous results from this section. □

This settles the big questions. We still need to discuss the remaining semantics (id, eg) but first shed a bit light on models of set theory that do not guarantee AC.

Proposition 6.16 (ZF, ¬AC)**.** *In ZF without choice for $\sigma \in \{na, pr, eg, id\}$ there are AFs without σ-extensions.*

Proof. Consider set of sets $X = (A_i)_{i\in\alpha}$ such that there is no choice function. Then the by Definition 6.12 defined AF F can not provide naive or preferred extensions, i.e. $na(F) = pr(F) = \emptyset$. Any cf- or ad-set in F is not maximal, for if it were maximal it would define a choice function for X. In AFs without preferred extensions ideal extensions would simply be sets that are maximal admissible. However under these circumstances such would resolve to be preferred extensions again and thus also $id(F) = \emptyset$. Finally, since $pr(F) = \emptyset$ implies $sm(F) = \emptyset$ the same argument works for eager semantics and thus $eg(F) = \emptyset$. □

Hence, for some models of ZF we even lack ideal and eager extensions. Formally they are defined as maximal admissible sets contained in each extension of their base semantics (preferred for ideal and semi-stable for eager). This definition leaves some space for the assumption of multiple extensions for some AFs. We make use of another fixed point operator with the aim of showing that in case the base semantics gives a non-empty extension set, its ideal derivate gives a unique extension.

Definition 6.17 (Backwards defense function)**.** Given AF F and set of arguments $S \subseteq A_F$, define

$$\mathscr{B}_F(S) = S \setminus \{x \in S \mid x_F^- \setminus S_F^+ \neq \emptyset\}.$$

Thus, $\mathscr{B}_F(S)$ takes a set of arguments S as input, eliminates all arguments this set does not defend and returns the result.

The intended purpose of this function is to operate on $\bigcap \sigma(F)$ for base semantics σ. Then subsequently (at each application of $\mathscr{B}_F$) we lose arguments until we eventually arrive at some admissible fixed point. Such fixed point is guaranteed already in ZF by the following lemma.

Lemma 6.18 (Backwards defense fixed point)**.** *In ZF, given AF F with conflict-free set of arguments $S \subseteq A_F$, then $\mathscr{B}_F$ has a fixed point in S ($\mathscr{B}_F^{\triangle}(S) \subseteq S$) and further*

1. *$\mathscr{B}_F^{\triangle}(S) \in ad(F)$,*
2. *and for any set $T \in ad(F)$ with $T \subseteq S$ we have $T \subseteq \mathscr{B}_F^{\triangle}(S)$.*

Proof. We may use Bourbaki-Witt (Theorem 2.26) and the sets $A_F \setminus S$ for construction of an increasing function and hence fixed point $\mathscr{B}_F^{\triangle}(S)$. This fixed point needs to be admissible by definition of $\mathscr{B}_F$ proving (1). For (2) consider some admissible set $T \subseteq S$ and argument $x \in T$. Since $T \subseteq S$ and T defends x we have that S also defends x and thus $\mathscr{B}_F(S)$ contains x. Since elimination of arguments via $\mathscr{B}_F(S)$ needs to happen at some discrete point of application we get $x \in \mathscr{B}_F^{\triangle}(S)$. □

Thus already in ZF, given that the base semantics delivers a non-empty extension set, its ideal derivate is uniquely defined. We highlight this insight with a proposition.

Proposition 6.19 (Unique definedness of ideal family semantics)**.** *In ZF, given AF F, conflict-free semantics σ with $\sigma(F) \neq \emptyset$ and the σ-ideal semantics τ, then $|\tau(F)| = 1$. If $\sigma(F) \subseteq co(F)$ then further for $\{S\} = \tau(F)$ we have $S \in co(F)$.*

Proof. The first claim essentially follows from Lemma 6.18. We only need to show the second claim, thus assume $\sigma(F) \subseteq co(F)$. Now for $S = \mathscr{B}_F^{\triangle}(\bigcap \sigma(F))$ assume for a contradiction $S \notin co(F)$ and use the defense operator for $S' = \mathscr{D}_F(S)$, where by assumption thus $S \subset S'$ holds. By Lemma 6.4 we get $S' \in ad(F)$. By definition for any $T \in \sigma(F)$ we have $S \subseteq T$ and $\mathscr{D}_F(T) = T$. By Lemma 6.3 then $S \subseteq T$ implies $S' = \mathscr{D}_F(S) \subseteq \mathscr{D}_F(T) = T$ and thus a contradiction to S being maximal admissible subset of all σ-extensions. □

As final result of this section we use our knowledge to round up our insights for existence of ideal/eager extensions in case of ZFC.

Theorem 6.20 (ZFC and id, eg)**.** *In ZFC for any AF F we have $|id(F)| = 1$ and $|eg(F)| > 0$; in case $sm(F) = \emptyset$ we get $eg(F) = pr(F)$, in case $sm(F) \neq \emptyset$ we get $|eg(F)| = 1$.*

Proof. Recall that in ZFC any AF provides preferred extensions, Theorem 6.15. Then the set $\bigcap pr(F) = \{x \in A_F \mid \forall S \in pr(F) \text{ we have } x \in S\}$ (see Definition 4.31) is conflict-free and we can thus apply Proposition 6.19 for a unique maximal admissible (and complete set) contained in S which by definition is the searched for ideal extension.

In case $sm(F) = \emptyset$, maximal admissible sets contained in all semi-stable extensions are simply maximal admissible and hence preferred extensions, i.e. we get $eg(F) = pr(F)$. In case $sm(F) \neq \emptyset$ again we make use of Proposition 6.19 and retrieve a unique eager extension. □

6.3 Variations of Choice

In this section we take a little detour into the surroundings of ZFC, models of set theory with variations of choice. Upon reading the previous section one might wonder why bother going for so much trouble when clearly a choice function is a very intuitive concept and can easily be given explicitly for arbitrary collection of sets. This however is a misleading intuition. For a more elaborate overview on the matter see Section 2.3 or [Moo12, Jec73, Kra02]. Also we will discuss one particular world without choice in Section 6.4.

Consider for instance the uncountable set of all real numbers $\mathbb{R}$. WO (an equivalent of AC) states that this set can be well-ordered, i.e. that we can address all real numbers by transfinite induction. This well-ordering however can not be made explicit by our measures. Indeed, Cantor's diagonal argument (Example 2.2) not only shows that $\mathbb{R}$ is uncountable but also that every explicit and countable listing of real numbers lacks most of the members of $\mathbb{R}$. Thus, in a very profound way AC is not constructive.

Example 6.21 (Choosing atoms, cf. Example 2.4)**.** Consider a collection of H_2O molecules (e.g. a glass of water). We interpret the H atoms as arguments and the O atoms as symmetric attacks between adjacent H atoms. Clearly the resulting AF F is odd-circle-free. But is it bipartite? If it was bipartite, then the necessary partition $A_F = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$ with attacks only between A_1 and A_2 coincidentally also represents two stable extensions of F: $A_1, A_2 \in st(F)$. By construction A_1 also gives a choice function, selecting exactly one H atom from each H_2O molecule. We do know that a bucket full of water contains only finitely many H_2O molecules, but there is no known procedure for selecting exactly one H atom from each molecule, yet alone for infinite collections of H_2O molecules. Thus there are AFs that are odd-circle-free, yet can only be regarded as bipartite in models of ZF incorporating at least some variation of choice.

We next take a look at variations of choice. Observe that finite choice, i.e. existence of a choice function for finite collections of non-empty sets, does not require AC but rather is a logical consequence of the property of being finite. Common variations of choice alter the arbitrary collection of arbitrary sets clause.

Definition 6.22 (AC(x,y))**.** For $x, y \in \{\alpha, {}^{<}\alpha, \infty \mid \alpha \textit{ some cardinal}\}$ we refer to AC(x,y) as the axiomatic *existence of a choice function for collections of x many sets of size y*. Here ${}^{<}\alpha$ refers to collections or sets of lesser cardinality than α, while ∞ refers to arbitrary sets.

Remark 6.23 (AC(x,y)). Consider above definition on variatons of choice. For instance

- AC(ω,∞) also called *countable choice* [Ber42] is the statement that each countable collection of arbitrary non-empty sets grants a choice function;
- AC$(\infty, {}^{<}\omega)$ also called *choice for finite sets* [Mos45, Szm47, BT60, Wiś72, Tru73] is the statement that any collection of non-empty and finite sets grants a choice function; and
- AC$(\omega, |\mathbb{R}|)$ also referred to as *countable choice for real numbers* [Myc64] is the statement that each countable collection of non-empty sets of real numbers grants a choice function.

Any statement AC(x,y) can be transformed to an argumentation related statement (for instance $\exists NA(a,b)$) via Definition 6.12. And apparently any statement AC(x,y) holds in ZFC. Also some statements hold already in ZF, such as AC$(\infty,1)$ or AC$({}^{<}\omega,\infty)$. Conversely however, we have that for instance countable choice and choice for finite sets are weaker than AC and incomparable with each other. In the context of abstract argumentation we remark that even finitary AFs require some variation of choice for existence of $na, pr, sg, sm, st, c2, s2$ extensions as witnessed by the corresponding AC$(\infty, {}^{<}\omega)$-AF of arbitrarily many finite (and thus finitary) connected components. As for AC$(\omega, |\mathbb{R}|)$, this is a weaker form of AC(ω,∞).

Another common variation of choice is to alter the choice function itself rather than the involved cardinalities. For instance multiple choice [Gau68] considers multi-functions but is not of interest in our context. Dependent choice [Ber42] on the other hand builds upon a specific relation between members of the set to choose from. For further reading we refer to Jech's all knowing text book on set theory [Jec06]. Jech in [Jec73] also dedicated a book specifically to AC which is quite pleasant to read. Also Herrlich in [Her06] dedicated a book to AC in which particularly the extensive chapters on *Disasters without Choice*, *Disasters with Choice* and *Disasters either way* are very enlightening.

6.4 Axiom of Determinateness

The claim of argumentation structures being constructive in nature is something that will pop up ever and again in debates or reviewing processes. Similarly the question of whether or not to consider AC as granted for argumentation purposes might at first sight seem superfluous. So far we did not present any AFs where existence of extensions for some semantics contradicts AC. To highlight the necessity of taking into account different perspectives we present this section. Here we introduce a class of AFs that are designed for particular models of set theory without AC. Claim: as a very flexible logical formalism argumentation can be expected to provide examples for any kind of logic problems. In this section we take this claim a step further by taking a closer look at the continuum (that is sets of the kind $\mathbb{R}$, 2^ω, or ω^ω) and problems that occur if we assume existence of a well-ordering for $\mathbb{R}$ (as granted by WO/AC). We first formally introduce the class of games we are interested in. It was first discussed in [MS62], our definitions are based on [Her06, Chapters 6 and 7] and [Jec06, Chapter 33].

Definition 6.24 (G_A)**.** Let $\alpha \leq \omega$ and $A \subseteq 2^\alpha$. The game G_A is played as follows:

Two players choose alternately consecutive elements $x_0, x_1, x_2, \ldots \in \{0,1\}$. Each player knows, whenever it is his turn, the tuple of previously chosen elements. The first player (i.e. the one choosing $x_0, x_2, x_4, \ldots$) wins if the resulting sequence belongs to A. Otherwise the second player (i.e. the one choosing $x_1, x_3, x_5, \ldots$) wins.

Definition 6.25 (Sequences)**.** For α-length sequence $x_0, x_1, x_2 \ldots$ we may equivalently use the notions $x_0 x_1 x_2 \ldots$ or $(x_i)_{i<\alpha}$. We use the symbol $\cdot$ to denote the empty sequence. Given two sequences $s = a_0 a_1 \ldots a_n$ and $\gamma = b_0 b_1 \ldots$ we define their association as $s \cdot \gamma = a_0 a_1 \ldots a_n b_0 b_1 \ldots$.

Example 6.26. Consider a game G_A where $A = \{1111, 1110, 1001, 1000\}$, and the statements Alice: 1, Bob: 0, Alice: 1, Bob: 0. We have as resulting sequence 1010 which is not a member of A and hence Bob wins the game.

Definition 6.27 (Strategy and Determination)**.** Consider as given a game G_A with $A \subseteq 2^\alpha$ for some $\alpha \leq \omega$. A *strategy* is a mapping $f : \{s \in 2^\beta \mid \beta < \alpha\} \mapsto \{0,1\}$. Given a sequence $s = (x_i)_{i<\alpha}$ and a strategy f we define the *first player application* of f to s as $f \circ s = f(\cdot) \cdot x_1 \cdot f(f(\cdot) \cdot x_1) \cdot x_3 \cdot \ldots$, and complementary the *second player application* of f to s as $s \circ f = x_0 \cdot f(x_0) \cdot x_2 \cdot f(x_0 \cdot f(x_0) \cdot x_2) \cdot \ldots$. A strategy f is called a *winning strategy for the first player* provided that for any sequence $s = (x_i)_{i<\alpha}$ we have $f \circ s \in A$. Likewise a strategy g is called a *winning strategy for the second player* provided that for any sequence $s = (x_i)_{i<\alpha}$ we have $s \circ g \notin A$. The game G_A is called *determined* if one of the players has a winning strategy.

Example 6.28. Consider again the game G_A from Example 6.26. For a first player strategy with $f(\cdot) = 1$ and $f(ab) = b$ we have that for any sequence $s = (abcd)$ we get $f \circ (abcd) = 1bbd$. That is, we get $f \circ s \in \{1000, 1001, 1110, 1111\} = A$. In other words f is a winning strategy for the first player and thus Alice can always win this game.

A rather important and well-known result is that every finite game is determined. This can be seen by considering the game tree with root $\cdot$, transition edges for connected sequences and full plays as leaves. There assign winning states to the leaves and compute the winning state for each node upwards. This observation has lead to the following idea.

Definition 6.29 (Axiom of Determinateness, [MS62, Myc64])**.** The *axiom of determinateness* (AD) states that for any $A \subseteq 2^\omega$ the game G_A is determined.

For this section so far we have introduced another formalism. We now connect the formalism AD with the previous parts of this chapter on AC.

Lemma 6.30 (AC vs AD, [GS53], see [Jec06, page 628] for more details.)**.** *In ZF, assuming AC, there exists $A \subseteq 2^\omega$ such that the game G_A is not determined.*

Proof. First enumerate all 2^ω many strategies by WO. Then construct sets A and B by sequentially considering strategies f_i for $i \in 2^\omega$. We recursively add, first, a sequence $a_i = a_i \circ f_i$ to A such that a_i is not yet contained in B, and, second, a sequence $b_i = f_i \circ b_i$ to B such that b_i is not yet contained in A. To construct these sets A, B we require AC and the well-ordering of the strategies to ensure that we do not run out of sequences.

For the resulting game G_A the first player does not have a winning strategy, as for any strategy f_i we have ensured that the second player can force a sequence b_i not contained in A. Similarly for any strategy f_i there is sequence a_i the first player can enforce that is contained in A. Thus neither of the players has a winning strategy and G_A is not determined. $\square$

Observe that this means that already AC$(2^\omega, 2^\omega)$ contradicts AD. However by the following we have that AD implies a weak form of AC.

Lemma 6.31 (AD $\Rightarrow$ AC$(\omega, 2^\omega)$, [Myc64], see [Jec06, page 628] for more details)**.** *In ZF, granted AD, we have that AC$(\omega, 2^\omega)$ holds.*

Proof. We first elaborate on sequences in regards of their first player characteristics. For $n \in \omega$ define the sequence $\gamma_n = (a_i)_{i \in \omega}$ with $a_i = 1$ for $i < 2n$ and $a_i = 0$ otherwise. For sequence $\gamma = (x_i)_{i \in \omega}$ further define $\min_\gamma(0)$ as the index n of the first occurrence of $x_{2n} = 0$ in γ.

Assume $X = \{\emptyset \neq X_i \subseteq 2^\omega \mid i \in \omega\}$ as given. For sequence $\gamma = (x_i)_{i \in \omega}$ we define $n = \min_\gamma(0)$, and include γ in B whenever $x_1 x_3 x_5 \ldots \in X_n$. Define $A = 2^\omega \setminus B$ and the game G_A. Since the X_n are always non-empty the first player can not have a winning strategy. By AD thus the second player has a winning strategy f. Now observe that for $(y_i)_{i \in \omega} = s_n \circ f$ by construction we have $y_1, y_3, y_5, \ldots \in X_n$, i.e. $s_n \circ f$ serves as choice function for X. $\square$

Observation 6.32. Given set $A \subseteq 2^{\alpha}$ and game G_A for *potential winning strategies* it suffices to consider equivalence classes of strategies. That is, we call strategies f, g *equivalent first (or second) player strategies* if for all $\gamma \in 2^{\alpha}$ we have $f \circ \gamma = g \circ \gamma$ (or $\gamma \circ f = \gamma \circ g$).

AD has been widely discussed in the literature and is often considered a valuable alternative to AC. For further discussion, consequences of AD and the interplay with variations of AC we refer to [GS53, MS62, Myc64, Fen71, Jec73, Kle77, Her06, Jec06]. We now turn back to argumentation and present an AF reflecting the structure of above games.

In the next definition, aimed at game strategies, we use three different types of arguments:

1. First, we use arguments of the type A, B, and s for any partial sequence s. These arguments represent, respectively, a first player strategy, a second player strategy and a sequence possibly occurring for this particular strategy.

2. Second, we use arguments of the type $\overline{A}$, $\overline{B}$ and $\overline{s}$. These arguments are complimentary to type (1) in that they indicate their negation.

3. Finally, we use arguments S_{γ} for $\gamma \in 2^{\alpha}$. These arguments are needed to ensure that a strategy defends the player in question against all winning sequences of the other player.

For the attack relation we make use of several mechanisms, as explained in Remark 6.35. We have symmetric attacks between type (1) arguments of the same level, i.e. between A and B, and between $s \cdot 0$ and $s \cdot 1$ for partial sequence s. We have directional attacks from type (1) arguments to their type (2) counter arguments. We have directional attacks from type (2) arguments to irrelevant cases for inheritance of decisions made by type (1) arguments. The only other attacks are to type (3) arguments S_{γ}, which are self-attacking and attacked by $\overline{B}$ in case $\gamma \notin A$, by $\overline{A}$ in case $\gamma \in A$, and again by type (2) arguments making them irrelevant.

Definition 6.33 (The number game AF)**.** Given $\alpha \leq \omega$, a set $A \subseteq 2^{\alpha}$ and the game G_A, we define the *number game AF* $F_A = (A_A, R_A)$ with

$$
\begin{aligned}
&A_A = \{A, B, \overline{A}, \overline{B}\} \cup \{s, \overline{s} \mid n \in \omega, n \leq \alpha, s \in 2^n\} \cup \{S_{\gamma} \mid \gamma \in 2^{\alpha}\}, \textit{ and} \\
&R_A = \{(A,B), (B,A), (A,\overline{A}), (B,\overline{B}), (\overline{A},0), (\overline{A},1), (\overline{B},00), (\overline{B},01), (\overline{B},10), (\overline{B},11)\} \\
&\quad \cup \{(s\cdot 0, s\cdot 1), (s\cdot 1, s\cdot 0), (s\cdot 0, \overline{s\cdot 0}), (s\cdot 1, \overline{s\cdot 1}) \mid n < \alpha, s \in 2^n\} \\
&\quad \cup \{(\overline{s}, s\cdot 00), (\overline{s}, s\cdot 01), (\overline{s}, s\cdot 10), (\overline{s}, s\cdot 11) \mid n+1 < \alpha, s \in 2^n\} \\
&\quad \cup \{(S_{s\cdot\delta}, S_{s\cdot\delta}), (\overline{A}, S_{s\cdot\delta}), (\overline{s}, S_{s\cdot\delta}) \mid k \in \omega, 2k+1 \leq \alpha, s \in 2^{2k+1}, s\cdot\delta \in 2^{\alpha} \setminus A\} \\
&\quad \cup \{(S_{s\cdot\delta}, S_{s\cdot\delta}), (\overline{B}, S_{s\cdot\delta}), (\overline{s}, S_{s\cdot\delta}) \mid k \in \omega, 1 < 2k \leq \alpha, s \in 2^{2k}, s\cdot\delta \in A\}.
\end{aligned}
$$

Example 6.34. Consider the number game AF F_A for $A = \{1000, 1001, 1110, 1111\}$, depicted in Figure 6.2. For evaluation of stable semantics $st(F)$ consider candidate argument set S with $B \in S$. Then 10 and 11 are defended against $\overline{B}$ and at least one of them needs to be member of S, assume w.l.o.g. $11 \in S$. Now arguments 1110 and 1111 are defended and one of them needs to be member of S, assume w.l.o.g. $1111 \in S$. Now we get that $S_{1111} \notin S_F^+$ since its sole attackers $\overline{B}, \overline{11}, \overline{1111}$

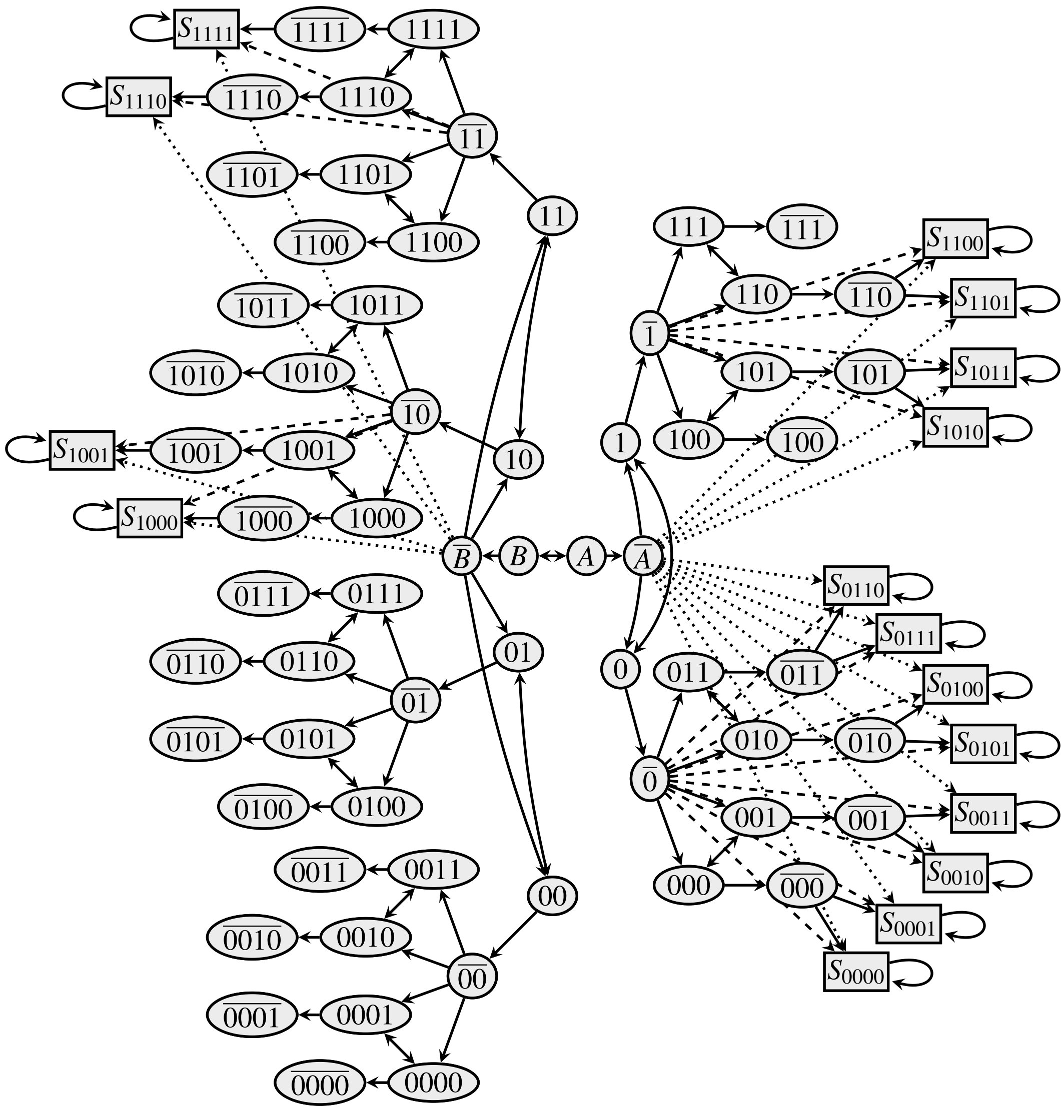


Figure 6.2: The number game AF according to Definition 6.33 for 2^4 game with A-set $\{0000, 0001, 0100, 0101\}$, cf. Example 6.34.

are attacked by S and hence S can not be a stable extension. Similarly we get that the only stable extension of F_A is $S = \{A, 1, \overline{0}, \overline{000}, \overline{001}, \overline{010}, \overline{011}, 100, \overline{101}, 111, \overline{110}\} \cup \{\overline{B}, \bar{s}, \overline{s \cdot t} \mid s,t \in 2^2\}$. Observe that S also represents a winning strategy for player I for the game G_A as discussed in Example 6.28, in fact it is even the only winning strategy of this game.

Remark 6.35 (The mechanisms behind F_A). The in Definition 6.33 given AF F_A has several properties of interest. We generalize the following three principles. (1) The only cycles occur for the self-attacking arguments S_γ, for the symmetric conflict $[A,B]^{cnf}$ and for partial sequences s as symmetric conflicts $[s \cdot 0, s \cdot 1]^{cnf}$. We call these 2-cycles *decision points*.

(2) Assign a cardinality to arguments from A_A, with $|\alpha| = |\overline{\alpha}|$, $|A| = |B| = 0$, $|x_1 x_2 \ldots x_n| = n$ and $|S_\gamma| = |\gamma| + 1$. Then the only remaining attacks $(\alpha, \beta)^{att}$ are directed and such that either $\beta = \overline{\alpha}$ or $|\alpha| < |\beta|$. We call this observation the *directionality principle* of F_A.

(3) Given preferred extension $S \in pr(F_A)$ where w.l.o.g. $A \in S$, by this directionality principle we have $\overline{B} \in S$ and for each $0 < k \in \omega$, $2k \leq \alpha$, $t \in 2^{2k}$ also $\bar{t} \in S$. Similarly for any decision point $[s \cdot 0, s \cdot 1]^{cnf}$ where w.l.o.g. $s \cdot 0 \in S$ we have $\overline{s \cdot 1} \in S$ and for each $k \in \omega$, $|s| < 2k+1 \leq \alpha$, $t \in 2^{2k}$ also $\overline{s \cdot 1 \cdot t} \in S$. We call this observation the *inheritance property* of F_A.

Lemma 6.36. *Consider as given a set $A \subseteq 2^{\alpha}$, the game G_A and the AF F_A. Each $S \in pr(F_A)$ accurately describes one equivalence class of strategies (cf. Observation 6.32) and vice versa.*

Proof. We make use of the three properties from Remark 6.35. Exactly one of A, B is member of S, marking S as a potential first player winning strategy in case $A \in S$, and as a potential second player winning strategy in case $B \in S$. Observe that each decision point $[s \cdot 0, s \cdot 1]^{cnf}$ of F_A (Remark 6.35) corresponds to one mapping $f(s)$. The inheritance property of F_A, together with its matching Observation 6.32, then delivers the claim. $\square$

Theorem 6.37 (F_A and G_A)**.** *In ZF, given $A \subseteq 2^{\omega}$, the following are equivalent:*

$$G_A \textit{ is determined} \qquad\qquad \textit{and} \qquad\qquad st(F_A) \neq \emptyset.$$

Proof. ($G_A \Rightarrow F_A$): First assume AD and thus determinateness of G_A with winning strategy f. Lemma 6.36 delivers a matching $S \in pr(F_A)$. The inheritance property ensures $\alpha, \overline{\alpha} \in S^*$ for $\alpha \in \{A, B\} \cup \{s \mid n \in \omega, n \leq \alpha, s \in 2^n\}$, it remains to show that $S_\gamma \in S^*$ for $\gamma \in 2^{\alpha}$.

Define $B = 2^{\alpha} \setminus A$ as the winning set of the second player and observe that $A \in S$ implies $\overline{B}$ attacking each S_γ for $\gamma \in A$ and similarly $B \in S$ implies $\overline{A}$ attacking each S_γ for $\gamma \in B$. For symmetry reasons w.l.o.g. we hence need only consider arguments S_γ for $\gamma \in B$ and first player winning strategy f and matching preferred extension S with $A \in S$.

Assume $S_\gamma \notin S^+$ for some $\gamma \in B$. Then for any $k \in \omega$, $2k+1 \leq \alpha$, $s \in 2^{2k+1}$ we have $\bar{s} \notin S$ and by inheritance hence $s \in S$. The decision points involved in construction of S however reflect choices of f. We thus have $f \circ \gamma = \gamma$ and hence f can not be a first player winning strategy.

($F_A \Rightarrow G_A$): Consider some $S \in st(F_A)$ as given and assume w.l.o.g. $A \in S$. By Lemma 6.36 we can derive a matching class $\mathbb{F}$ of first player strategies and assume some $f \in \mathbb{F}$.

Given $\gamma \in 2^{\alpha}$, assume for a contradiction that $f \circ \gamma \in A$ (where w.l.o.g. $\gamma = f \circ \gamma$) holds. By construction we have $S_\gamma \in S^+$. By inheritance and directionality (Remark 6.35) hence there is $k \in \omega, 2k < \alpha, s \in 2^{2k}$ such that w.l.o.g. $\gamma = s \cdot 1 \cdot \delta$, $\overline{s \cdot 1} \in S$ and $s \cdot 0 \in S$ hold. With $[s \cdot 0, s \cdot 1]^{cnf}$ being an active decision point of F_A for extension S we have that by Lemma 6.36 the sequence $f \circ \gamma$ has $s \cdot 0$ as initial sequence. But then, with $f(s) = 0$, $s \cdot 1 \cdot \delta = \gamma \neq f \circ \gamma$. $\square$

Scenarios, as highlighted by Lemma 6.30, are only possible in models of set theory where AC does not hold. The purpose of this result in particular and this section in general is to emphasize that in abstract argumentation we should be careful whether or not to assume AC/$\exists$*NA*/$\exists$*PR*.

The use of different semantic bombs (i.e. suitable sub-AFs instead of the self-attacking arguments S_γ) in Definition 6.33 leads to further similar results. By directionality of construction these are possible only for sufficiently directional semantics. In particular we either make use of admissibility or directionality for the following corollary.

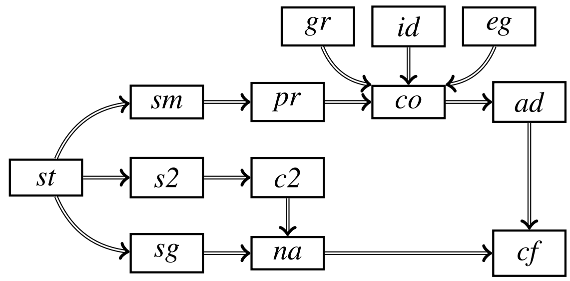


Figure 6.3: Subset Relations as highlighted in Proposition 6.39.

Corollary 6.38 (F_A^σ)**.** *In ZF, given $A \subseteq 2^\omega$, for $\sigma \in \{st, sm, c2, s2\}$ there are AFs F_A^σ such that the following are equivalent:*

$$G_A \text{ is determined} \qquad \text{and} \qquad \sigma(F_A^\sigma) \neq \emptyset.$$

For preferred semantics such constructions are possible as well, however due to Theorem 6.15 the equivalence would be with "AD or AC holds". Due to the indirect attacks from Definition 6.33 we can not apply this construction directly to stage semantics. For a different approach (i.e. a translation) for stage semantics we refer to Definition 7.11 and Theorem 7.14.

6.5 Relations between Semantics

In this section we discuss and summarize relations between the introduced argumentation semantics. First, we formally present the subset relations in terms of σ-extensions always being τ-extensions.

Proposition 6.39. *In ZF, given arbitrary AF F, the following hold:*

- $st(F) \subseteq sm(F) \subseteq pr(F) \subseteq co(F) \subseteq ad(F) \subseteq cf(F)$;
- $gr(F), id(F), eg(F) \subseteq co(F)$;
- $st(F) \subseteq sg(F) \subseteq na(F) \subseteq cf(F)$;
- $st(F) \subseteq s2(F) \subseteq c2(F) \subseteq na(F)$.

These results are depicted in Figure 6.3, where a directed path from semantics σ to semantics τ indicates that every σ-extension is also a τ-extension.

The results from Proposition 6.39 are fairly well known in the literature and typically one of the first questions of interest when considering a new semantics. Due to our claim of applicability to models of set theory where only ZF is granted we still provide the following proof, partially consisting of references to earlier results.

Proof. Admissibility of *ad*-based semantics *sm*, *pr*, *co*, *gr*, *id*, *eg*, as well as conflict-freeness of all semantics under consideration follows by definition. Due to monotonicity of the defense function (Lemma 6.3), admissibility of $\mathscr{D}_F(S)$ for $S \in ad(F)$ (Lemma 6.4), and maximal admissibility of preferred extensions we get $pr(F) \subseteq co(F)$. Lemma 6.18 (completeness of backwards defense fixed points) together with Proposition 6.19 (uniqueness of ideal semantics for non-empty base semantics) and the observation that ideal semantics default to preferred semantics in cases where the base semantics collapses we receive completeness of gr, id and eg.

Range-maximality implies subset-maximality and hence delivers $sm(F) \subseteq pr(F)$, $s2(F) \subseteq c2(F)$, $sg(F) \subseteq na(F)$. By definition ($c2 = na_{SCC}$) we have $c2(F) \subseteq na(F)$. Stable extensions are the ultimate range-maximal extensions and thus for $st(F) \neq \emptyset$ we get $st(F) = sm(F) = sg(F)$ while of course in case $st(F) = \emptyset$ anyway $\emptyset \subseteq \sigma(F)$ for any defined semantics σ always holds.

Finally, regarding the relation between st and $s2$ semantics again observe that any stable extension S is by definition ultimately range-maximal. This means that for the SCC-reduct $G = \mathscr{R}_S^{\triangle}(F)$ we have the following situation. First, S is still conflict-free in G. Some arguments that are attacked by S in F (i.e. the ones that were member of a different SCC at some point) are deleted in G, but S still attacks all arguments $A_G \setminus S$. Thus $S \in st(G) = sg(G)$ and $S \in s2(F)$. $\square$

Regarding a more detailed relation between semantics we now proceed by considering an extension of one kind as given and investigating relations to another semantics. The following proposition is in essence a collection of previous results.

Proposition 6.40 (Manipulating cf and ad sets)**.** *In ZF for any AF F the following hold:*

1. *for each $S \in cf(F)$, there is a unique maximal subset $S' \subseteq S$ with $S \in ad(F)$;*
2. *for each $S \in ad(F)$, there is a unique minimal superset $S' \supseteq S$ with $S' \in co(F)$.*

And in ZFC for any AF F the following hold:

3. *for each conflict-free set $S \in cf(F)$, there is some extending $S' \in na(F)$ with $S \subseteq S'$;*
4. *for each admissible set $S \in ad(F)$, there is some extending $S' \in pr(F)$ with $S \subseteq S'$;*
5. *for each complete set $S \in co(F)$, there is some extending $S' \in pr(F)$ with $S \subseteq S'$.*

Proof. (1) and (2) refer to unique definedness of fixed points for conflict-free sets as starting points of, respectively the backwards defense function (Lemma 6.18) and the defense function (Lemma 6.6). (3), (4) and (5) are references to Lemma 6.9. $\square$

Very similar but with slightly different purpose we will make use of the following proposition. It is important though to note that these statements do not work in ZF without choice, as then even existence of some naive extension does not tell us anything about sub-AFs where naive semantics might collapse.

Proposition 6.41 (Defining relationship between cf and na)**.** *In ZFC the following hold:*

1. *Given $\mathbb{S} = na(F)$, we can define $cf(F)$ as set of all subsets of members of $\mathbb{S}$, $cf(F) = \{S \in \wp(\bigcup\bigcup\mathbb{S}) \mid \exists T \in \mathbb{S}, S \subseteq T\}$.*

2. *Given $\mathbb{S} = cf(F)$, we can define $na(F)$ as maximal elements of $\mathbb{S}$, $na(F) = \{S \in \mathbb{S} \mid \nexists S' \in \mathbb{S}$ with $S \subset S'\}$.*

Proof. The second statement is merely a different wording of Proposition 6.40 (3). For the first statement we first observe that the given definition is axiomatically well formed. To conclude, subsets of (here maximal) conflict-free sets of course are again conflict-free. □

As observed with Lemma 6.18 and Proposition 6.40 (1) the ideal family of semantics leads to a unique extension set whenever the respective base semantics does not collapse. With the following proposition, following up on Proposition 6.39, we set these unique extension sets into relation.

Proposition 6.42 (Relationship between unique status semantics)**.** *In ZF for any AF F the following hold:*

1. *$|gr(F)| = 1$;*

2. *if $pr(F) \neq \emptyset$ then $|id(F)| = 1$ and for $S \in gr(F)$, $T \in id(F)$ we have $S \subseteq T$;*

3. *if $sm(F) \neq \emptyset$ then $|eg(F)| = |id(F)| = 1$ and for $S \in gr(F)$, $T \in id(F)$, $U \in eg(F)$ we have $S \subseteq T \subseteq U$.*

Proof. Follows immediately from previous observations. □

We now shift focus to AF classes and equivalence of semantics. For a listing of the classes of interest we refer to Remark 3.18 and Figure 6.3. We are thus interested in various fine grainings between bipartite, odd-circle-free and well-founded AFs as the most general classes on the one hand and symmetric loop-free AFs as the most special class on the other hand. Before presenting our Theorem 6.45 on comparability of AF classes we first give two helpful lemmata.

Lemma 6.43. *In ZF, for any non-empty limited controversial AF F there is a non-empty admissible set $T \in ad(F)$.*

Proof. Since $F \neq (\emptyset, \emptyset)$ there is an argument $x \in A_F$ such that there is no $y \in A_F$ which is controversial with respect to x. Considering members of directed paths towards x as predecessors of x, this immediately also means that the same (non-existence of controversial arguments with respect to) also holds for all predecessors of x. Now define the function $f : X \mapsto X \cup (X^-)^-$ (as a means for Bourbaki-Witt starting with $\{x\}$).

Observe that for any X the set $f(X)$ by definition defends X. The function f is inflationary (Definition 2.27) as required for Theorem 2.26, thus the fixed point $f^\triangle(X)$ exists for any $X \subseteq A_F$. Consider $T = f^\triangle(\{x\})$. Since these predecessors of x can not be controversial with each other or with x by above observation, and by construction of f (adding defenders only) we have $T \in cf(F)$ and thus $T \in ad(F)$. □

This lemma shows that non-empty limited controversial AFs may not provide empty preferred extensions, $pr(F) \neq \{\emptyset\}$. Observe that the empty extension set $pr(F) = \emptyset$ is not excluded by this result. The following describes the nature of admissible sets from a local perspective.

Lemma 6.44. *In ZF, assume some AF F and admissible set $S \in ad(F)$, then S is locally stable, i.e. $S \in st(F|_{S^1_F})$.*

Proof. Since S defends itself in F we have $S^- \subseteq S^+$ and thus $S^1_F = S^*_F$. Naturally conflict-free sets are still conflict-free in restrictions of any AF. Since we precisely restrict F to the range of S the claim immediately follows. □

Theorem 6.45 (Comparability for AF classes)**.** *It is mostly known [Dun95, CDM05, Dun07] in the finite case, and we assert in arbitrary models of ZF that*

1. *for symmetric AF F we have $cf(F) = ad(F)$, $na(F) = pr(F) = c2(F)$, and $sg(F) = sm(F) = s2(F)$;*
2. *for symmetric loop-free AFs na, pr, sg, sm, c2, s2, st coincide;*
3. *for well-founded AFs gr, id, eg, co, pr, sg, sm, st, s2, c2 coincide;*
4. *limited controversial AFs are coherent, i.e. st, sm and pr coincide;*
5. *for uncontroversial AF F we have $s2(F) = sg(F) = st(F)$;*
6. *circle-free AFs are super-coherent, i.e. st, sg, c2 and pr coincide;*
7. *for cycle-free AF F we have $c2(F) = s2(F) = st(F)$.*

Proof. (1) regarding the relationship between *cf* and *ad* is obvious by definition. This further implies relationships between *na* and *pr* as well as between *sg* and *sm*. For the interlocking of *na* with *c2* as well as *sg* with *s2* simply observe that for symmetric AFs any connected component is already strongly connected. (2) accurately describes Lemma 6.14, (3) here is a reference to Lemma 5.43. It thus remains to talk about (4), (5) and (6).

Dung showed coherence of limited controversial AFs in [Dun95, Theorem 33] under the precondition that an AF provides a preferred extension, i.e. by assuming ZFC. For ZFC we hence need to take additional care.

Concerning (4) assume a limited controversial AF F as given. If there is no preferred extension, then by Proposition 6.39 also semi-stable and stable semantics collapse. Now take into account $S \in pr(F) \setminus st(F)$. First consider Lemma 6.44 and hence $S \in st(F|_{S^1_F})$. This however means that any preferred extension of the complement of S^1_F in F extends S as admissible extension in F and thus $\{T \in pr(F) \mid S \subseteq T\} = \{S \cup T' \mid T' \in pr(F|_{A_F \setminus S^1_F})\} = \{S\}$. By existence of non-empty admissible sets in limited controversial AFs (Lemma 6.43) however we have $pr(F_{A_F \setminus S^1_F}) = \{\emptyset\}$ only for $A_F = S^1_F = S^*_F$ and thus $S \in st(F)$.

For (5) assume some uncontroversial AF F as given. First recall Proposition 6.39 and thus $st(F) \subseteq sg(F)$ and $st(F) \subseteq s2(F)$ and observe that uncontroversial AFs are odd-cycle-free and thus lack self-attacking arguments. This means that for $\sigma \in \{sg, s2\}$ and $S \in \sigma(F) \setminus st(F)$ there is some argument $x \in A_F \setminus S^*$, where remarkably x is not member of any odd-cycle.

Now consider the following construction:

$$T_0 = \{x\} \qquad S_i = S \setminus T_i^+ \qquad T_{i+1} = T_i \cup (S^+ \setminus (S_i \cup T_i)^+)$$

The aim here is to construct a set $U = S_\triangle \cup T_\triangle$ that is superior to the σ-extension S, i.e. for stage semantics we want $S^* \subset U^*$ and additionally for stage2 semantics we want that S does not precede U. To this end T_i, starting from $\{x\}$, gradually collects arguments attacked by S, yet not attacked anymore upon removal of arguments attacked by T_i, while S_i reflects the shrinking set of arguments originally contained in S. By recursive definition T_i defines an inflationary function and Bourbaki-Witt delivers existence of fixed points $S_\triangle$ and $T_\triangle$ and thus definedness of $U = S_\triangle \cup T_\triangle$. By definition we have $S_F^* \subset U_F^*$.

For stage semantics it remains to show that U is conflict-free. Assume for a contradiction that $[U]_F^{cnf}$. Since S is conflict-free by assumption then $\left[U, T_\triangle\right]_F^{cnf}$. By definition of T_i we further know that $S_\triangle$ does not attack $T_\triangle$, while by definition of $S_\triangle$ also $T_\triangle$ does not attack $S_\triangle$. This means $\left\{S_\triangle, S_\triangle\right\}^{ind}$, $\left\{S_\triangle, T_\triangle\right\}^{ind}$, $\left\{T_\triangle, S_\triangle\right\}^{ind}$; and thus $[U]_F^{cnf} = \left[T_\triangle\right]_F^{cnf}$. Hence for some i, j we need T_j attacking T_i, i.e. there is $y \in T_i$ such that $y \in T_j^+$. By recursive construction then the initial argument x both indirectly defends and indirectly attacks y, i.e. x is controversial with respect to y in contradiction to F being uncontroversial. Thus it holds that $\{U\}_F^{ind}$.

For stage2 semantics observe that regarding SCC $C_x = SCC_F(x)$ for each $y \in S \cap x^+$ we have that either $C_x = SCC_F(y)$ or C_x precedes $SCC_F(y)$. In other words, since $x \notin S^*$, S can not be a stage extension of $\mathscr{R}_S(F)$. Thus $st(F) = sg(F) = s2(F)$.

For (6), consider some circle-free AF F. By above and the fact that every circle-free AF is also uncontroversial (Remark 3.18) it remains to show that $c2(F) = st(F)$. Now observe that for circle-free AFs any SCC consists of exactly one argument. For $S \in c2(F)$ it follows that $\mathscr{R}_S(F) = F|_S$. This in particular means that $S^* = A_F$ and thus $S \in st(F)$.

For (7), finally, observe that since F is cycle-free each SCC-component consists of exactly one argument. Hence for any $S \subseteq A_F$ and $(A_S, R_S) = \mathscr{R}_S(F)$ (see Definition 4.8) we have $R_S = \emptyset$ and thus fixed point $\mathscr{R}_S^\triangle(F) = \mathscr{R}_S(F)$. By definition for $S \in cf(F)$ further $S \subseteq A_S$ holds. Now assume $S \in c2(F)$, i.e. in this case $S \in na(\mathscr{R}_S(F))$ and thus $A_S = S$, $\{S\} = na(\mathscr{R}_S(F))$. This means that for $x \in A_F \setminus S$ we have $x \in S_F^+$, indicating that also $S \in st(F)$ holds. By the relations from Proposition 6.39 the assertion follows. $\square$

As for the comparison of *pr* and *c2* extensions in bipartite AFs we consider the following example.

Example 6.46. Consider the bipartite AF F with $A_F = \{1, 2, 3, 4, 5, 6\}$ and $R_F = \{(1,2), (2,3), (3,4), (4,5), (5,6), (6,1)\}$ as depicted in Figure 6.4. Since F consists of only one SCC we have

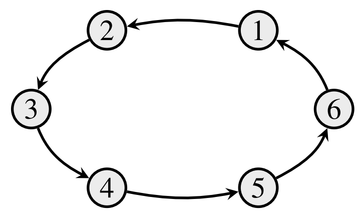


Figure 6.4: Bipartite AF illustrating differences between *c2* and *pr* semantics, cf. Example 6.46.

$c2(F) = na(F)$ and thus $S = \{2,5\} \in c2(F)$. Further since S does not defend itself against the attacks $(1,2),(4,5)$ we have that $S \notin ad(F)$ and thus $S \notin pr(F)$. In other words for this bipartite AF *c2* and *pr* do not coincide.

Regarding semantic AF classes for now we are only interested in super-coherent AFs (Definition 5.40), where coincidence of *st*, *pr* and *c2* is known. We first give the following corollary regarding subset relations from Proposition 6.39.

Corollary 6.47. *In ZF for any super-coherent AF F we have* $pr(F) = sm(F) = st(F) = sg(F) = s2(F) = c2(F)$.

Example 6.48. Consider the AF $F = (\{x\},\{(x,x)\})$ (the minimal *st* bomb from Example 6.1) with $st(F) = \emptyset$ and $\sigma(F) = \{\emptyset\}$ for all other semantics σ considered. This example is thus witness for $sg(F) = pr(F) = c2(F)$ with different *st* extension set.

In Section 7.4 we elaborate on the fact that coincidence of general subsets of $\{st, sg, pr, c2\}$ do not yet lead to super-coherence. However we now give the following satisfying proposition for models $\exists PR$.

Proposition 6.49. *In ZF for any AF F with* $st(F) = c2(F) = pr(F) \neq \emptyset$ *already super-coherence follows.*

6.6 Conclusions

Our previous work can be seen as blueprint for this chapter, in particular Section 6.2. In [Spa15c] we draw attention to the close relationship between existence of preferred/naive extensions and AC. The detailed work on argumentation in ZF from Section 6.1 as well as the alternative world of ZF with AD from Section 6.4 and the semantic relations from Section 6.5 are the major enhancements of this chapter.

Regarding related literature, in [Dun95] we find arbitrary AFs yet without distinction between set theoretic principles. The axiomatic account from [CA05] superficially seems close to our research, but is mostly about providing guidelines for structured argumentation. In [Wey11] model theoretic methods are used to solve abstract argumentation questions. More closely related to our work though, Friedman in [Fri11] investigates graph theoretic properties in the light of set theoretic tools.

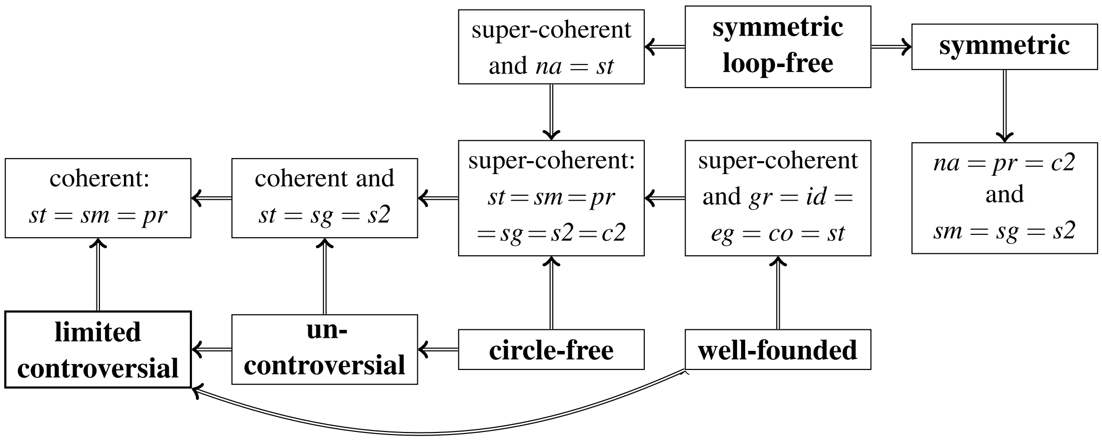


Figure 6.5: Syntactic AF classes in regards of ZF (possibly without AC) in context of equivalence relations between semantics. In addition to the selection of classes from Figure 3.5 we have depicted occurrences $=_i \sigma_i$ indicating coincidence of semantics σ_i.

Regarding future work, promising open problems pop up when looking at the game-AF from Definition 6.33. As pointed out above, this AF is not well-founded, yet bears some similarities to well-founded AFs. We flesh out the following two definitions as remarkable observations.

Definition 6.50 (SCC-founded AFs)**.** An AF F is called *SCC-founded*, if each induced sub-AF $G = F|_B$ (for some $B \subseteq A_F$) provides an initial SCC. That is, there is $C \in SCC(G)$ such that $C_G^- \subseteq C$.

Definition 6.51 (SCC-(super-)coherence)**.** An AF F is called *SCC-(super-)coherent*, if each induced sub-AF $G = F|_B$ (for some $B \subseteq A_F$) is (super-)coherent.

Be sure to observe though, that the game-AF is not SCC-coherent (only its induced subgraph without sequence arguments S_s is). The reason (and by the way also the reason for this AF not being limited controversial) is that we need to make use of stable bombs, here self-attacking arguments S_s. As immediate observation we derive that SCC-(super-)coherent AFs (as well as limited controversial AFs) do not provide self-attacking arguments. However, regarding these classes of AFs we state the following question.

Open Question 6.52. *What can we say about existence questions for or relations between semantics for SCC-founded and SCC-(super-)coherent AFs?*

Chapter 7

Collapse

All of old. Nothing else ever. Ever tried. Ever failed.
No matter. Try again. Fail again. Fail better.

Samuel Beckett, Worstward Ho (1983) [BTS83]

In this chapter we take a closer look at the collapse of abstract argumentation semantics. To be more precise we investigate models of set theory (ZF and ZFC) and various classes of AFs (circle-free, cycle-free, limited controversial, symmetric, loop-free) with focus on AFs that do not provide any σ-extensions for some semantics σ. This chapter is based on [Spa15c, Spa16b] augmented by our investigations of argumentation without choice and the intuitive concept of σ-bombs with derived AF-manipulation.

Example 7.1 (Stable collapse)**.** Consider the AF F depicted in Figure 7.1 with $A_F = \{1,2,3,4\}$ and $R_F = \{(1,1),(1,2),(2,1),(2,3),(3,2),(3,4),(4,3),(4,4)\}$. Apparently this AF is finite and symmetric. We have as conflict-free sets $cf(F) = \{\emptyset,\{2\},\{3\}\}$. Neither $\{2\}$ nor $\{3\}$ are stable, e.g. $\{2\}^*_F = \{1,2,3\} \subset A_F$. Thus this AF does not provide any stable extensions.

By definition none of the considered semantics but stable semantics collapses for finite AFs. Further, as pointed out in Section 6.1 *cf*, *ad*, *co* and *gr* never collapse for models of ZF. Naturally, in this chapter we thus investigate infinite AFs and the remaining semantics. In Section 7.1 we discuss examples from the literature and variations thereof with the aim of building some intuition regarding collapse. In Section 7.2 we elaborate on the significance of collapse and disasters/opportunities resulting from collapsing AFs. In Section 7.3 we dive deeper into AF restrictions (or classes) that allow for occurrences of collapses. In Section 7.4 we connect previous results and draw relations between AF classes and collapses for different semantics. Finally, in Section 7.5 we conclude, relate to the literature and highlight the most important results.

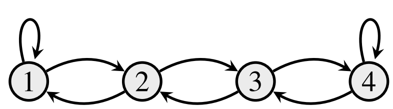


Figure 7.1: Finite and symmetric AF without stable extensions, cf. Example 7.1.

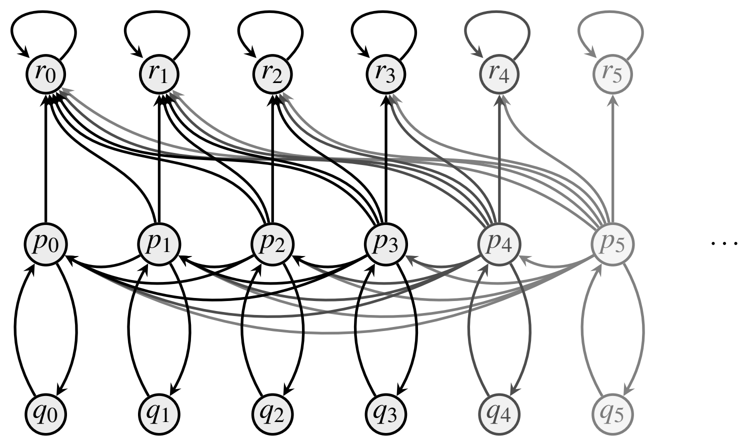


Figure 7.2: A first example without semi-stable or stage extensions, cf. Example 7.2.

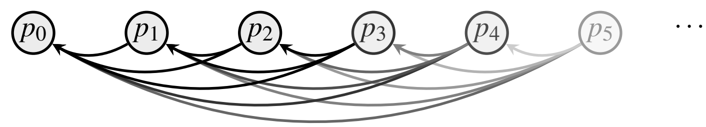


Figure 7.3: Minimal AF without stage, stage2 and cf2 extensions, cf. Example 7.3.

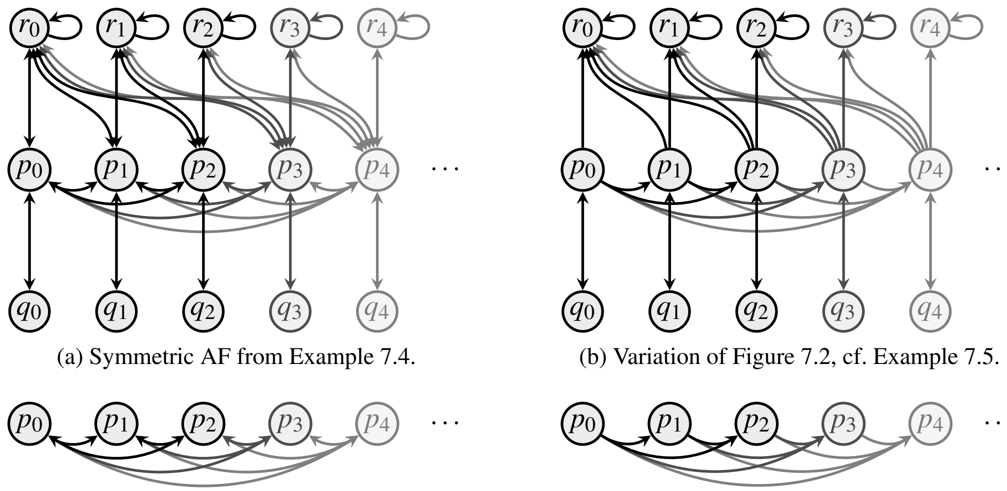
7.1 Preliminary Examples

We now discuss examples first introduced into abstract argumentation in [Ver03] (Examples 7.2 and 7.3) and variations thereof (Examples 7.4 and 7.5).

Example 7.2. Consider the AF F illustrated in Figure 7.2 with $A_F = \{p_i, q_i, r_i \mid i \in \mathbb{N}\}$ and $R_F = \{(p_i, q_i), (q_i, p_i), (p_i, r_i), (r_i, r_i) \mid i \in \mathbb{N}\} \cup \{(p_i, p_j), (p_i, r_j) \mid j < i\}$. We have as only *pr* and *na* extensions $S = \{q_i \mid i \in \mathbb{N}\}$ and the sets $S_n = (S \cup \{p_n\}) \setminus \{q_n\}$ for $n \in \mathbb{N}$. For $i < j$ it holds that $S^* \subset S_i^* \subset S_j^*$. So in effect for any *pr* or *na* extension there is another one of larger range and thus *sm* and *sg* collapse. Regarding *c2* and *s2* semantics however the set $S = \{q_i \mid i \in \mathbb{N}\}$ is an extension. This is easily verified by the SCC-reduct of S in F which resolves to $\mathscr{R}_S(F) = F|_{\{r_i, q_i \mid i \in \mathbb{N}\}}$, with S as only maximal conflict-free and thus stage extension.

Example 7.3. Contained as a subframework in Example 7.2 is the AF F, as illustrated in Figure 7.3, with $A_F = \{p_i \mid i \in \mathbb{N}\}$ and $R_F = \{(p_i, p_j) \mid j < i\}$. Here the only admissible set is the empty set and hence $pr(F) = sm(F) = \{\emptyset\}$. The singletons p_i are conflict-free and even serve as naive sets. For stage semantics, however, given $S_i = \{p_i\}$, we have that for instance S_{i+1} has larger range and thus *sg* collapses. So for this AF *sg* collapses but *sm* does not. Regarding *c2* and *s2* observe that the only candidates for extensions are the singletons p_i. For the SCC-reduct of these singletons then $G = \mathscr{R}_{\{p_i\}}(F) = (\{p_j \mid i \leq j < \omega\}, \emptyset)$, an AF without attacks and infinitely many arguments. Clearly then $\{p_i\}_G^* = \{p_i\}$ and thus both *s2* and *c2* collapse.

Observe that removing any finite amount of arguments from the AF from Example 7.3 leaves an isomorphically equivalent AF, i.e. there is a bijection respecting attacks. It can thus be



(a) Symmetric AF from Example 7.4.

(b) Variation of Figure 7.2, cf. Example 7.5.

(c) Symmetric restriction, cf. Example 7.5.

(d) Reversed restriction, cf. Example 7.5.

Figure 7.4: Variations of the AF F from Example 7.2 as described in Examples 7.4 and 7.5.

seen as a minimal *sg* collapse. We now discuss minor modifications of Example 7.2 and the restriction from Example 7.3.

Example 7.4. Regarding the AF F from Example 7.2 consider its symmetrization $G = sym(F)$ depicted in Figure 7.4a. For this AF *sg* and *sm* still collapse, while *c2* does not. Since for symmetric AFs *s2* and *sg* coincide (Theorem 6.45) also *s2* collapses. However, observe that the restriction to the p_i, $H = G|_{\{p_i | i \in \mathbb{N}\}}$ depicted in Figure 7.4c now represents an infinite symmetric clique of arguments, and thus an AF where each $\{p_i\}$ is a stable and thus a *sg*, *sm*, *c2*, *s2* extension.

Taking a closer look at above examples it appears that for instance for stage semantics the collapse in question might be due to an infinite clique of controversial arguments. With the following example we point out that for the initial example the direction of the attacks in the clique does not matter.

Example 7.5. Consider the AF F from Example 7.2. We now reverse the attacks between the p_i, that is, we construct AF G with $A_G = A_F$, $R_G = (R_F \setminus \{(p_i, p_j) \mid j < i\}) \cup \{(p_j, p_i) \mid j < i\}$ as depicted in Figure 7.4b. Again for this AF still *sm* and *sg* collapse. In other words the direction of the attacks between the p_i does not matter. Observe that the restriction $H = G|_{\{p_i | i \in \mathbb{N}\}}$ depicted in Figure 7.4d now represents an AF where $\{p_0\}$ serves as sole *st*, *sm*, *sg*, *s2*, *c2* extension.

Remark 7.6. For the above Examples we have collected extensional results in Table 7.1. For line σ and column x: an entry $\emptyset$ means that σ collapses for x, any other entry means that there is at least some σ-extension. Observe that occasionally the only existing extension is $\emptyset$ (i.e. extension set $\{\emptyset\}$) or the singleton $\{p_0\}$ (i.e. extension set $\{\{p_0\}\}$. Further we use as abbreviations $P = \{\{p_i\} \mid i \in \mathbb{N}\}$, $Q = \{\{q_i \mid i \in \mathbb{N}\}\}$ and $P \times Q = Q \cup \{\{q_i \mid i \in \mathbb{N}, i \neq j\} \mid j \in \mathbb{N}\}$.

	Figure 7.2	Figure 7.3	Figure 7.4a	Figure 7.4c	Figure 7.4b	Figure 7.4d
st	$\emptyset$	$\emptyset$	$\emptyset$	P	$\emptyset$	$\{\{p_0\}\}$
sg	$\emptyset$	$\emptyset$	$\emptyset$	P	$\emptyset$	$\{\{p_0\}\}$
sm	$\emptyset$	$\{\emptyset\}$	$\emptyset$	P	$\emptyset$	$\{\{p_0\}\}$
na	$P \times Q$	P	$P \times Q$	P	$P \times Q$	P
pr	$P \times Q$	$\{\emptyset\}$	$P \times Q$	P	$P \times Q$	$\{\{p_0\}\}$
c2	$P \times Q$	$\emptyset$	$P \times Q$	P	$P \times Q$	$\{\{p_0\}\}$
s2	$P \times Q$	$\emptyset$	$\emptyset$	P	$P \times Q$	$\{\{p_0\}\}$
id	$\{\emptyset\}$	$\{\emptyset\}$	$\{\emptyset\}$	$\{\emptyset\}$	$\{\emptyset\}$	$\{\{p_0\}\}$
eg	$P \times Q$	$\{\emptyset\}$	$P \times Q$	$\{\emptyset\}$	$P \times Q$	$\{\{p_0\}\}$

Table 7.1: Collapse and existence for selected semantics and the AFs from Figures 7.2–7.4, cf. Remark 7.6.

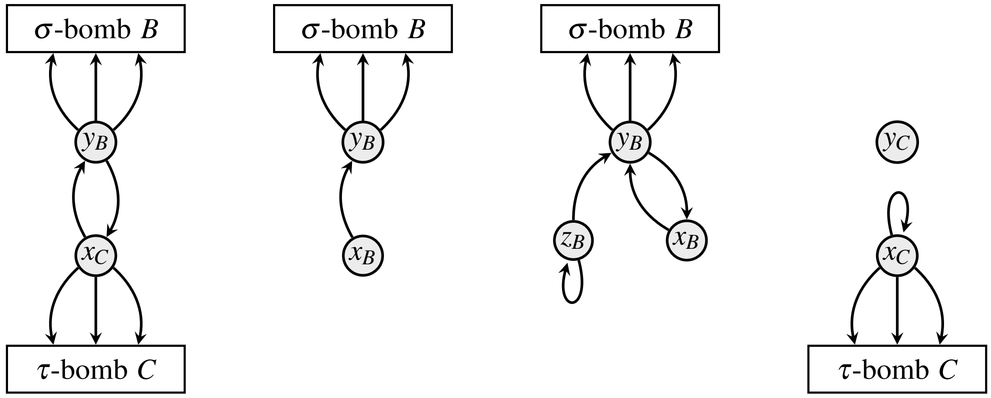
7.2 Significance of Collapse and Semantic Bombs

The intuition of the word *collapse* is that existence of such AFs is problematic for modular approaches, i.e. by component independence (Definition 5.7) if G and H do not share any arguments ($G \cap H = \emptyset$) and σ collapses for G, then σ also collapses for $F = G \cup H$ regardless of possible σ-extensions of H. In this section we further elaborate on this intuition and highlight disasters introduced in tandem with the possibility of collapse.

Example 7.7 (A stable game of bombs)**.** Consider some arbitrary AF F and the following two-player game. At each move player I selects some argument $x \in A_F$, we say that I places a *bomb* on field x. These bombs consist of two components, one liquid and one solid, and the only possibility to defuse them is to separate the components by either pouring the liquid to all successors x^+ of x or moving the solid to some predecessor $y \in x^-$ and pouring the liquid to all successors y^+ of y (including x). Once defused, liquids and solids irrevocably remain in place for the remainder of the game and will react to further additions of the other component.

Consequently a move of player II is to defuse the bomb placed by I. Player II loses the game if at some point in time a bomb explodes, otherwise she wins. The connection with stable semantics is as follows. Player I selects an argument and asks player II whether this should be accepted or attacked. In any case at each turn player II has to mark an argument as accepted. Since accepting two contradicting arguments would result in an explosion, a winning strategy for player II means a conflict-free set of accepted arguments. Since player I can shift focus to any argument and thus force player II to label (see Definition 4.17) this argument as *in* (solid component) or *out* (liquid component) player II has a winning strategy if and only if $st(F) \neq \emptyset$.

We use this playful example as a motivation for further investigations. In a very broad sense, given some AF F, a stable bomb is a sub-AF $G = F|_{A_G}$ such that there is no winning strategy for player II for AF G, i.e. such that $st(G) = \emptyset$. The AF from Example 7.1 serves as such a *st*-bomb. Also any self-attacking argument represents a *st*-bomb. The remarkable thing about such bombs is that in the game from Example 7.7 they need to (at least partially) be labelled as *out* for the full AF to provide a *st* extension. Stable semantics does not know of an *undecided*



(a) $s(\sigma,\tau,B,C)$ (b) $d(\sigma,B)$ (c) $a(\sigma,B)$ (d) $l(\tau,C)$

Figure 7.5: An illustration of the crashing manipulators from Definition 7.9.

label. Similarly induced sub-AFs with σ-collapse need to be taken care of for any semantics σ. This insight applies to any semantics allowing collapses and leads to the following definition.

Definition 7.8 (Semantic bomb)**.** Given semantics σ, an AF F is called a σ-*bomb* if it collapses on F, i.e. if $\sigma(F) = \emptyset$.

As a first application of semantic bombs we introduce crashing manipulators to be used for the automatic creation of AFs distinguishing collapse and existence for different semantics.

Definition 7.9 (Crashing Manipulators)**.** Given semantics σ, τ, and further σ-bomb B and τ-bomb C with $B \cap C = \emptyset$, define the following generic AFs:

1. *symmetric crashing manipulator*, illustrated in Figure 7.5a, $F = s(\sigma,\tau,B,C)$, $A_F = A_B \cup A_C \cup \{x_C, y_B\}$ and $R_F = R_B \cup R_C \cup \{(x_C, y_B), (y_B, x_C)\} \cup \{(y_B, x), (x_C, y) \mid x \in A_B, y \in A_C\}$;

2. *directed crashing manipulator*, illustrated in Figure 7.5b, $F = d(\sigma, B)$ with $A_F = A_B \cup \{x_B, y_B\}$ and $R_F = R_B \cup \{(x_B, y_B)\} \cup \{(y_B, x) \mid x \in A_B\}$;

3. *admissible crashing manipulator*, illustrated in Figure 7.5c, $F = a(\sigma, B)$ with $A_F = A_B \cup \{x_B, y_B, z_B\}$ and $R_F = R_B \cup \{(x_B, y_B), (y_B, x_B), (z_B, y_B), (z_B, z_B)\} \cup \{(y_B, x) \mid x \in A_B\}$;

4. *looped crashing manipulator*, illustrated in Figure 7.5d, $F = l(\tau, C)$ with $A_F = A_C \cup \{x_C, y_C\}$ and $R_F = R_C \cup \{(x_C, x_C)\} \cup \{(x_C, y) \mid y \in A_C\}$.

For the manipulators, if the exact nature of the bombs is not of importance we may also abbreviate $s(\sigma,\tau)$, $d(\sigma)$, $a(\sigma)$, $l(\tau)$.

The intuition behind above manipulators is to force one semantics to attack one bomb and another semantics to attack another. This forcing is then supposed to result in vastly different semantic evaluations. The extreme case is a semantic bomb for both semantics ($B = C$), while for the crashing manipulator one semantics collapses and the other does not.

Proposition 7.10. *Given semantics $\sigma, \tau \in \{st, sg, sm, na, pr, c2, s2, id, eg\}$, it holds that*

1. *for $F = s(\sigma, \tau, B, C)$ we have that $\sigma(F) = \{\{y_B\} \cup S \mid S \in \sigma(C)\}$ and $\tau(F) = \{\{x_C\} \cup T \mid T \in \tau(B)\}$, i.e. $\bigcup \sigma(F) \cap \bigcup \tau(F) = \emptyset$;*

2. *if $\tau \in \{sg, na\}$ and $\sigma \notin \{sg, na\}$, then σ collapses on $d(\sigma)$ while τ does not;*

3. *if $\tau \in \{sg, na, c2, s2\}$ and $\sigma \in \{sm, pr, id, eg\}$, then σ collapses on $a(\sigma)$ while τ does not;*

4. *if $\sigma \in \{sm, pr, id, eg\}$ and $\tau \in \{st, sg, na, c2, s2\}$, then τ collapses on $l(\tau)$ but σ does not.*

Proof. For (1) w.l.o.g. observe that for $S \in \sigma(F)$ with $x_C \in S$ we have $S|_{A_B} \in \sigma(B)$, i.e. due to σ-collapse of B no such S exists. Further semantics *st*, *sm*, *pr*, *c2*, *s2* need to accept at least one of x_C, y_B, while for *sg*, *na* we again have to evaluate B and C if neither x_C nor y_B are accepted. For $\sigma \in \{id, eg\}$ observe that σ-collapse implies *pr*-collapse. The only chance for non-empty extension sets are thus as claimed above.

For (2) observe that for all semantics from the selection but *sg* and *na* any σ-extension $S \in \sigma(d(\sigma))$ necessarily has $x_B \in S$. By the same argument as above we get $\sigma(d(\sigma)) = \{\{x_B\} \cup S \mid S \in \sigma(B)\} = \emptyset$. For *na* also $\{y_B\}$ serves as extension, and with $y_B^* = \{y_B\} \cup A_B$ it requires a stable extension of F to outperform $\{y_B\}$, which necessarily again contains x_B.

For (3) recall the similar Example 4.19. We have that y_B is not contained in any admissible set. For $S \in \sigma(F)$ we thus have $x_B \in S$, and consequently also $S \cap A_B \in \sigma(B)$. But then immediately $\sigma(F) = \emptyset$. Conversely observe that for τ the admissibility-disabling attack from z_B to y_B is of no importance and the *na* extension $T = \{y_B\}$ apparently provides a valid τ extension.

For (4) first observe that none of the arguments from C can be contained in an admissible set in $F = l(\tau)$. Consequently we get $sm(F) = pr(F) = id(F) = eg(F) = \{\{y_C\}\}$. There can be no stable extension in F since x_C is attacked only by itself. For the remaining semantics now observe that $\tau(F) = \tau(F|_{A_C \cup \{y_C\}})$ since x_C does not influence the semantic evaluation here. $\square$

Another application of bombs is to exploit the incapacity of stable semantics to label arguments as undecided. As mentioned before semantic bombs similarly require attacks from the outside. The following definition makes use of this observation by providing a very generic translation from stable semantics for infinite AFs.

Definition 7.11 (Bomb-shadow)**.** Given semantics σ and AF-bomb B such that $\sigma(B) = \emptyset$, define the σ *bomb-shadow* modifications on arbitrary AFs F as $B_\sigma(F)$ by addition of bombs for each argument $x \in A_F$. That is we first define $|A_F|$ many renamings of B, for each $x \in A_F$ a distinct AF $B(x) \equiv B$, where for $x \neq y$ we have $B(x) \cap B(y) = \emptyset$ and $B(x) \cap F = \emptyset$. Further define the σ bomb-shadow $G = b_\sigma(F)$ as

$$A_G = A_F \cup \bigcup_{x \in A_F} A_{B(x)} \qquad R_G = R_F \cup \bigcup_{x \in A_F} R_{B(x)} \cup \{(y, b), (x, b) \mid x \in A_F, y \in x_F^-, b \in A_{B(x)}\}$$

Example 7.12. Consider the AF F from Example 7.1 and its bomb-shadow $G = B_\sigma(F)$ as depicted in Figure 7.6a. For instance for $\sigma = st$ and $B = (\{x\}, \{(x,x)\})$, w.l.o.g. $A_G = A_F \cup \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ and $R_G = R_F \cup \{(\bar{1},\bar{1}), (\bar{2},\bar{2}), (\bar{3},\bar{3}), (\bar{4},\bar{4}), (1,\bar{1}), (2,\bar{1}), (2,\bar{2}), (2,\bar{3}), (3,\bar{2}), (3,\bar{3}), (3,\bar{4}), (4,\bar{4})\}$. Stable semantics requires all self-attacking arguments to be attacked by any extension. Stable extensions may not contain self-attacking arguments. Now $\{2,3\}$ would qualify but is not conflict-free either due to the bidirectional attack between 2 and 3. But 2 does not attack $\bar{4}$ (or 4) and 3 does not attack $\bar{1}$ (or 1) and thus there is no stable extension in G.

Remark 7.13. Compared to Definition 3.21 the *st*-bomb $B = (\{x\}, \{(x,x)\})$ for above definition of the *st* bomb-shadow B_σ accurately (that is isomorphically) describes the range shadow $shad^*$.

Theorem 7.14 (Bomb-shadow as translation)**.** *Given AF F, semantics $\sigma \in \{sg, na, c2, s2\}$ and σ bomb-shadow $G = B_\sigma(F)$, we have that $\sigma(G) = st(G) = st(F)$.*

Proof. First observe that if there is no stable extension in F, then there is no naive set in F attacking all arguments $x \in A_F$ and thus there is no σ-extension in G attacking all $B(x)$-bombs. Then however σ-evaluation of G also means σ-evaluation of some $B(x)$ and thus collapse of G under σ. We further assume $st(F) \neq \emptyset$.

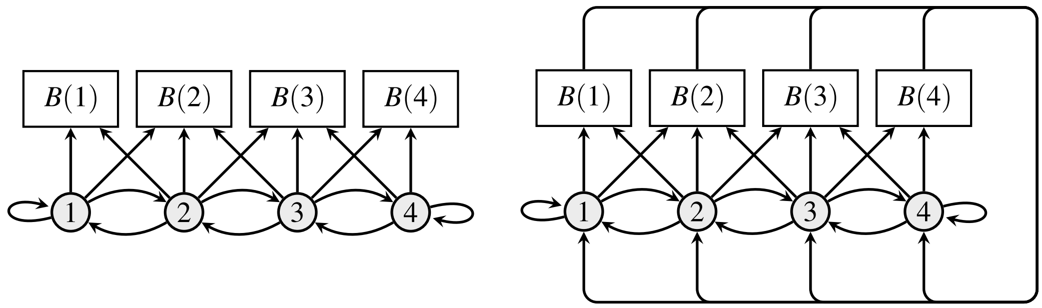
Now observe that there are no attacks from $A_G \setminus A_F$ to A_F, meaning that stable extensions of G necessarily project to stable extensions of F. Since stable extensions $S \in st(F)$ have $S^*_F = A_F$ and by construction of the bomb-shadow this further implies $st(F) = st(G)$.

Now recall that for any AF any *st*-extension is always also a σ-extension (Proposition 6.39), implying $st(G) \subseteq \sigma(G)$. Since there are stable extensions in G, these are the only range-maximal conflict-free sets and thus automatically the stage extensions. By directionality for $\sigma \in \{c2, s2\}$ evaluation of $G|_{A_F}$ precedes evaluation of $G|_{A_G \setminus A_F}$ meaning any $S \in \sigma(F) \setminus st(F)$ would lead to further evaluation of some bomb $B(x)$ for some undecided argument $x \in A_F$. And similarly any naive extension of F that is not a stable extension results in necessary evaluation of some $B(x)$ in G, ultimately leading to collapse for *na*-bomb B. Thus $\sigma(G) = st(G)$. □

Observe that the bomb shadow as defined above does not work for preferred and semi-stable semantics. This is due to admissibility resulting in local stable extensions (compare Lemma 6.44 or [GG07] for motivation of semi-kernels in digraphs). On a side note, there is no translation from *st* to *eg*, *id*. For *id* this is due to this semantics always either collapsing or providing exactly one extension. For *eg* we have multiple extensions only if *sm* collapses, which however in turn implies collapse of *st*. We use [Spa13, Translation 3.1.70] to build a different exact translation.

Definition 7.15 (Admissible Bomb-shadow)**.** Given semantics σ, σ-bomb B, initial AF F and its σ bomb-shadow $G = B_\sigma(F)$, we define the *admissible bomb-shadow* of F as $H = B^{ad}_\sigma(F)$ with $A_H = A_G$ and $R_H = R_G \cup \{(x,y) \mid x \in A_G \setminus A_F, y \in A_F\}$.

Example 7.16. Following up on Example 7.12, consider the AF F and for preferred semantics and the collapsing AF B from Example 7.19 the admissible bomb-shadow $G = B^{ad}_\sigma(F)$. Consider



(a) $G = B_\sigma(F)$, Definition 7.11 (b) $H = B^{ad}_\sigma(F)$, Definition 7.15

Figure 7.6: The bomb-shadows applied to the AF from Example 7.1.

$S \in pr(F)$. Assume $S \cap A_F \neq \emptyset$ and thus w.l.o.g. $2 \in S$. Now observe that $B(4)$ is not attacked by 2 but actually defends itself against the attacks from 3 and 4. Then however we need $S|_{A_{B(4)}} \in pr(B(4))$, which due to the attacks from $B(4)$ to argument 2 means that we need $pr(B(4)) = \{\emptyset\}$. However we have $pr(B(4)) = \emptyset$ which means that $S \notin pr(G)$.

Now assume $A_F \cap S = \emptyset$. Then either $S = \emptyset$ (which in ZFC actually is the case) or $S \cap A_{B(i)} \neq \emptyset$ which again leads back to the collapse of $B(i)$ and thus for some models of ZF (i.e. models where B collapses for pr) we have $pr(G) = \emptyset$.

Before we apply this enhanced bomb-shadow for a translation from *st* to *pr* and *sm* we interlope an on its own interesting detail. Since existence of *cf* and *ad* sets is always granted in ZF the only possible reason for a σ-collapse is too much variations to choose from. For instance for maximal conflict-free (naive) sets we need an infinite amount of distinguished sets to prevent an implicitly given choice function.

Lemma 7.17 (Collapse in detail)**.** *In ZF, consider semantics $\sigma \in \{sg, sm, na, pr, c2, s2, id, eg\}$ and some AF F with σ-collapse $\sigma(F) = \emptyset$. It holds that*

1. *there is an infinite amount of pairwise disjoint conflict-free sets;*
2. *for $\sigma \in \{sm, pr, id, eg\}$ there is an infinite amount of pairwise distinct admissible sets.*

Theorem 7.18 (Admissible bomb-shadow as translation)**.** *Given semantics $\sigma \in \{sm, pr\}$ and σ-bomb B, then for the admissible bomb-shadow $H = B^{ad}_\sigma(F)$ we have $\sigma(H) = st(H) = st(F)$.*

Proof. Given Lemma 7.17, there are arguments in each $B(x)$ for $x \in A_F$ that are acceptable with respect to admissible semantics. This means that any preferred extension of H (note that by Proposition 6.39 semi-stable extensions are preferred as well) either attacks all $B(x)$ or is attacked by some $B(x)$. Since all arguments from all $B(x)$ now attack all arguments from A_F we have that the collapse of B is multiplied and hence $\sigma(H) = st(F)$. $\square$

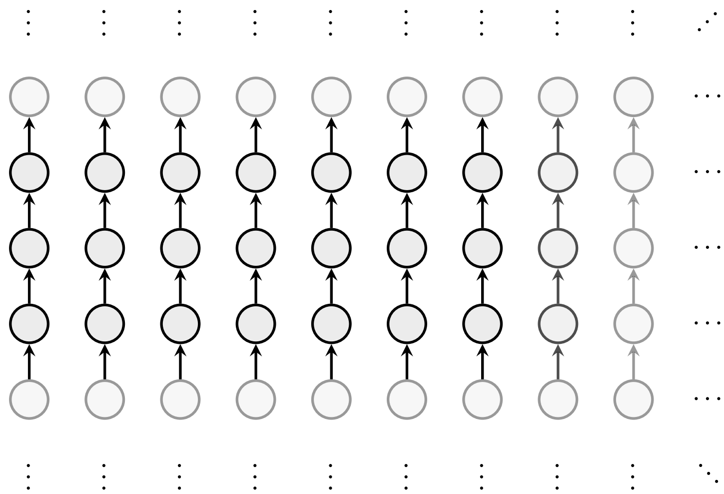


Figure 7.7: Finitary, circle-free, and planar AF which collapses for semantics of interest unless some variation of AC is granted, cf. Example 7.19.

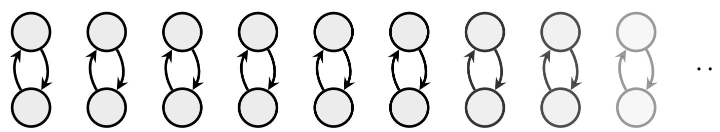


Figure 7.8: Finitary, symmetric, odd-circle-free and planar AF which collapses for semantics of interest unless some variation of AC is granted, cf. Example 7.20.

7.3 Refinement of AF classes

In this section we systematically investigate the collapse of argumentation semantics for AF classes of interest. To this end as a first step we take a look into the differentiation between ZF and ZFC. Recall that in ZF semantics *cf*, *ad*, *gr*, *co* never collapse, while in ZFC additionally semantics *na*, *pr*, *id*, *eg* never collapse. As pointed out in Theorem 6.15, AC and existence of naive/preferred extensions for arbitrary AFs are equivalent formulations (i.e. axioms) for models of ZF. The following two examples emphasize that regarding existence of extensions some variation of choice is already necessary for all (but *cf*, *ad*, *co*, *gr*) semantics of interest and circle-free, symmetric, loop-free, finitary, and planar AFs.

Example 7.19 (Circle-free collapse in ZF)**.** Consider the AF F illustrated in Figure 7.7 consisting of α many connected components, where each component is isomorphic to the AF $(\mathbb{Z}, \{(i, i+1) \mid i \in \mathbb{Z}\})$. It is easy to see that for any such component there are exactly two preferred extensions and further that there are no circles in this finitary AF F. However for infinite α existence of *pr*, *na* and thus *sg*, *sm*, *st*, *c2*, *s2*, *id*, *eg* extensions essentially provides a choice function. Thus there are models of ZF where $\sigma \in \{st, sg, sm, na, pr, c2, s2, id, eg\}$ collapses.

Example 7.20 (Symmetric odd-circle-free collapse in ZF)**.** Consider the AF F derived from Definition 6.12 for a collection of sets of two members. That is, as illustrated in Figure 7.8, there

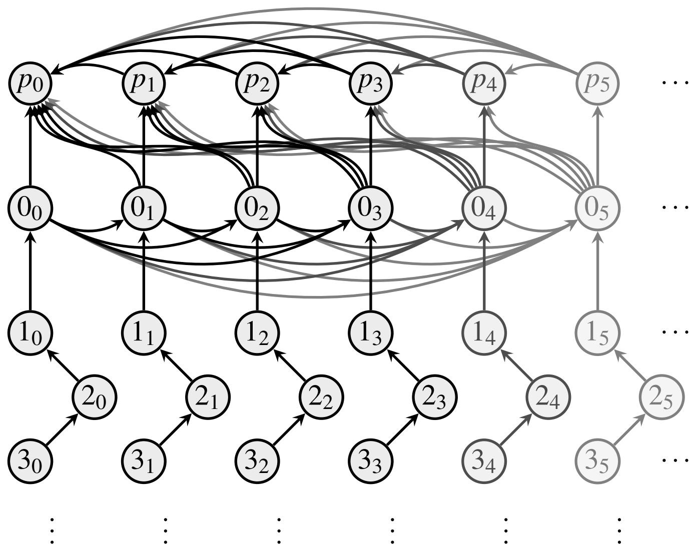
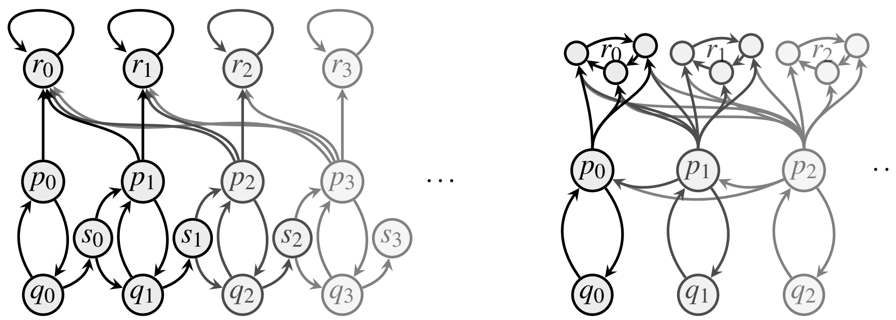


Figure 7.9: A cycle-free AF collapsing for *st*, *sg*, *sm*, *s2*, and *c2* semantics, cf. Example 7.21.

are α many distinct components, each of which is isomorphic to the AF $(\{x,y\},\{(x,y),(y,x)\})$. Clearly this AF is symmetric, loop-free and finitary. Now observe that for infinite α existence of *na*, *pr* and thus *sg*, *sm*, *st*, *c2*, *s2*, *id*, *eg* extensions is equivalent to the assumption of a choice function for the initial collection of pairs. Also compare Russell's socks, Example 2.4. Thus there are models of ZF such that some symmetric loop-free finitary AFs collapse for semantics $\sigma \in \{st, sg, sm, na, pr, c2, s2, id, eg\}$.

Taking a look at the AF class relations depicted in Figure 3.5 it appears that the only cases where we did not provide examples of collapse in ZF for the semantics of interest are bipartite or well-founded AFs. Lemma 5.43 is witness to well-founded AFs being very uncontroversial and always providing a unique non-empty (for non-empty AF) extension for all semantics of interest. In Theorem 8.6 we show that a very similar statement holds for bipartite AFs, where the necessary partition enables construction of extensions without AC. We thus turn to investigation of collapse in ZFC. Recall that with AC every AF provides naive and preferred extensions (Theorem 6.15). In ZFC we show perfection for limited controversial, finitary and symmetric AFs (Theorems 8.8, 8.9 and 8.16) for semi-stable and stage semantics. Regarding collapse we thus investigate symmetric, cycle-free, odd-cycle-free, loop-free and planar AFs. As a first remark observe that Example 7.4 is witness to symmetric AFs sometimes collapsing for stage, semi-stable and stage2 semantics. The next example deals with cycle-freeness.

Example 7.21 (Cycle-free collapse)**.** Consider the AF F as depicted in Figure 7.9. First observe that for the sequence of maximal admissible sets $S_i = \{0_i, 2_i, 4_i \ldots\} \cup \{1_j, 3_j, 5_j \ldots \mid j \neq i\}$ we have $S_i^* \subset S_j^*$ for all $i < j$. Further observe that the p_i as well as the 0_i are pairwise in conflict and thus any conflict-free set S contains at most one of each, w.l.o.g. $p_i, 0_j \in S$. But now $S^* \subset S^*_{\max i, j+1}$ and hence F collapses for *sg*, *sm* and *st*.



(a) No infinite clique, cf. Example 7.22 (b) No self-attacking arguments

Figure 7.10: Lower and upper variations of Example 7.2.

For *c2* consider some $S \in na(F)$, where clearly for at most one pair of $i < j$ we have $0_i, p_j \in S$. Thus the SCC-reduct $G = \mathscr{R}_S(F)$ removes all p_k for $k < j$ and all 0_k for $k > i$ but leaves all p_k for $k > j$ intact. In fact for $k > j$ we have $p_k \in A_G$, $p_k \notin S$ and further $(p_k)^-_G = \emptyset$, meaning that S can not be a *na* extension of G and thus neither a *c2* extension of F. Finally, with *c2* collapsing then also *s2* collapses (Proposition 6.39).

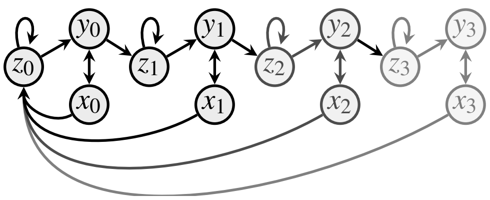
Observe that all in ZFC collapsing AFs presented in this chapter so far are variations or restrictions of Example 7.2. Looking at examples of collapse in ZFC so far the infinite cliques are evident. With the next examples we attempt to get rid of as many non-finitary arguments (arguments attacked by infinitely many other arguments) as possible while still providing collapsing AFs. First we provide two more variations of Example 7.2, altering the lower structure between p_i and q_j to get rid of the infinite clique, or altering the upper structure of the r_i to get rid of the self-attacks.

Example 7.22. Consider the AF $F = (A,R)$ from Example 7.2. We replace the attacks between the p_i with an infinite chain of admissibility (illustrated in Figure 7.10a) $F' = (A \cup \{s_i \mid i \in \mathbb{N}\}, R')$ where $R' = (R \setminus \{(p_i, p_j) \mid i, j \in \mathbb{N}\}) \cup \{(q_i, s_i), (s_i, p_{i+1}), (s_i, q_{i+1}) \mid i \in \mathbb{N}\}$.

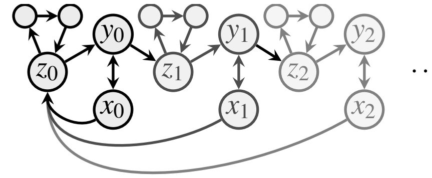
Now observe that the only preferred extensions are $S_q = \{q_i \mid i \in \mathbb{N}\}$ and for each $n \in \mathbb{N}$ the sets $S_n = \{q_i, p_n, s_j \mid i < n, j \geq n\}$. Here p_n defends s_n, and accepting s_n for completeness reasons means that we will accept each s_j for $j > n$. Again for $i < j$ we have $S^*_q \subset S^*_i \subset S^*_j$, and hence the collapse of semi-stable semantics.

For stage semantics, on the other hand, we need to consider more candidates, as also $S_p = \{p_i \mid i \in \mathbb{N}\}$ and any feasible combination between p_i, q_j and s_k serve as naive extensions. Now take some $S \in na(F)$ as given. If there is a maximal $n \in \mathbb{N}$ with $p_i \notin S$ for $i > n$, then S_{n+1} as defined above has larger range than S. Hence assume that for each $n \in \mathbb{N}$ there is some $i > n$ with $p_i \in S$. We conclude that for some $m \in \mathbb{N}$ we have both $s_m \notin S^*$ and one of $p_{m+1} \in S$ or $q_{m+1} \in S$. We construct $T = \{q_j \mid j \leq m\} \cup (S \cap \{p_i, q_i, s_i \mid i > m\})$. By construction $S^* \subset T^*$, and hence stage semantics collapses for this AF as well.

Regarding *c2* and *s2* semantics observe that arguments r_i are neglectable. As easily seen thus $S_q \in c2(F)$ and $S_q \in s2(F)$ hold.



(a) Only one non-finitary argument



(b) Loop-free and planar modification

Figure 7.11: AF illustrating the collapse of semi-stable semantics, cf. Example 7.24.

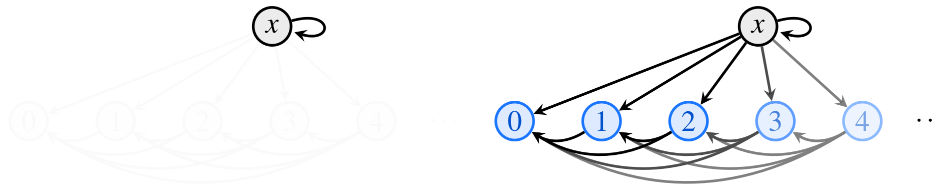
Example 7.23. Consider the AF F as illustrated in Figure 7.10b, i.e. the loop-free modification of Example 7.2. In terms of *st*, *sm* and *pr* extensions this modification does not alter the result. However, the *na* extensions now contain additional arguments from the upper odd cycles. This does not interfere with *c2* or *s2* extensions, i.e. in ZFC these two semantics still do not collapse. For stage semantics observe that any naive extension $S \in na(F)$ still contains at most one p_i and thus among the upper odd cycles r_j for $i < j$ always only two out of three arguments are in range of S. Clearly there is a naive extension containing p_j bigger in range than S and thus *sg* semantics collapses.

The modifications of Example 7.2 can be grouped in several approaches. We might want to replace the self-attacks with odd cycles (Example 7.23) and in case of admissibility based semantics undefendable arguments, or with other structures ensuring that not arbitrarily many of the arguments p_i can be included in some extension, such as in Example 7.21 or Example 7.23. Regarding the clique among the p_i we may arbitrarily alter the direction of the attacks as elaborated on in Examples 7.5 and 7.4; but again we may use more elaborate modifications as in Example 7.22. Even the 2-cycle between the p_i and q_i can be replaced by an infinite chain of arguments as in Example 7.21. Finally all of these modifications are compatible in the sense that there are similar AFs incorporating any selection of modifications.

Among our ZFC-examples only the following example incorporates a substantially different approach to collapse. Now that we have seen a vast amount of examples illustrating how close we can get to finitariness while keeping the collapse for stage semantics, we take one step further for semi-stable semantics.

Example 7.24 (Not quite finitary collapse)**.** Consider the AF $F = (A,R)$ illustrated in Figure 7.11a, with $A = \{x_i, y_i, z_i \mid i \in \mathbb{N}\}$ and $R = \{(z_i,z_i),(z_i,y_i),(x_i,y_i),(x_i,z_0),(y_i,x_i),(y_i,z_{i+1}) \mid i \in \mathbb{N}\}$. Observe that only z_0 violates the finitary condition here.

We have as only preferred extensions $S_x = \{x_i \mid i \in \mathbb{N}\}$ and the sets $S_n = \{x_i, y_j \mid j \le n, i > n\}$ for each $n \in \mathbb{N}$. Again for $i < j$ we have $S_x^* \subset S_i^* \subset S_j^*$ and hence semi-stable semantics collapses. For stage semantics, however, the set $S_y = \{y_i \mid i \in \mathbb{N}\}$ is maximal in range, as only $z_0 \notin S_y^*$, but attacking z_0 means including x_j for some j and thus not attacking z_{j+1}. For *c2* and *s2* semantics observe that this AF provides exactly one SCC, meaning $c2(F) = na(F)$ and $s2(F) = sg(F)$.



(a) AF F (b) AF G

Figure 7.12: Stage extension might not exist in finitely superseded AFs.

Finally observe that this AF is planar and replacement of the self-attacks with 3-cycles (as illustrated in Figure 7.11b) does not alter its collapsing nature.

As last modifications of Example 7.2 we consider adding additional arguments, examples for finitely/finitarily superseded AFs (see Definition 3.13). Recall that an AF F supersedes another AF $G \supseteq F$ (with $F = G|_{A_F}$) if for each $y \in A_G$ there is some $x \in A_F$ such that $x^- \subseteq y^-$ and $y^+ \subseteq x^+$. Finite/finitary superseding then refers to F being finite/finitary. Now consider the following example, a modification of Example 7.3 with an additional argument.

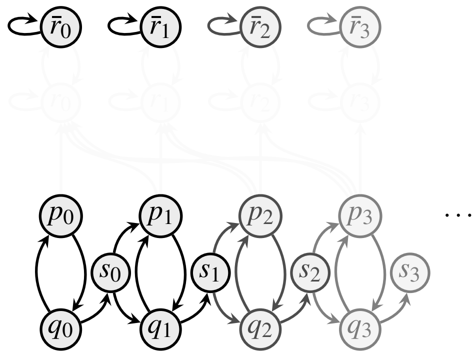
Example 7.25. Consider AFs F and G with $A_F = \{x\}$, $R_F = \{(x,x)\}$ as illustrated in Figure 7.12a and $A_G = A_F \cup \mathbb{N}$, $R_G = A_F \cup \{(x,i) \mid i \in \mathbb{N}\} \cup \{(j,i) \mid i < j \in \mathbb{N}\}$ as illustrated in Figure 7.12b. Here for any $i \in A_G$ we have that x supersedes i. First, x attacks any argument in G and thus the outward neighborhood of i is subset of the outward neighborhood of x. And second, the only attacker of x is x itself which attacks any i and thus also the inward neighborhood of x is subset of the inward neighborhood of all arguments from G. Finally, by similarity to Example 7.3 the finitely superseded AF G collapses for st, sg, $c2$, and $s2$ while $sm(G) = \{\emptyset\}$.

Remark 7.26 (Finite superseding and collapse). Observe that the construction of Example 7.25 is very arbitrary. That is, given any AF F, its modification G with $A_G = A_F \cup \{x\}$ and $R_G = R_F \cup \{(x,x),(x,y) \mid y \in A_F\}$ is finitely superseded by the AF $(\{x\},\{(x,x)\})$. For semantics that do not imply admissibility (i.e. cf, na, sg, $c2$, $s2$) this modification does not alter the semantic evaluation. For all other semantics $\sigma \in \{ad, co, gr, pr, st, sm, id, eg\}$ it does. Namely, regardless of the semantic evaluation $\sigma(F)$ we have $st(G) = \emptyset$ and for the others $\sigma(G) = \{\emptyset\}$.

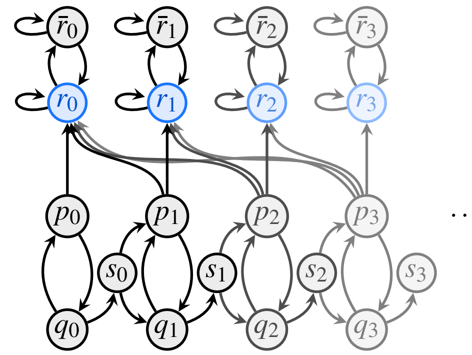
Above remark gives a strong insight into the mechanics of superseding AFs. With the following lemma we highlight that these mechanics prohibit collapse of ad-based semantics in finitely superseded AFs.

Lemma 7.27 (Superseding and completeness [CO14, Theorem 1]). *In ZF, given AFs $F \subseteq G$ where F supersedes G, we have that $co(G) = \{\mathscr{D}_G(S) \mid S \in co(F)\}$, $co(F) = \{S' \cap A_F \mid S' \in co(G)\}$ and $|co(G)| = |co(F)|$.*

Proof. In this special case we abstain from providing a detailed proof despites the (as compared to [CO14]) additional claim of the result holding in ZF. This is simply because the acclaimed functions (intersection and $\mathscr{D}_G$, cf. Definition 4.6) do not require AC. □



(a) AF F



(b) AF G

Figure 7.13: Finitarily superseded AF G with *sm* collapse, cf. Example 7.29.

Observe that above lemma essentially says that G is a faithful modification of F for semantics $\sigma \in \{co, pr, gr, id, eg\}$, cf. Definition 5.24.

Corollary 7.28. *Given AFs $F \subseteq G$ where F finitely supersedes G, we have that $sm(F) \neq \emptyset$.*

Proof. This result merely requires existence of semi-stable extensions in finite AFs (Theorem 8.2) and the observation that any *sm*-collapse requires an infinite amount of *pr* extensions (compare Theorem 7.37), or collapse of *pr*. This however can not be the case due to the *pr* extensions of G corresponding one to one to the *pr* extensions of F (Lemma 7.27) which are limited by the finite number of in F available arguments. □

The last example of this section then is for the collapse of semi-stable semantics again. Corollary 7.28 is witness for semi-stable semantics never collapsing for finitely superseded AFs, we are thus left with finitarily superseded AFs. The following example can be seen as modification of Example 7.24.

Example 7.29. Consider the AF F depicted in Figure 7.13a and its embedding modification G from Figure 7.13b. First observe that regarding collapse G provides the same results as Example 7.24, i.e. it collapses for *st*, *sm* and *sg*. Further observe that F is finitary (and planar) by definition. Finally observe that F supersedes G with any r_i being superseded by $\bar{r}_i$.

7.4 Relations between Semantics

In this section we relate semantics in terms of collapse. The first obvious observation regards given relations between semantics (cf. Proposition 6.39). Namely, if for every AF F we have $\sigma(F) \subseteq \tau(F)$ then any τ-bomb represents a σ-bomb as well. Thus Figure 6.3 gives an indication (with the arrows interpreted in the opposite direction) whether a collapse of one semantics is expected to influence the collapse of another.

Proposition 7.30 (Collapse relations)**.** *Given AF F, we have that $pr(F) = \emptyset \implies sm(F) = \emptyset$; $na(F) = \emptyset \implies c2(F) = \emptyset \wedge sg(F) = \emptyset$; $c2(F) = \emptyset \implies s2(F) = \emptyset$; and $sm(F) = \emptyset \vee s2(F) = \emptyset \vee sg(F) = \emptyset \implies st(F) = \emptyset$.*

It remains to elaborate on any other necessary relations. First recall Example 7.1 where *st* collapses while all other semantics of interest do not. Then observe that *gr*, *co*, *ad*, *cf* semantics never collapse in ZF (Theorems 6.2 and 6.7). We now present a remarkable correlation.

Theorem 7.31 (Collapse of *pr*, *id*, *eg*)**.** *Given semantics $\sigma, \tau \in \{pr, id, eg\}$ and any AF F such that $\sigma(F) = \emptyset$ holds, then also $\tau(F) = \emptyset$ holds.*

Proof. By Theorems 6.15 and 6.20 σ does not collapse in ZFC, we hence deal with a model of ZF without AC. By Proposition 6.19 further collapse of *id* further implies collapse of *pr*. Similarly collapse of *eg* implies collapse of *sm*. If *sm* collapses for AF F however we have that $eg(F) = pr(F)$ and hence collapse of *eg* implies collapse of *pr*. Now, given collapse of *pr*, by Proposition 7.30 we have collapse of *sm*. If the base semantics of some ideal family semantics τ collapses by definition we have $\tau(F) = pr(F)$ and thus the claim follows. □

This however is as far as relations between collapse of semantics of interest go. We thus proceed pointing out counterexamples for other combinations.

Theorem 7.32 (Collapse not related)**.** *Given AF F, we have that $\sigma(F) = \emptyset \not\Longrightarrow \tau(F) = \emptyset$ for*

1. *$\sigma \in \{sm, sg\}$ and $\tau \in \{pr, id, eg, s2, c2, na\}$;*

2. *$\sigma = s2$ and $\tau = c2$;*

3. *$\sigma \in \{sm, pr, id, eg, s2, c2\}$ and $\tau \in \{sg, na\}$;*

4. *$\sigma \in \{sm, pr, id, eg\}$ and $\tau \in \{s2, c2, sg, na\}$;*

5. *$\sigma \in \{s2, c2, sg, na\}$ and $\tau \in \{sm, pr, id, eg\}$.*

Proof. For (1) consider Example 7.2 where *sm* and *sg* collapse while *pr*, *s2*, *c2*, and *na* do not. For (2) consider Example 7.4 where *s2* collapses while *c2* does not.

For (3-5) we make use of crashing manipulators from Definition 7.9 and their implications from Proposition 7.10. To this end we need exemplary collapsing AFs. In the case of ZF (without AC) we may use Examples 7.19 or 7.20 for the general collapse of semantics $\sigma, \tau \in \{st, sm, pr, id, eg, s2, c2, sg, na\}$. In the case of ZFC (and thus existence of pr, id, eg, na extensions) we may use Example 7.21 for the general collapse of $\sigma, \tau \in \{st, sm, s2, c2, sg\}$. Thus assume some bomb F simultaneously for σ and τ as given. Then the directed crashing manipulator $d(\sigma, F)$ reflects (3), the admissible crashing manipulator $a(\sigma, F)$ reflects (4), and the looped crashing manipulator $l(\sigma, F)$ reflects (5). □

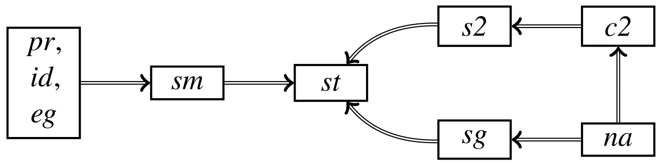


Figure 7.14: Relations between semantics regarding their collapse, cf. Remark 7.34.

Remark 7.33. In Example 6.48 we highlight that coincidence of *pr*, *sg*, and *c2* semantics does not yet imply super-coherence ($st = pr = sg = c2$, Definition 5.40). Theorem 7.32 now lets us construct witnessing AFs for the other combinations ($st = pr = sg$, $st = pr = c2$, and $st = sg = c2$) without super-coherence.

Remark 7.34. Gathered results now let us draw a concise picture of relations between collapse of different semantics, built on the relations between different semantics from Figure 6.3. Observe that in the corresponding Figure 7.14 an arrow from semantics σ to τ (or inclusion in the same box) means that σ-collapsing AFs are always also τ-collapsing. Further we do not include semantics co, gr, ad, cf since those never collapse in ZF.

With Figure 7.14 we have revealed the collapse relations for considered semantics and arbitrary AFs. With witnessing examples being referenced in the proof of Theorem 7.32 we further point out the significance of the symmetric crashing manipulator.

Example 7.35. Consider semantics σ, τ, σ-bomb B with $\tau(B) \neq \emptyset$ and τ-bomb C with $\sigma(C) \neq \emptyset$. We make use of the symmetric crashing manipulator $G = s(\sigma, \tau, B, C)$ and give the additional modification F with $A_F = A_G \cup \{\bar{x}_C, \bar{y}_B\}$ and $R_F = R_G \cup \{(x_C, \bar{x}_C), (\bar{x}_C, \bar{x}_C), (y_B, \bar{y}_B), (\bar{y}_B, \bar{y}_B)\}$.

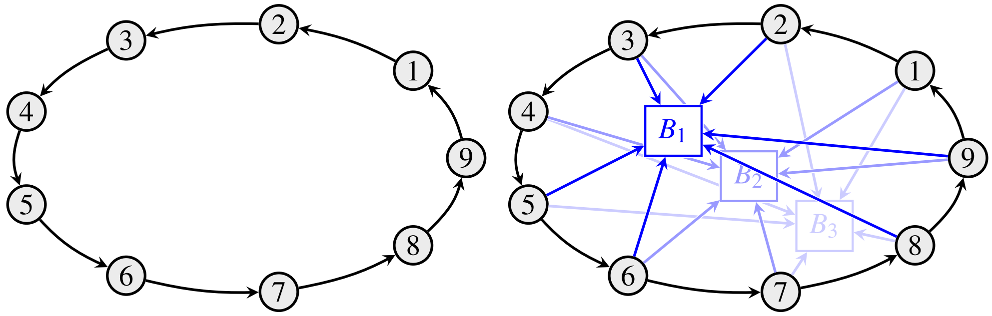
Corollary 7.36 (Differences between semantic evaluation)**.** *If there is no directed path from semantics σ to semantics τ in Figure 7.14 then there are AFs F such that the extension sets are disjoint ($\sigma(F) \cap \tau(F) = \emptyset$), the sets of acceptable arguments are incomparable ($\bigcup \sigma(F) \setminus \bigcup \tau(F) \neq \emptyset$ and $\bigcup \tau(F) \setminus \bigcup \sigma(F) \neq \emptyset$), and even for any σ-extension S and τ-extension T we may have incomparability and range-incomparability ($S^*_F \setminus T^*_F \neq \emptyset$ and $T^*_F \setminus S^*_F \neq \emptyset$).*

Proof. It suffices to observe that for the AF F from Example 7.35 and extensions $S \in \sigma(F)$, $T \in \tau(F)$ we have $y_B \in S$ and $x_C \in T$. Thus w.l.o.g. also for any extension set $U \in \sigma(F) \cup \tau(F)$ we have $\bar{y}_B \in U^*_F$ if and only if $U \in \sigma(F)$ and thus $U \notin \tau(F)$. □

In Lemma 7.17 we highlight that in ZF collapse of some semantics implies an infinite amount of extensions for some other semantics. In ZFC we further know that *na*, *pr*, *id* and *eg* never collapse (see Theorem 6.15). We now give more semantic relations as follows.

Lemma 7.37 (Collapse in more detail)**.** *In ZFC consider semantics $\sigma/\tau \in \{sm/pr, sg/na\}$. If for some AF σ collapses, then there is an infinite amount of τ-extensions.*

Proof. Observe that semantics $\sigma \in \{sm, sg\}$ result from range-comparison of their base semantics. If there is only a finite amount of τ-extensions, then there is only a finite amount of extensions for range-comparisons and thus a maximal (= σ-extension) guaranteed. □



(a) 9-cycle (b) Modification with $c2$-bombs B_i

Figure 7.15: The nine-cycle of arguments and its $sg \to c2$-translated version, cf. Example 7.38.

7.5 Conclusions

In general for abstract argumentation of course we want to avoid collapse, we attempt to restrict investigated AFs to those providing extensions and hence most often focus on properties that ensure this attempt to be successful. To this end however it is important to know under which circumstances a collapse can occur. One benefit of work dedicated to collapse thus is to provide exemplary AFs that might cause problems. In this chapter we even go a step further and provide constructive results enabled by some semantics failure to provide extensions. One application of this work thus is the following example sketching a $sg \Rightarrow c2$ translation.

Example 7.38 (Realizing sg extension sets for $c2$)**.** Consider the AF F depicted in Figure 7.15 with $A_F = \{1,2,\dots 9\}$ and $R_F = \{(i,i+1) \mid 0 < i < 9\} \cup \{(9,1)\}$, i.e. the directed cycle of nine arguments. We have as stage extensions the nine sets $S_i = \{i, i+2 \bmod 9, i+4 \bmod 9, i+6 \bmod 9\}$, each omitting only one argument in range. Since F is strongly connected for $c2$ semantics we get the additional extensions $T_1 = \{1,4,7\}, T_2 = \{2,5,8\}, T_3 = \{3,6,9\}$.

Assume some AF G with $A_G = A_F$ as given and assume $c2(G) = sg(F)$. By maximal conflict-freeness of $c2$ we derive conflicts $[i, i+1 \bmod 9]_G^{cnf}$ for $i \in A_G$ and independencies $\{i, j\}^{ind}$ for all other combinations of $i, j \in A_G$. With w.l.o.g. $S_1 = \{1,3,5,7\}$ being an extension not containing arguments 8 and 9 by directionality of $c2$ this means $(7,8)_G^{att}$ and $(1,9)_G^{att}$. Hence G again is strongly connected and provides the unwanted $c2$ extensions T_1, T_2, T_3.

In [Spa13] we have a translation from sg to st, eliminating unwanted extensions one by one which we can facilitate to achieve the same for $c2$ semantics. That is, we use AF H with $A_H = A_F \cup B_1 \cup B_2 \cup B_3$ and $R_H = R_F \cup R_1 \cup R_2 \cup R_3 \cup \{(x,b) \mid i \in \{1,2,3\}, x \in A_F \setminus T_i, b \in B_i\}$, where (B_i, R_i) refer to pairwise disjoint $c2$-bombs.

The first SCC of this AF H consists of all arguments from A_F, which, together with $S_i \setminus T_j \neq \emptyset$ for any $i \in A_F$, $j \in \{1,2,3\}$, means that S_i attacks all bombs and thus is a $c2$ extension of H. The other na extensions of $SCC_H(1)$ are the sets T_j. However each such set provides a $c2$ extension only if the respective bomb (B_j, R_j) does.

Collapse in the literature is usually discussed upon introduction of some semantics. Since Dung [Dun95] does not explicitly mention his use of AC the only collapses in his original work are for *st* semantics. With the introduction of *sm* and *sg* semantics in [Ver03] first examples of infinite collapses appear. Although the technical report [Cam05a] of [Cam05b] bears *collapse* in its title, it is rather about justification states and other problems of instantiation and application and not actually related to this work.

Remark 7.39 (The ups and downs of collapse, cf. Remark 8.30). The possibility of some semantics to collapse can be seen as providing additional expressiveness. This approach was merely touched, particularly in Section 7.2, and thus is a key aspect of collapse to be exploited by future research. For instance Example 7.38 illustrates this issue for *c2* semantics.

A downside of collapse is a loss of comparability. For finite AFs for instance we have that there is always some *sg* extension containing the *gr* extension, any *pr* extension is contained in some *sg* extension, and similar. Theorem 7.32 shows that for arbitrary AFs by the power of collapse a lot of these relations are not valid anymore.

As shown in this chapter considering infinite AFs leads to maximal diversity in terms of collapse. For instance in the finite case for any *pr* extension S there is a *na* extension T with $S \subseteq T$. By Theorem 7.32 in the infinite case (and without AC) this statement is no longer valid.

In this chapter we put focus on properties one by one. Imminent future research issues connect these properties. For instance considering Theorem 7.32 observe that the directed crashing manipulator $d(\sigma)$ respects cycle-freeness. That is, the results gained by this manipulator applied to cycle-free bombs are valid for cycle-free AFs as well. Similarly for symmetric AFs apparently *sg* and *s2* semantics (and thus collapsing AFs) coincide. Further the intuition behind finitely/finitarily superseded AFs can be used for symmetric/biparite/and other supersedings. It remains to be seen how combined AF classes integrate with collapse comparisons in more detail. As an exemplary AF we provide the following *pr*-compact and *sm*-collapsing example.

Example 7.40. Consider the AF F depicted in Figure 7.16 with $A_F = \{a_i, b_i, p_i, r_i \mid i \in \mathbb{N}\}$ and

$$\begin{aligned} R_F = \{&(a_i, p_j), (p_j, a_i), (a_i, b_i), (b_i, a_{i+1}), (b_i, p_{j+3}) \mid j \in \mathbb{N}, i = j \text{ div } 3\} \\ \cup \{&(r_k, r_{k+1}), (r_{k+1}, r_{k+2}), (r_{k+2}, r_k) \mid k \equiv 0 \text{ mod } 3\} \\ \cup \{&(p_k, p_l) \mid k \text{ div } 3 = l \text{ div } 3\} \\ \cup \{&(p_i, r_j) \mid j \leq i, i \equiv j \text{ mod } 3\}. \end{aligned}$$

We have as preferred extensions the set $\{a_i \mid i \in \mathbb{N}\}$ and sets S_i^j where $i = j$ div 3 with $a_k \in S_i^j$ for $k < i$, $b_k \in S_i^j$ for $k > i$, $p_j \in S_i^j$ and for $k \leq j$ and $k \equiv j$ mod 3 also $r_k \in S_i^j$. Thus for any argument $x \in A_F$ there is $S_i^j \in pr(F)$ with $x \in S_i^j$. On the other hand for *sm* semantics observe that S_{i+1}^{j+3} has bigger range than S_i^j meaning that *sm* collapses.

Remark 7.41. Regarding possible collapse of semantics with respect to AF classes we have gathered our results in Table 7.2. Here a checkmark in line x and column σ refers to existence

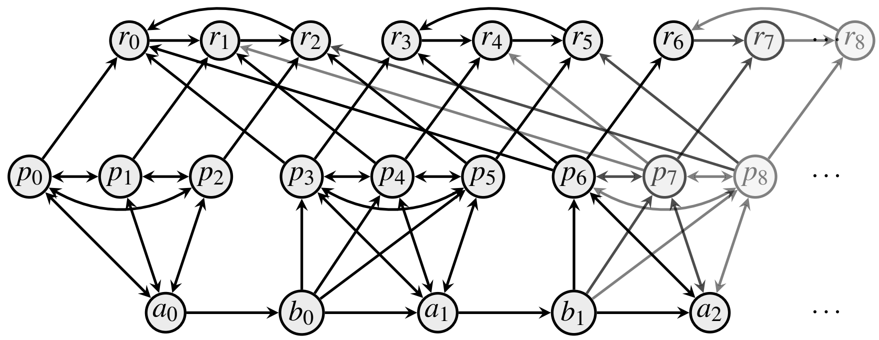


Figure 7.16: AF that is *pr*-compact, whith σ-collapse for $\sigma \in \{sg, sm, st\}$, cf. Example 7.40.

	na	*pr*	*id*	*eg*	*st*	*sg*	*sm*	*c2*	*s2*
cycle-free	ZF	ZF	ZF	ZF	✓	✓	✓	✓	✓
circle-free	ZF	ZF	ZF	ZF	ZF	ZF	ZF	ZF	ZF
symmetric	ZF	ZF	ZF	ZF	✓	✓	✓	ZF	✓
symmetric loop-free	ZF	ZF	ZF	ZF	ZF	ZF	ZF	ZF	ZF
finitary	ZF	ZF	ZF	ZF	✓	ZF	ZF	?	?
planar	ZF	ZF	ZF	ZF	✓	?	✓	?	?
finitely superseded	ZF	ZF	ZF	ZF	✓	✓	-	✓	✓
finitarily superseded	ZF	ZF	ZF	ZF	✓	✓	✓	✓	✓

Table 7.2: Collapse results gathered in this chapter (and partially in Chapter 8), cf. Remark 7.41.

of σ-bombs for x AFs, the term ZF indicates that a collapse is possible in ZF but (as shown in Chapter 8) not in ZFC, a question mark indicates that we know collapsing AFs without AC, but do not know about the status in ZFC. The dash for finitely superseded AFs and *sm* semantics refers to Corollary 7.28 and impossibility of collapse in that case.

Considering Lemmata 7.17 and 7.37 on the correlation between collapse of *sg*, *sm* and the infinite number of extensions for the base semantics we may wonder whether similar results are possible for *c2* and *s2* semantics. Clearly, if for some AF F we have $S \in na(F) \setminus \sigma(F)$, then $S \notin na(\mathscr{R}_S(F))$ and thus there is an argument $x \in A_F$ with $x^+ \cap S \neq \emptyset$ but $x^- \cap S = \emptyset$. In the case of such x repeatedly belonging to different SCCs we can construct an infinite amount of distinguished *na* extensions. The difficulty however arises with the SCC-reduct also being able to recursively break SCCs. We still feel confident enough to state the following.

Conjecture 7.42. *In ZF, given AF F such that $c2(F) = \emptyset$, then $|na(F)| \geq \omega$ holds.*

Very close to this conjecture and already conjectured in [BS15] is the relation between *c2* semantics and finitary AFs. We take another look at the level of finitariness for collapse of *s2* semantics in the following example.

Example 7.43. Consider the variation of AF G from Example 7.5 as AF F depicted in Figure 7.17 where $A_F = \{r_i, p_i \mid i \in \mathbb{N}\} \cup \{o\}$ and $R_F = (R_G \cap A_F \times A_F) \cup \{(r_{i+1}, r_i) \mid i \in \mathbb{N}\} \cup \{(r_0, o), (o, o), (o, p_0)\}$. Since p_0 attacks any p_i, which in turn attacks r_i from which

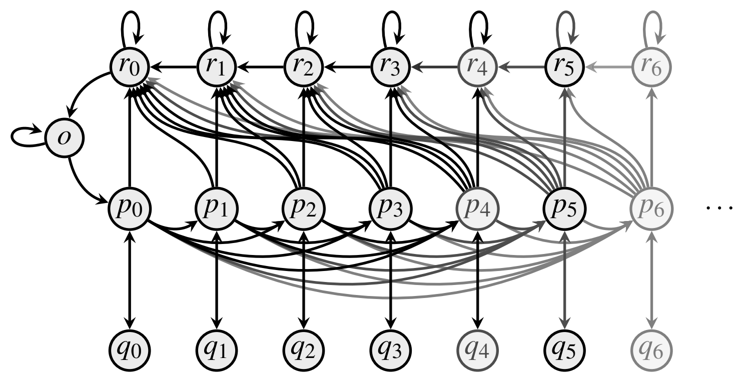


Figure 7.17: Variation of Example 7.2 illustrating non-finitariness of *s2* semantics, cf. Example 7.43.

we have a directed path $r_{i-1} \dots r_0$, which attacks o which attacks p_0 this AF is strongly connected. Thus $sg(F) = s2(F) = \emptyset$ as the evaluation for sg is not influenced by this modification.

Comparing extension sets and realizability recall bomb-shadow modifications Definition 7.11 and 7.15, translating stable semantics (and collapse examples) to any other semantics allowing collapse. With Example 7.24 we have pointed out that for semi-stable semantics one non-finitary argument suffices for a collapse to occur. For stage semantics we refer to Section 8.4 and our hypothesis (Conjecture 8.23) that any *sg*-collapse requires an infinite amount of non-finitary arguments. Example 7.43 is witness to a modification technique relating *s2* and *sg* semantics in the question of non-finitary arguments. For *c2* semantics in particular we proclaim a much more graph theoretic approach. First, observe that all in ZFC *c2*-collapsing examples so far had the AF from Example 7.3 as isomorphic sub-AF. Now consider the following example.

Example 7.44. Consider the AF F depicted in Figure 7.18 with $A_F = \mathbb{N} \cup \{i^j, j_i \mid i < j \in \mathbb{N}\}$ and $R_F = \{(j, j_i), (j_i, i^j), (i^j, i) \mid i < j \in \mathbb{N}\}$.

($ad(F) = \{\emptyset\}$): Assume $i \in S \in ad(F)$ for some $i \in \mathbb{N}$ and thus $i^j \notin S$ for all $i < j$. Since $\left(i^j, i\right)_F^{att}$ we need to defend i against i^j with the only possible defender j_i, hence $j_i \in S$. Conflict-freeness of S further implies $j \notin S$. Now similarly, for the defense of j_i against j we need $j^k \in S$ for some $j < k$ and $k_j \notin S$. But now we know ($i < j < k$) that $k \notin S$ and thus S can not defend itself against k_j. Consequently assume $i \notin S$ for all $i \in \mathbb{N}$ and $S \in ad(F)$. For $i < j$ then $i^j \notin S$ holds as we can not defend i^j against j_i, and similarly (no defense against j) $j_i \notin S$, i.e. $S = \emptyset$.

($st(F) = \emptyset$): Observe that $st(F) \subseteq ad(F)$ always holds. Above result ($\bigcup ad(F) = \emptyset$) combined with $A_F \neq \emptyset$ immediately yields $st(F) = \emptyset$.

($c2(F) = s2(F) = \emptyset$): Observe that F is cycle-free. By Theorem 6.45 (7) then collapse of *st* already implies collapse of *c2* and *s2*.

($sg(F) = \emptyset$): Assume $S \in na(F)$ as given. Since $st(F) = \emptyset$ we have $A_F \setminus S_F^* \neq \emptyset$. Define $A_n = \{i, j, i^j, j_i \mid i < j \leq n\}$. We proceed to show that S is not range-maximal.

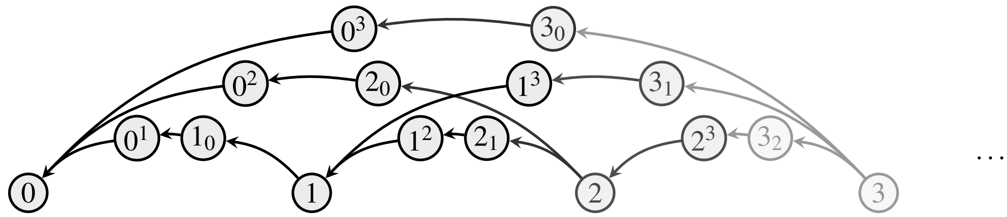


Figure 7.18: AF without stage, stage2 and cf2 extensions, cf. Example 7.44.

First, assume $n \in \mathbb{N}$ with $n \in A_F \setminus S_F^*$ and hence $A_n \setminus S_F^* \neq \emptyset$. We use $S|_n = (S \setminus A_n) \cup \{n\} \cup \{i^n \mid i < n\}$ and observe that $S|_n \in na(F)$ and with $A_n \subseteq (S|_n)_F^*$ also $S_F^* \subset (S|_n)_F^*$. This means that any assumed range-maximal naive extension S needs $n \in S_F^*$ for each $n \in \mathbb{N}$.

Second, assume $m < n \in \mathbb{N}$ with $m, n \in S$. Since $n_m \in n_F^+$ then $n_m \notin S$ and $m^n \notin S_F^*$. For $S|_n$ as defined in the first case thus again $S|_n \in na(F)$ and $S_F^* \subset (S|_n)_F^*$ hold. Hence, any assumed range-maximal naive extension S has at most one $n \in \mathbb{N}$ as member $n \in S$ and $A_n \subseteq S_F^*$.

Finally, combining the first and second observation we have that for most $n \in \mathbb{N}$ (all but possibly one) any potential range-maximal naive extension S needs $n \in S_F^+$ and thus $n^i \in S$ for some $n < i$. But then there is $i_n \in A_F$ with $i_n \notin S_F^*$. We use $S|^n = (S|_n \setminus \{n^i \mid n < i \in \mathbb{N}\}) \cup \{i_n \mid n < i \in \mathbb{N}\}$. By construction again $S|^n \in na(F)$ and $S_F^* \subset (S|_n)_F^*$ hold.

With the following remark we use the insights from working on above example to draw connections to a graph theoretic field of research. That being said, be aware that minors [RS83] traditionally are defined for graphs and there is no standard for the digraph definition.

Remark 7.45 (Minors). Some given AF F is said to have an AF G as *minor* if G is isomorphic to some AF F' resulting from F by (in any order) removing arguments, removing attacks and contracting edges (i.e. removing some attack by merging adjacent arguments).

The AF F from Example 7.44 has the AF G from Example 7.3 as minor. This is fairly obvious by considering the merging operator $n' = \{i_n \mid i < n\} \cup \{n\} \cup \{n^j \mid n < j\}$.

Further, all *sg*-collapsing AFs discussed so far have Example 7.3 as minor. For instance for the AF from Figure 7.10a, first, remove all arguments q_i, s_i as well as attacks (r_i, r_i). We then retrieve the result by contracting the attacks $(p_i, r_i) = n$.

Observe that Example 7.44 results from Example 7.3 by replacing all attacks with indirect attacks and two interstitial arguments. This modification can be seen as faithful in that odd-length path replacements of attacks result in the same relationship between adjacent arguments regarding attack/defense.[1] Further observe that even-length path replacement can lead to very different semantic evaluation. For AF F with $A_F = \mathbb{N} \cup \{i^j \mid i < j \in \mathbb{N}\}$ and $R_F = \{(j, i^j), (i^j, i) \mid i < j \in \mathbb{N}\}$ we have $st(F) = \{\mathbb{N}, \{i^j \mid i < j \in \mathbb{N}\}\}$. That is: AFs with Example 7.3 as minor might not be collapsing for any of the semantics of interest.

However observe that for any *sg*-collapsing AF F the $\subseteq$-poset $\{S_F^* \mid S \in na(F)\}$ does not provide any maxima. In a certain sense this means that each naive set is range-defeated by some

[1] It is easy to see, but not of significance here, that for stable semantics this type of odd-length path replacement indeed is a faithful translation in the sense of Definition 5.24.

other naive set, also compare Theorem 8.16 for a special class of AFs without *sg*-collapse. Such range-defeating chains can be used to define argument clusters, which leads us to the following conjecture.

Conjecture 7.46 (Stage collapsing AFs)**.** *For any given AF F with $sg(F) = \emptyset$ we have the AF G from Example 7.3 as minor.*

Known examples do not hint at any significant difference between *sg* and *s2* semantics regarding collapse and further *c2*-collapse implies *s2*-collapse. However, since the available examples do not provide much actual insight, for *c2*/*s2* we conclude this chapter with the following open questions.

Open Question 7.47. *Are there finitary AFs witnessing collapse of s2 in ZFC? Are there AFs not having Example 7.3 as minor and witnessing collapse of c2?*

Chapter 8

Perfection

> This is a talk about something and naturally also a talk about nothing.
> ...
> And how is this done? Done by making something which then goes in and reminds us of nothing. It is important that this something be just something, finitely something; then very simply it goes in and becomes infinitely nothing.
>
> John Cage, Lecture on Something [Cag59]

This chapter should be seen as an extended version of [Spa16b], with not yet published results as well as results from [BS15] incorporated. The inspiration (and naming) stems from *Kernel-Perfection* [BD90], a vivid research area established for digraphs. As noted in Lemma 4.14 stable extensions of some AF F are exactly the kernels of the corresponding digraph F^{-1}. Consequently we will also reference digraph results.

With σ-perfection (Definition 5.39) we label AFs F such that for any induced sub-AF $G = F|_{A_G}$ we have $\sigma(F) \neq \emptyset$. A natural modification of this definition might further require $\sigma(F) \neq \{\emptyset\}$, which we do not. Since we allow the empty AF $(\emptyset, \emptyset)$ for $A_G = \emptyset$ we always have $\sigma(G) = \{\emptyset\}$. This is a technical hindrance we could circumvent with slightly different definitions. However, for the time being, density of this thesis is evidence that for the purpose of providing novel research it suffices to leave that extra effort aside.

As compared to common graph theory literature [GN84, DM93, Tom90] we face two main differences. One is the range of semantics. Although technically there is a fine collection of digraph semantics (cf. Definition 2.38) the property of all induced subgraphs providing an extension is, to the best of our knowledge, considered only for kernels and hence the notions of perfection and kernel-perfection are often used synonymously. The other difference is our focus on set theoretic principles and thus consideration of models without AC. Further, digraphs are often defined as loop-free and sometimes even finite entities. Given that stable-perfection is already non-trivial in finite loop-free scenarios, such approaches are reasonable. As stated in Theorem 8.2 for our selection of semantics, finite AFs are not very interesting.

Stable semantics collapses already for the self-attacking singleton $F = (\{x\}, \{(x,x)\})$ (cf. Example 6.1), which deems us the main reason research on perfection casts aside with self-attacks. It might be observed (for instance Examples 7.23 or 7.21) that the event $\sigma(F) = \emptyset$, i.e. the collapse of semantics σ on AF F, is not vastly affected by assuming or not assuming loop-freeness. By our general definitions we hence include self-attacking arguments in our considerations.

The remainder of this chapter is organized as follows. In Section 8.1 we discuss perfection in scenarios of ZF where AC might not be given, while in Section 8.2 we additionally assume AC. In Section 8.3 we highlight the case of finitary AFs with our proof technique for existence of semi-stable and stage extensions, first published in [BS15]. In Section 8.4 we emphasize the special qualities of stage semantics in respect of perfection, as first published in [Spa16b]. Finally, in Section 8.5 we conclude, highlight achievements and relate to the literature.

8.1 Perfection without Choice

In this section we establish an argumentation scenario of perfection in a world without choice. That is, we emphasize cases of semantics σ and AFs F where AC or equivalent notions such as Zorn's Lemma are not necessary for σ-perfection of F. To this end we start with the most basic semantics, i.e. the semantics we did not provide collapsing examples for in Chapter 7.

Theorem 8.1 (Perfection in ZF)**.** *In ZF for semantics $\sigma \in \{cf, ad, co, gr\}$ any AF is σ-perfect.*

Proof. Theorems 6.2 and 6.7 justify existence and thus perfection. □

Witnessed by Example 6.1 stable semantics might collapse even for finite AFs. The following highlights that this can not be the case for all other semantics under consideration.

Theorem 8.2 (Finite perfection)**.** *For semantics $\sigma \in \{na, pr, sg, sm, c2, s2, id, eg\}$ any finite AF is σ-perfect.*

Proof. By Lemma 7.17 collapse of σ implies infinitely many different *cf* sets. Since the number of *cf* sets for any AF F is limited by $|\wp(A_F)|$ this can not be the case for finite AFs. □

Stable semantics is the only one among Dung's initial set of semantics [Dun95] to collapse in the finite case and inspired introduction of several new semantics possibly collapsing in the infinite case [Ver03, CV10, BGG05]. Unsurprisingly *st* semantics plays a very special role for perfection in general. Recall Section 6.5 on the relations between semantics and derive the following profound result.

Lemma 8.3 (Stable-perfection)**.** *Given st-perfect AF F, then F is also σ-perfect for $\sigma \in \{cf, ad, co, sg, sm, c2, s2, na, pr, gr, id, eg\}$ and further $|gr(F)| = |id(F)| = |eg(F)| = 1$.*

Proof. Observe that it suffices to show existence of σ-extensions for F to deduce the same result for all induced sub-AFs. By Proposition 6.39 any *st* extension is also a τ-extension for $\tau \in \{cf, ad, co, sg, sm, c2, s2, na, pr\}$. Finally, with non-empty *co*, *pr*, and *sm* extension sets by Proposition 6.19 thus unique definedness of $gr(F)$, $id(F)$ and $eg(F)$ follows. □

Already to be found in [Dun95] is the case of well-founded AFs (Definition 3.13), where every directed path necessarily provides a first argument in line. Our main achievement here is to highlight that AC is not necessary and to include several additional semantics. For this (and for several of the results to come) we make use of induced sub-AFs often inheriting AF classes of the original AF.

Theorem 8.4 (Well-founded perfection in ZF)**.** *In ZF any well-founded AF is σ-perfect for semantics $\sigma \in \{st, sg, sm, c2, s2, na, pr, id, eg\}$.*

Proof. Since induced sub-AFs of well-founded AFs are in turn well-founded this claim almost follows from Lemma 5.43 (on super-coherence of well-founded AFs). It remains to show *na*-perfection, which follows from Lemma 8.3 or the plain observation that *st* extensions of course are maximal conflict-free. □

Finally, as last result of this section we discuss the case of bipartite AFs. The attentive reader of Chapter 7 might have observed that we did not provide any collapsing bipartite AFs. The following lemma and theorem elaborate on the reasons why with a given partition dividing the attacks we can already construct σ-extensions for any semantics σ of interest.

Lemma 8.5 (Bipartite AFs)**.** *In ZF for any bipartite AF F with $A_F = B \dot{\cup} C$, $\{B\}_F^{ind}$, $\{C\}_F^{ind}$, and complete extension $S_0 \in co(F)$ we have that $S = S_0 \cup (B \setminus S^+)$ is a stable extension: $S \in st(F)$.*

Proof. First take a look at complete extensions. By definition we have $A_F = S_0 \cup S_0^+ \cup X$, where $X^{\pm} \cap S_0 = \emptyset$ and for each $x \in X$ even $x^- \cap X \neq \emptyset$. In more detail X can not be in conflict with S_0, since completeness facilitates admissibility, i.e. attackers of S_0 need to be attacked by S_0 (and thus are member of S_0^+). Any $x \in X$ needs to be attacked by X. Otherwise it is defended by S_0 and thus $x \in \mathscr{D}_F(S_0)$, i.e. S_0 would not be complete.

Now for the set S we add arguments from B that are compatible with S_0 and thus member of $B \setminus S^+$. Again, for any $x \in B$ with $(x, S_0)_F^{att}$ also $(S_0, x)_F^{att}$ holds by completeness of S_0. The remaining argument set X then is divided into $X \cap B$ and $X \cap C$ with attacks in between these components. That is, arguments $x \in X \cap C$ are attacked by some argument $y \in X \cap B$. And hence S is conflict-free and $S^* = S_0 \cup S_0^+ \cup (X \cap B) \cup (X \cap C) = A_F$. □

Theorem 8.6 (Bipartite perfection in ZF)**.** *In ZF any bipartite AF F ($A_F = B \dot{\cup} C$, $\{B\}_F^{ind}$, $\{C\}_F^{ind}$) is σ-perfect for semantics $\sigma \in \{st, sg, sm, c2, s2, na, pr, id, eg\}$.*

Proof. Observe that induced sub-AFs of bipartite AFs are in turn bipartite (for $G = F|_A$ use the partition $B \cap A$ and $C \cap A$). By Theorem 6.7 any AF provides a grounded (and thus complete) extension S_0. By Lemma 8.5 then there is a stable extension. □

8.2 Perfection with Choice

First, in light of Section 8.1, taking a look at Figure 3.5 observe that for the remaining AF classes and semantics Examples 7.19 (finitary, circle-free, planar) and 7.20 (finitary, symmetric, odd-circle-free, planar) provide matching collapses for some models of ZF (without AC). It thus becomes evident that for the remaining results of this chapter some variation of AC (cf. Section 6.3) is required. Regarding the question of which variation however we will not go into detail. As first result of this section we take into account general perfection in case of ZFC, that is semantics where for arbitrary AF we can always construct extensions granted a choice function. Very similar (at least to how we approached the related equivalence of AC and existence of preferred extensions in Section 6.2) loop-free symmetric digraphs, referenced to as graphs in Definition 2.29, allow for a very general existence statement in ZFC.

Theorem 8.7 (Perfection in ZFC)**.** *In ZFC for semantics $\sigma \in \{na, pr, id, eg\}$ any AF is σ-perfect. And further any symmetric loop-free AF is τ-perfect for $\tau \in \{st, sg, sm, c2, s2\}$.*

Proof. Theorem 6.15 establishes the equivalence of AC with existence of *na* and *pr* extensions and thus *na*- and *pr*-perfection in ZFC. This theorem also justifies τ-perfection of symmetric loop-free AFs. Theorem 6.20 (existence of *id* and *eg* extensions in ZFC) similarly establishes *id*- and *eg*-perfection. □

Above we have a statement on symmetric loop-free AFs. The remaining semantics of interest thus are *st*, *sg*, *sm*, *c2*, *s2* for AFs that are not symmetric and loop-free. It is only natural to try and approach sub classes of graphs, namely symmetric AFs and loop-free AFs. For loop-free AFs evidence suggests that letting go of self-attacks on its own does not alter the chance for collapse for the semantics of interest (Examples 7.3 and 7.23). Further, as shown in Example 7.4 semantics *st*, *sg*, *sm*, *s2* might collapse for symmetric AFs. In terms of symmetric AFs we are thus only left with *c2* semantics and the following theorem.

Theorem 8.8 (Symmetric perfection in ZFC)**.** *In ZFC any symmetric AF is c2-perfect.*

Proof. By Proposition 6.39 *c2* semantics agrees with *na* semantics for symmetric AFs. Theorem 8.7 then delivers the claim. □

Looking at the relations between AF classes illustrated in Figure 3.5 observe that, regarding models of ZFC, with Example 7.21 we provide a cycle-free AF with σ-collapse for $\sigma \in \{st, sg, sm, c2, s2\}$. Naturally this AF is also odd-cycle-free. By discussing the limited controversial case (see Definition 3.17 for the sharp definition stemming from [Dun95]) we can then also conclude perfection for circle-free, odd-circle-free and uncontroversial AFs. The following result again is an extension of Dung's work in the axiomatic sense as well as for the matter of semantics.

Theorem 8.9 (Limited controversial perfection in ZFC)**.** *In ZFC any limited controversial AF is σ-perfect for semantics $\sigma \in \{st, sg, sm, c2, s2\}$.*

Proof. By Theorem 6.45 we have that any limited controversial AF F is coherent, that is $st(F) = pr(F)$. By Theorem 6.15 in ZFC any AF (and thus also the limited controversial AF

F) provides *pr* extensions. Thus $st(F) \neq \emptyset$. Now observe that induced sub-AFs of limited controversial AFs are in turn limited controversial. Then, by Lemma 8.3 σ-perfection of F already follows. □

Remark 8.10 (Finitely/finitarily superseded AFs). With Example 7.25 we have provided a finitely superseded AF that collapses for *st*, *sg*, *c2*, *s2*. Example 7.29 represents a finitarily superseded AF with *sm*-collapse. Finitely superseded AFs do provide *sm* extensions (Corollary 7.28) yet do not ensure perfection as witnessed by Remark 7.26 with any collapsing input AF F that is finitely superseded by the AF $(\{x_F\}, \{(x_F, x_F)\})$.

We have given a perfection result for finite AFs with Theorem 8.2. Taking another look at Figure 3.5 and above results it thus appears that the only unknown territory is regarding finitary and planar AFs. We dedicate the following section to finitary AFs.

8.3 Finitary AFs

In this section we focus on the class of finitary AFs and provide perfection results for *sg* and *sm* semantics. As pointed out in Examples 7.19 and 7.20 finitary AFs might collapse for *sg* and *sm* in models of ZF without at least some variation of AC. Consequently, for this section we consider only models of ZFC.

Finitary AFs have been conjectured to be *sm*-perfect in [CV10], which was proven first in [Wey11] using a model theoretic approach. Partially because we found the latter proof hard to follow, partially because we were convinced that if correct a direct proof via transfinite induction should be possible we came up with the techniques highlighted in this section. On the one hand this section serves as proof for *sm*-perfection and *sg*-perfection of finitary AFs (first published in [BS15]). On the other hand the techniques devised for our proof provide substantial insight into the mechanics of the discussed semantics.

When dealing with range-maximal extensions in infinite AFs as seen in the examples from Chapter 7 we might deal with sets of sets of arguments that keep growing in size with respect to their range. For being able to handle constructions of this kind we introduce the following two definitions. The intuition for the first definition is that we want to be able to say something about arguments and sets occurring (un)restricted in collections of extensions. For the second definition we focus on the idea of infinitely range-growing sets of extensions.

Definition 8.11 (Keepers, Outsiders, Keeping Sets and Compatibility)**.** Consider some AF F. For $\mathscr{E}$ a set of sets of arguments we call $\mathscr{E}^* = \bigcup_{E \in \mathscr{E}} E^*$ the *range* of $\mathscr{E}$ and for some argument $a \in \mathscr{E}^*$ we say that:

- a is a *keeper* of $\mathscr{E}$ if it occurs range-unbounded in $\mathscr{E}$, i.e. for any $E_1 \in \mathscr{E}$ with $a \notin E_1$ there is some $E_2 \in \mathscr{E}$ such that $a \in E_2$ and $E_1^* \subseteq E_2^*$;
- a is an *outsider* of $\mathscr{E}$ if it is not a keeper of it, i.e. there is some $E_1 \in \mathscr{E}$ with $a \notin E_1$ such that for any $E_2 \in \mathscr{E}$ with $E_1^* \subseteq E_2^*$ we also have $a \notin E_2$.

Furthermore for a set $A \subseteq \mathscr{E}^*$ we say that:

- A is a *keeping set* of $\mathscr{E}$, or *kept* in $\mathscr{E}$, if it occurs range-unbounded in $\mathscr{E}$, i.e. for every $E_1 \in \mathscr{E}$ with $A \not\subseteq E_1$ there is some $E_2 \in \mathscr{E}$ such that $A \subseteq E_2$ and $E_1^* \subseteq E_2^*$.
- A is called *compatible* with $\mathscr{E}$ if every finite subset of A is kept in $\mathscr{E}$, i.e. for $B \subseteq A$ with $|B| < \omega$ we have that B is a keeping set of $\mathscr{E}$.

Definition 8.12 (Range Chain, Chain Range, Induced AF)**.** Consider some AF F. A set of sets of arguments $\mathscr{E}$ is called a *range chain* if for any $E_1, E_2 \in \mathscr{E}$ we have $E_1^* \subseteq E_2^*$ or $E_2^* \subseteq E_1^*$, again the range of $\mathscr{E}$ (the *chain range* $\mathscr{E}^*$) is defined as $\mathscr{E}^* = \bigcup_{E \in \mathscr{E}} E^*$.

Now for a given range chain $\mathscr{E}$ we will consider the by $\mathscr{E}$ *induced* AF $F|_{\mathscr{E}}$:

$$F|_{\mathscr{E}} = (\mathscr{E}^*, \{(a,b) \mid a,b \in \mathscr{E}^*, (a,b) \in R_F\} \cup \{(b,b) \mid b \textit{ outsider of } \mathscr{E}\})$$

Observe that naturally finite range chains or chains that have a maximum will not be of interest to us. Also observe the implicit transitivity, i.e. for $E_1, E_2, E_3 \in \mathscr{E}$ from $E_1^* \subset E_2^*$ and $E_2^* \subset E_3^*$ it follows that also $E_1^* \subset E_3^*$. Thus a range chain by definition gives a well-ordering on the equivalence class of elements with equal range.

In the case of *sm* and *sg* extensions we deal with semantics that sometimes are seen as weaker forms of *st* semantics. In this sense we think of range chains that range-cover the whole framework, or in other words reduce frameworks to arguments being relevant (Definition 8.12) to some range chain only. We next deal with the question whether some argument or sets of arguments are part of some stable extension. The intuition is that we want to recursively cover the full range of some AF. The following definition helps in discussing the recursion step.

Definition 8.13 (Unresolved Range)**.** Consider as given some AF F, a range chain $\mathscr{E}$ such that $F|_{\mathscr{E}} = F$, and a set $A \subseteq \mathscr{E}^*$. We define the *unresolved range* of A as the set A^u that as a next step has to be resolved if A is to be subset of a stable extension. A^u thus consists of arguments endangering A without defense, as well as arguments attacked by A^* but not by A.[1] Also see Figure 8.1 for an illustration.

$$A^u = \left\{b \notin A^* \mid (b,A)^{att}\right\} \cup \left\{a \notin A^* \mid (A^*,a)^{att}\right\}$$

Lemma 8.14. *Consider as given some finitary AF F, some range chain $\mathscr{E}$, such that $F|_{\mathscr{E}} = F$, and some with $\mathscr{E}$ compatible set $A \subseteq \mathscr{E}^*$. Then there is some with $\mathscr{E}$ compatible set $B \subseteq \mathscr{E}^*$ such that $A \subseteq B$ and $A^u \subseteq B^*$, we have $A^* \cup A^u \subseteq B^*$.*

Proof. First observe that for every finite set $C \subseteq A^* \cup A^u$ there has to be a finite set D such that $C \cap A \subseteq D$ and $C \cap A^u \subseteq D^*$. This is due to the finitary condition and the definitions. For every finite set of arguments there are only a finite number of sets that have at most this range, but since the chain $\mathscr{E}$ is unbounded in F there is at least one. Furthermore if D resolves $A_1 \cup A_2$ then D resolves A_1 and A_2. By transfinite induction on the size of D we can show that there is a set with the desired properties. $\square$

[1] Observe that not in general, but in case of $\{A\}_F^{ind}$ and thus the desired use case, we have $A^* \setminus A = A^+$.

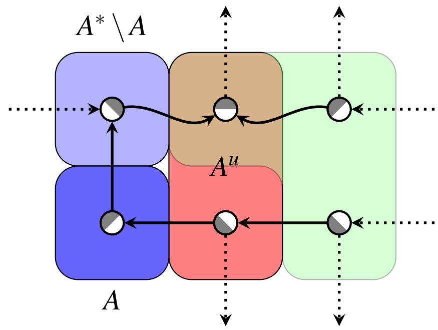


Figure 8.1: An illustration of unresolved range A^u (Definition 8.13). Observe that the rightmost area characterizes all arguments that can resolve A^u, when incorporating A.

Proposition 8.15. *For any finitary AF F, $|sm(F)| \geq 1$ and $|sg(F)| \geq 1$.*

Proof. Take some finitary AF F, and $\sigma = pr$ or $\sigma = na$, and $\sigma^* = sm$ or $\sigma^* = sg$ respectively. We show that for any range chain $\mathscr{E} \subseteq \sigma(F)$ there is some σ-extension E that covers the full chain range, i.e. $\mathscr{E}^* \subseteq E^* \in \sigma(F)$. By then applying Zorn's Lemma it follows that $\mathscr{E}$ also contains at least one range-maximal set or in other words a σ^*-extension.

To this end for any range chain $\mathscr{E} \subseteq \sigma(F)$, we proceed with the following steps (1) – (5) using transfinite recursion to find an upper bound A with $\mathscr{E}^* \subseteq A^*$ such that there is some $E \in \sigma(F)$ with $A \subseteq E$. Step 1: consider only relevant arguments of F; step 2: recursion start, motivation and intuition; step 3: successor step, augment by resolving keeper sets or compatible keepers; step 4: limit step, collect successor steps; step 5: remarks, conflict-freeness and range-completeness.

1. Consider only relevant arguments of F: As presented in Definition 8.12 we will make use of some AF $F|_{\mathscr{E}}$ that contains only arguments from the range of $\mathscr{E}$, plus all outsiders are self-attacking. If we retrieve a conflict-free (admissible) set A such that A contains only keepers of $\mathscr{E}$ and spans the whole range, $A^* = \mathscr{E}^*$, we can as stated in Zorn's Lemma (see Definition 2.21) retrieve a σ-extension that covers the whole chain range. Clearly every stable extension of $F|_{\mathscr{E}}$ serves this purpose. In the following we thus construct a stable extension and consider some AF F where $F|_{\mathscr{E}} = F$.

2. Define the recursion start: As recursion start we use the set $A_0 = \{a\}$ for some keeper a of $\mathscr{E}$. In each step we augment this set in a clever way, by choosing compatible sets that either cover the unresolved range or some arbitrary compatible keeper.

3. Successor Steps, $\alpha = \beta + 1$: Assume some compatible set A_β. If A_β has unresolved range $A^u_\beta \neq \emptyset$ we choose a compatible set $A_\alpha \supset A_\beta$ such that $A^u_\beta \subset A^*_\alpha$. As stated in Lemma 8.14 such a set exists, but we might need the axiom of choice to find one. If on the other hand $A^u_\beta = \emptyset$ we pick some compatible keeper $a \notin A_\beta$, such that $A_\alpha = A_\beta \cup \{a\}$ is compatible with $\mathscr{E}$.

4. Limit Steps, α: Consider a range chain $\{A_i\}_{i<\alpha}$ where for any $i < j$ we have $A_i \subseteq A_j$ and all A_i are finitely compatible. We define $A_\alpha = \bigcup_{i<\alpha} A_i$, implicitly using the axiom of choice. By definition A_α is compatible with $\mathscr{E}$, for otherwise there would be some $B \subseteq A_\alpha$, $|B| < \omega$ that is not kept in $\mathscr{E}$, but then due to the construction it follows that already $B \subseteq A_i$ for some $i < \alpha$, in contradiction to the successor step.

5. Conflict-freeness and range-completeness: Conflict-freeness follows from compatibility, range-completeness follows from definition of unresolved range and successor/limit steps resolving this issue. Latest at each limit step, A_α becomes admissible and independent from arguments that are not member of A^*_α, i.e. if $(a,A_\alpha)^{att}$ then $(A_\alpha,a)^{att}$, and if $(A^*_\alpha,a)^{att}$ then $(A_\alpha,a)^{att}$, and if $(a,b)^{att}$ where $b \in A^*_\alpha$ then $(A_\alpha,b)^{att}$.

Having showed that every range chain of σ-extensions has an upper bound in $\sigma(F)$ using Zorn's lemma we now conclude that there is a range-maximal σ-extension, in other words a σ^*-extension. □

Theorem 8.16 (Finitary perfection in ZFC)**.** *In ZFC for semantics $\sigma \in \{sg, sm\}$ any finitary AF is σ-perfect.*

Proof. Considering existence of σ-extensions for any finitary AF F (Proposition 8.15) it remains to observe that induced sub-AFs of finitary AFs are in turn finitary. □

8.3.1 A Note on *c2* and *s2* Semantics

The two remaining semantics which have defied any attempt of solving w.r.t. the problem of existence in case of finitary AFs are *c2* and *s2* semantics [BGG05, DG16], also see Definition 4.26. Given SCC-recursiveness, we have to face some difficulties in drawing conclusions with respect to infinite or finitary AFs. If every subframework does have an initial SCC (which is guaranteed for finite AFs), i.e. some strongly connected subframework that is not attacked from the outside (also see Definition 6.50), then obviously this AF provides a $\sigma 2$-extension as soon as every single component provides a σ-extension. If on the other hand there is no initial SCC, things become more complicated and, in particular especially due to the recursive definitions, not that easy to handle. So for now we go with the following conjecture.

Conjecture 8.17. *For any finitary F, $|c2(F)| \geq 1$ and $|s2(F)| \geq 1$.*

8.4 Collapse-Resistance of Stage Semantics

In this section we ask the question for other conditions granting perfection for the semantics of interest and give some answers, mostly for stage semantics. In Section 8.5 we further highlight conditions for stable semantics from the literature, or kernel-perfection to be precise. What makes stage semantics particularly interesting in this context is its simple definition. Stable extensions are conflict-free sets where each argument is either labelled in or out. Semi-stable

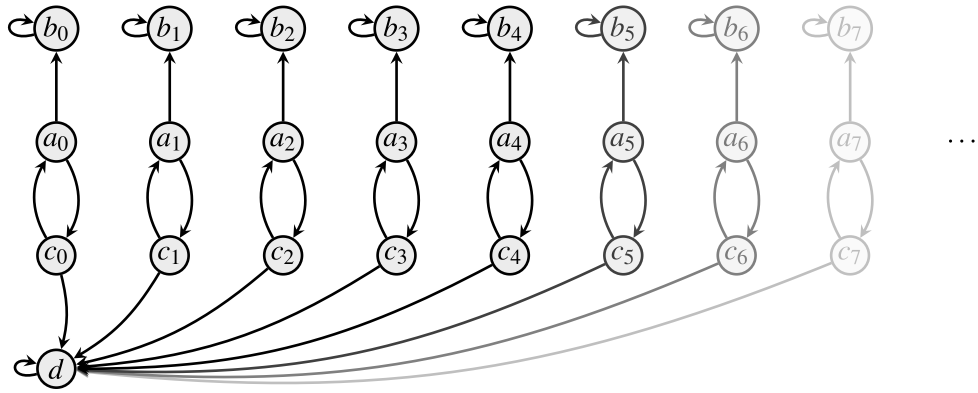


Figure 8.2: Unbounded range-chain for *sg*-perfect AF as discussed in Example 8.18.

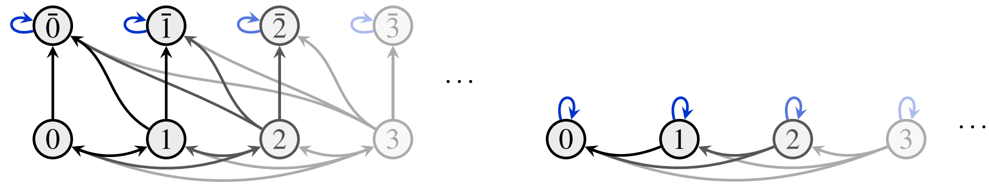
extensions then are admissible sets where undecided arguments are minimal. Similarly stage extensions are conflict-free sets where undecided arguments are minimal. While this difference might seem subtle, in [Spa13] we highlight that already in the finite case the requirement of admissibility adds substantial expressiveness. Notably semi-stable semantics provides mechanisms that allow extensions S where undecided arguments x might not be in conflict, that is where $\{S \cup \{x\}\}^{ind}$ holds. Due to maximal conflict-freeness this can not be the case for stage semantics. This property might be seen as a weakness of stage semantics and has been facilitated in that way for instance in [DS17, DDLW15].

In this section in particular, as published in [Spa16b], we emphasize maximal conflict-freeness of stage semantics as a strength in terms of collapse resistance. The for [Spa16b, Theorem 7] given proof (Stage Perfection Characterization) however has weak points we address with the following example.

Example 8.18. Consider the AF F as illustrated in Figure 8.2 with $A_F = \{a_i, b_i, c_i \mid i \in \mathbb{N}\} \cup \{d\}$ and $R_F = \{(b_i, b_i), (a_i, b_i), (a_i, c_i), (c_i, a_i), (c_i, d) \mid i \in \mathbb{N}\} \cup \{(d, d)\}$. Observe that this AF does not have a stable extension as for $B = \{b_i \mid i \in \mathbb{N}\}$ for any naive extension S only one of $d \in S^*_F$ and $B \subseteq S^*_F$ holds. Then $S_n = \{a_i \mid i \leq n\} \cup \{c_j \mid j > n\}$ is an unbounded range-chain as $S^*_i \subset S^*_j$ for any $i < j$ and $\bigcup S_n = A_F$. However we have $\{a_i \mid i \in \mathbb{N}\}$ and $\{c_n\} \cup \{a_i \mid i \in \mathbb{N}, i \neq n\}$ as stage extensions and this AF even is *sg*-perfect.

Further consider the AF G with $A_G = A_F \cup \{e\}$ and $R_G = R_F \cup \{(d, a_i), (d, b_i), (d, c_i) \mid i \in \mathbb{N}\}$. Similarly $(S_i)_{i \in \mathbb{N}}$ serves as unbounded range-chain. Now however even the collection of keepers of this chain (i.e. the set $S = (a_i)_{i \in \mathbb{N}}$) is not a stage extension, as $S^*_F \subset \{e\}^*_F$ holds.

In [Spa16b] we claim that for any unbounded range-chain involving some fixed argument x in a given AF there is some unbounded chain already in a proper sub-AF not containing x. The AF F from Example 8.18 gives an unbounded range-chain $(S_i)_i$ that is not unbounded for $x = d$. The AF G further illustrates that even more lax definitions of conversion (or limit points using keeping sets) might not be of use as there the set S is not range-comparable to the chain-range of $(S_i)_i$. We have to admit that we could neither come up with counterexamples to [Spa16b,



(a) Looping infinitely many arguments (b) Delooping infinitely many arguments

Figure 8.3: AFs illustrating finite limits of (de)looping *sg*-perfect AFs, cf. Example 8.22.

Theorem 7] nor with a working proof. What we did come up with is a weaker version of the theorem, investigating additional self-attacking arguments.

Lemma 8.19 (Adding Arguments, Effects on Range I)**.** *In ZFC, consider as given AFs F, G with $G = F|_{A_F \setminus \{x\}}$ and $(x,x)^{att}_F$. For any $S \in sg(G)$ there is $T \in sg(F)$ with $S^*_F \subseteq T^*_F$.*

Proof. Since $(x,x)^{att}_F$ it holds that $na(F) = na(G)$ and for $S \in na(G)$ we have $S^*_F \setminus \{x\} = S^*_G$. For $S \in sg(G)$ not to be *sg* extension of F there has to be some $T \in na(F)$ with $S^*_F \subset T^*_F$. Then however we need $T^*_F = S^*_F \cup \{x\}$ and thus $S^*_G = T^*_G$, i.e. T is a *sg* extension of G as well. $\square$

Lemma 8.19 put in other words, with additional self-attacking arguments we may lose some *sg* extensions but only if there are other *sg* extensions defeating them. It is not possible to gain additional range-chains with finitely many self-attacking arguments. This immediately yields the following result.

Corollary 8.20 (Delooping *sg*-perfect AFs)**.** *For any sg-perfect AF F and finite argument set $X \subseteq A_F$ the modification G with $A_G = A_F$ and $R_G = R_F \cup \{(x,x) \mid x \in X\}$ is still sg-perfect.*

In light of this result it seems natural to ask the dual question of removing self-attacks in *sg*-perfect AFs. Although our insights strongly suggest this approach to be fruitful it still requires more fine-graining of our notions of keepers and outsiders. For now we thus present the following conjecture.

Conjecture 8.21 (Looping *sg*-perfect AFs)**.** *For any sg-perfect AF F and finite argument set $X \subseteq A_F$ the modification G with $A_G = A_F$ and $R_G = R_F \setminus \{(x,x) \mid x \in X\}$ is still sg-perfect.*

We intercede with examples highlighting that the finite condition in above corollary and conjecture is important.

Example 8.22. Looping: As illustrated in Figure 8.3a consider the AF F with $A_F = \{i, \bar{i} \mid i \in \mathbb{N}\}$ and $R_F = \{(i,j),(j,i),(j,\bar{i}) \mid i < j \in \mathbb{N}\}$ and its modification G with $A_G = A_F$ and $R_G = R_F \cup \{(\bar{i},\bar{i}) \mid i \in \mathbb{N}\}$. That is, G results from F by looping infinitely many arguments. We have $sg(F) = \{\{n\} \cup \{\bar{i} \mid n \neq i \in \mathbb{N}\} \mid n \in \mathbb{N}\}$ while $sg(G) = \emptyset$.

Delooping: As illustrated in Figure 8.3b consider the AF F with $A_F = \mathbb{N}$ and $R_F = \{(j,i),(\bar{i},\bar{i}) \mid i < j \in \mathbb{N}\}$ and its modification G with $A_G = \mathbb{N}$ and $R_G = R_F \setminus \{(i,i) \mid i \in \mathbb{N}\}$. That is, G results from F by delooping infinitely many arguments. We have $sg(F) = \{\emptyset\}$ while $sg(G) = \emptyset$.

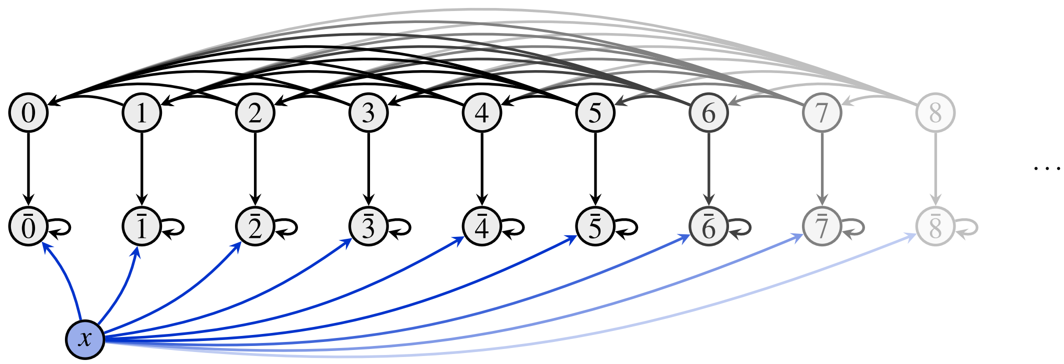


Figure 8.4: Adding an argument to provoke a collapse for *sg* semantics, cf. Example 8.25.

The attentive reader might have observed that above conjecture leads to a more general notion of Lemma 8.19. This more general notion is precisely [Spa16b, Theorem 7] which we hereby also give as conjecture.

Conjecture 8.23 (Stage Perfection Characterization)**.** *In ZFC, consider as given AFs F, G with $G = F|_{A_F \setminus \{x\}}$. If G is sg-perfect then so is F.*

One potential way of proving this conjecture hence is to consider *sg*-perfect AFs and investigate the effects of removing self-attacks. Another more straightforward way would be to take a closer look at range-posets and range-chains. Work on this matter (the dynamics of keepers and outsiders in *sg*-perfect AFs) has led us to the following conjecture.

Conjecture 8.24 (Range in *sg*-perfect AFs)**.** *In ZFC, consider as given some sg-perfect AF F and $C \in cf(F)$. Then there is $S \in sg(F)$ with $C_F^* \subseteq S_F^*$.*

The principle of Conjecture 8.24 is that in *sg*-perfect AFs we assume any possible (conflict-free) range to be covered by some *sg* extension. In the finite case this is a rather simple observation. In the general infinite case any AF with *sg*-collapse in ZFC serves as counter-example. For *sg*-perfect AFs this question again seems to require more refined notions regarding range-chains. Consequently we proclaim this conjecture as another method of arriving at Conjecture 8.23.

When first considering Conjecture 8.23 one might think why we require *sg*-perfect AFs and not merely AFs with non-empty *sg* extension set. The following example highlights that the latter is not a sufficient condition.

Example 8.25. Consider the AFs F, G depicted in Figure 8.4 with $A_F = \{i, \bar{i} \mid i \in \mathbb{N}\} \cup \{x\}$, $R_F = \{(i, \bar{i}), (\bar{i}, \bar{i}), (x, \bar{i}) \mid i \in \mathbb{N}\} \cup \{(j, i) \mid i < j \in \mathbb{N}\}$ and $G = F|_{A_F \setminus \{x\}}$. These AFs represent an initial *sg*-collapse (among arguments i), avoided by range modulators (arguments $\bar{i}$) and reinforced by range annihilator (argument x). We thus have $sg(G) = na(G) = \{\{i\} \mid i \in \mathbb{N}\}$ while $sg(F) = \emptyset$.

Lemma 8.19 is a helpful replacement of [Spa16b, Theorem 7] in that it still suffices to arrive at the following result. Recall that in symmetric AFs *cf* and *ad* and thus *sg* and *sm* coincide, and that symmetric loop-free AFs (see Theorem 8.7) are *sg*-perfect in ZFC.

Corollary 8.26. *Symmetric AFs with finitely many self-attacking arguments are sg/sm-perfect.*

To close up this section we now proceed to briefly investigate planarity. Planarity is most formally defined via graph minors (cf. Remark 7.45), where one condition is that no planar graph has a complete graph (edges between any pair of arguments) of five arguments as minor. Since the AF from Example 7.3 is a complete graph of ω many arguments we may use Conjecture 7.46 to motivate the following conjecture.

Conjecture 8.27. *Planar AFs are sg-perfect.*

8.5 Conclusions

In this section we conclude our investigations in perfection, highlight possible future research questions and relate to the literature. Regarding abstract argumentation, perfection results are implicitly given in [Dun95], conjectured in [CV10] and proven in [Wey11], with the observation that several AF-properties carry over to sub-AFs. As compared to these references the benefit of our work (besides new results) is to include set theoretic principles, systematically categorize AF classes and propagate perfection as a principle. Perfection in general and the investigation of infinite AFs in particular is more related to research on digraphs and kernels [GN84]. For an overview on kernel-perfection we recommend [BD90, GG07, GG16]. Kernel-perfection results often deliver elaborate constructions that might or might not be of obvious applicability in the light of abstract argumentation. We decided to present the following as referenced results, adapted and transformed for our purposes. Observe that *st*-perfection by Lemma 8.3 implies σ-perfection for all other semantics σ of interest. Further observe that formally all results from the following theorem are originally stated for loop-free structures (per Definition of digraphs). For the first two results however loop-freeness immediately follows by the conditions.

Theorem 8.28 (Perfection-results from the literature)**.** *The following hold in ZFC:*

- *A finitary AF F is st-perfect if and only if every finite induced sub-AF $G = F|_{A_G}$, $|A_G| < \omega$ provides a st extension. [DM93]*
- *An AF F is st-perfect if every non-empty induced sub-AF $G = F|_{A_G}$, $A_G \neq \emptyset$ provides a non-empty admissible set $S \in ad(G)$, $S \neq \emptyset$. [GN84]*
- *A loop-free AF F is st-perfect if every cycle of odd length is symmetrical. [Ric46]*
- *A finite loop-free AF F where for $(a,b)^{att}_F$ and $(b,c)^{att}_F$ it already follows that $(a,c)^{att}_F$ is st-perfect and all st extensions have the same cardinality. [Kön36]*

	co	*na*	*pr*	*st*	*sg*	*sm*	*c2*	*s2*	*gr*	*id*	*eg*
well-founded	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
bipartite	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
finite	✓	✓	✓	-	✓	✓	✓	✓	✓	✓	✓
limited controversial	✓	AC	AC	AC	AC	AC	AC	AC	✓	AC	AC
symmetric loop-free	✓	AC	AC	AC	AC	AC	AC	AC	✓	AC	AC
finitary	✓	AC	AC	-	AC	AC	?	?	✓	AC	AC
symmetric	✓	AC	AC	-	-	-	AC	-	✓	AC	AC
planar	✓	AC	AC	-	?	-	?	?	✓	AC	AC
finitely superseded	✓	AC	AC	-	-	-	-	-	✓	AC	AC
finitarily superseded	✓	AC	AC	-	-	-	-	-	✓	AC	AC
arbitrary	✓	AC	AC	-	-	-	-	-	✓	AC	AC

Table 8.1: Perfection results of this chapter, cf. Remark 8.29.

In this chapter we have given novel existence conditions for various semantics, discussed the concept of perfection in the context of argumentation and elaborated on its meaning as well as drawn a comprehensive picture of perfection conditions. In particular stage semantics appears to provide an inductively powerful resistance against collapse as highlighted in Lemma 8.19. Conjecture 8.23 extends this resistance by stating that for any rule guaranteeing stage-perfection and any stage-collapsing AF there should be an infinite amount of violations of this rule.

Remark 8.29 (Perfection Results, cf. Table 8.1). We present most of the perfection results gathered in this chapter collected in Table 8.1. Here a checkmark in line x and column σ means that x AFs are σ-perfect in ZF. The term AC means that we get σ-perfection in ZFC but not in all models of ZF. A dash means that even with AC we do not get perfection. A question mark means that we have counterexamples in ZF (see Chapter 7) but do not know yet about ZFC.

Remark 8.30 (The downs and ups of perfection). In comparison with Remark 7.39 on the ups and downs of collapse naturally perfection is the dual question. That is, since there is no collapse for induced sub-AFs of σ-perfect AFs, we can not make use of the additional expressiveness granted by the power of collapse. On the other hand, we gain comparabilities. For instance in ZFC every *pr* extension S is contained in some *na* extension T, $S \subseteq T$. Similarly for finite AF F (also see Conjecture 8.24) each *sm* extension S is range-covered by some *sg* extension T, $S_F^* \subseteq T_F^*$. Such relations, where properties from the finite case do carry over to the infinite case, are among the interesting topics for future research regarding perfection.

As obvious future research questions there are several other semantics out in the wild to be considered. Further results from graph theory on kernel-perfection can deliver additional immediate results for stable-perfection. It might also prove rather useful to consider classes of finitely generated infinite argumentation frameworks (see for instance [BCDG13]). Finally, also other syntactic AF-properties or combinations of such might be of interest in terms of σ-perfection.

Part III

Conflict and Expressiveness

Chapter 9

Necessity

> Work diligently. Diligently.
> Work patiently and persistently. Patiently and persistently.
> And you're bound to be successful.
> Bound to be successful.
>
> Satya Narayan Goenka

S.N. Goenka was a Burmese-Indian teacher of Vipassanā meditation. His teachings are rather popular nowadays, not least for the widespread and affordable accessibility of his 10-day residential courses. Scholars of Vipassanā meditation à la Goenka share the common skill of upon reading above quote immediately also hearing Goenka's sonorous voice from the inside. The technique of Vipassanā can be defined as "Looking into something with clarity and precision, seeing each component as distinct and separate, and piercing all the way through so as to perceive the most fundamental reality of that thing" [GG11, p. 21]. And that is precisely what we aim at in this chapter. Looking into conflicts of argumentation with clarity and precision, distilling necessities and attempting to perceive the most fundamental reality of their nature.

In this chapter we discuss syntactic and semantic conflicts (Definitions 3.7 and 4.43) in general and semantic conflicts that are necessarily syntactic (Definition 5.35) in particular. This chapter is a concise investigation on the relations between syntax and semantics in that we flesh out what it takes for a conflict to become necessary and thus also what it takes for a syntactic conflict not to be necessary. This chapter is based on research the author presented details of at several occasions such as [Spa16a, Spa16c]. While the full matter has not been published before, it can be described as intuitive (or insider) knowledge allowing important (counter) examples facilitated for instance in [BDL$^+$16]. The overall assumption is that we are provided with a fixed extension set. That is, either we are given an AF F and semantics σ with $\mathbb{S} = \sigma(F)$, or some semantics σ and a σ-realizable extension set $\mathbb{S}$.

In Section 9.1 we investigate the required definitions in first examples to develop some intuitive knowledge. In Section 9.2 we take a look at syntactic/semantic modifications and how we might facilitate such formalisms to approach the question of necessary conflicts. In Section 9.3 we use signatures to show general results of necessity for all semantics of interest. In Section 9.4 we apply our results of general necessity to an exemplary extension set and

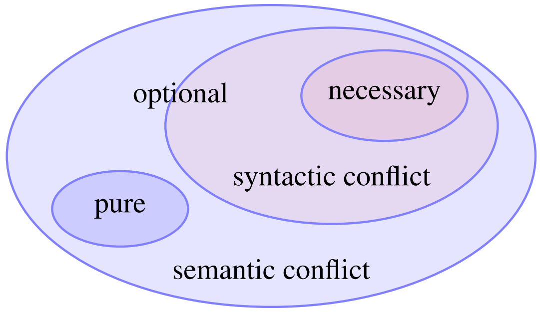


Figure 9.1: A Venn-diagram illustrating different levels of conflict.

further take a look at compact (see Definition 5.45) AFs and restricted necessities therein. In Section 9.6 we conclude and relate to the literature.

9.1 Definitions and Observations

Before going deeper we recall basic observations and definitions as needed. For any extension set $\mathbb{S}$ argument (sets) x and y are said (see Definition 4.43) to be in semantic conflict $[x,y]^{cnf}_{\mathbb{S}}$ if they are not independent, they are independent $\{x,y\}^{ind}_{\mathbb{S}}$ if each of their arguments $x' \in x^0$, $y' \in y^0$ appears together in some extension $S \in \mathbb{S}$: $\{x',y'\} \subseteq S$.[1] If for each σ-realizing AF F of $\mathbb{S}$ (with $\sigma(F) = \mathbb{S}$), we have a syntactic conflict $[x,y]^{cnf}_{F}$ (see Definition 3.7) then the conflict $[x,y]^{cnf}_{\mathbb{S}}$ is called necessary. It is called pure if it is always realized as syntactic independence $\{x,y\}^{ind}_{F}$ (see Definition 5.35). Semantic conflicts that are neither necessary nor pure are called optional. Further, a conflict $[x,y]^{cnf}_{\mathbb{S}}$ can also be a necessary attack $(x,y)^{att}_{F}$ for all σ-realizing AFs F, and each necessary attack obviously is also a necessary conflict. Conflicts hence are defined on pairs of sets of arguments or pairs of arguments or combinations thereof and can be classified as illustrated in Figure 9.1.

While this thesis in general discusses the wide range of semantics $\sigma \in \{cf, ad, na, pr, st, sm, sg, co, gr, id, eg, c2, s2\}$ (see Section 4.2), here we focus mainly on multi-status I-maximal semantics (that is $na, pr, st, sm, sg, c2, s2$). From a practical perspective single-status semantics do not provide semantic conflicts and semantic conflicts of cf, ad, co are to some extent reflected by semantic conflicts of na and pr. With the following example we give a first introspection into the detailed matter of this chapter.

Example 9.1. Consider the AFs E,F,G,H,I with $A_E = A_F = A_G = A_H = \{x,y\}$, $A_I = \{x,y,z\}$ and $R_E = \{(y,y)\}$, $R_F = \{(x,y)\}$, $R_G = \{(y,x)\}$, $H = F \cup G$, $R_I = R_G \cup \{(z,z),(x,z),(z,y)\}$ as illustrated in Figure 9.3. We have $cf(E) = \{\emptyset,\{x\}\}$, $na(E) = \{\{x\}\}$, $cf(F) = cf(G) = cf(H) = cf(I) = \{\emptyset,\{x\},\{y\}\}$, $na(F) = na(G) = na(H) = na(I) = \{\{x\},\{y\}\}$.

AF E does not provide any necessary conflicts. Observe that for $\mathbb{S} = na(F) = na(G) = na(H)$ we have $[x,y]^{cnf}_{\mathbb{S}}$ as necessarily syntactic conflict, but we do not bother whether $(x,y)^{att}_{F}$

[1] For a more general applicability of *conflict* we define this relation to also cover arguments not contained in $\bigcup\mathbb{S}$. Thus rejected arguments are self-conflicting. This behaviour is debatable but not of relevance for this thesis. Indeed, the core results work for acceptable (sets of) arguments already, also see Observation 9.18.

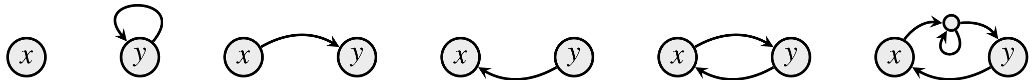


Figure 9.2: Illustration of necessary conflicts, cf. Example 9.1.

	E	F	G	H	I
na	$\{\{x\}\}$	$\{\{x\},\{y\}\}$	$\{\{x\},\{y\}\}$	$\{\{x\},\{y\}\}$	$\{\{x\},\{y\}\}$
st	$\emptyset$	$\{\{x\}\}$	$\{\{y\}\}$	$\{\{x\},\{y\}\}$	$\emptyset$
sg,$c2$,$s2$	$\{\{x\}\}$	$\{\{x\}\}$	$\{\{y\}\}$	$\{\{x\},\{y\}\}$	$\{\{x\},\{y\}\}$
pr,sm	$\{\{x\}\}$	$\{\{x\}\}$	$\{\{y\}\}$	$\{\{x\},\{y\}\}$	$\{\emptyset\}$

Table 9.1: Semantic evaluations of the AFs from Example 9.1 and Figure 9.2.

or $(y,x)^{att}_G$ or as in H even both. Finally, observe that $S = \{x,y\}$ is semantically and syntactically in conflict with itself: $[S]^{cnf}_{\mathbb{S}}$, $[S]^{cnf}_F$, $[S]^{cnf}_G$ and $[S]^{cnf}_H$. We can conclude that the semantic attack $(S)^{att}_{\mathbb{S}} = (S,S)^{att}_{\mathbb{S}}$ is even necessary. We collect a semantic evaluation of these AFs in Table 9.1.

When looking at the definitions of (syntactic or semantic) conflicts it becomes evident that colloquially speaking supersets of conflicting sets are conflicting again. This observation turned upside down any syntactically conflicting sets contain singletons that already are in conflict. This however is immediate since conflict and independence are defined via pairs of arguments. For references we use the following lemma to refer to this insight.

Lemma 9.2 (Minimality of Conflicts)**.** *Consider AF F (or extension set $\mathbb{S}$) and argument sets x,y. If $[x,y]^{cnf}_F$ ($[x,y]^{cnf}_{\mathbb{S}}$) then there are arguments $v \in x, w \in y$ with $[v,w]^{cnf}_F$ ($[v,w]^{cnf}_{\mathbb{S}}$). Further for each such v,w and arbitrary $v \in x'$, $w \in y'$ we have $[x',y']^{cnf}_F$ ($[x',y']^{cnf}_{\mathbb{S}}$).*

Observe that minimality of conflicts is tight in the sense of conflict over attack. For instance the AF G from Figure 9.2c has $[x,y]^{cnf}_F$ but not $(x,y)^{att}_F$. However by definition of conflict we still get $\left(x^0 \cup y^0\right)^{att}_F$ for any AF F and syntactic conflict $[x,y]^{cnf}_F$. To comply with the literature [DDLW15, BDL+16] and for a more distinct vocabulary we introduce the following notion.

Definition 9.3 (Implicit and Explicit Conflicts)**.** Given some AF F and semantics σ, a semantic conflict $[x,y]^{cnf}_{\sigma(F)}$ is called *explicit* if it is syntactic, i.e. $[x,y]^{cnf}_F$, otherwise it is called *implicit*.

Remark 9.4. Comparing the notions of implicit and explicit conflicts to our notions of syntactic and semantic conflicts (see Definition 3.7 and 4.43) there are some similarities. For given AF and semantics by conflict-freeness of introduced semantics, explicit and syntactic conflicts always coincide. Syntactic conflicts are always semantic as well, while implicit and explicit conflicts are always distinct. We thus have semantic conflicts that are explicit and not implicit. Syntactic conflicts are defined for AFs. Semantic conflicts are defined for extension sets. Explicit and implicit conflicts are defined for AFs in conjunction with some semantics.

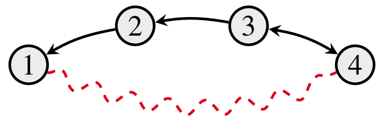


Figure 9.3: AF illustrating implicit and explicit conflicts, cf. Example 9.5.

With the following example (which is isomorphic to Examples 3.4 and 4.44) we highlight the possibility of implicit conflicts for *pr* semantics.

Example 9.5 (Implicit and Explicit conflicts)**.** Consider the AF F as depicted in Figure 9.3 with $A_F = \{1,2,3,4\}$ and $R_F = \{(2,1),(3,2),(3,4),(4,3)\}$ with minimal syntactic (and explicit) conflicts $[1,2]_F^{cnf}$, $[2,3]_F^{cnf}$ and $[3,4]_F^{cnf}$. We have $pr(F) = \{\{1,3\},\{2,4\}\}$ and thus the implicit (and semantic) conflict $[1,4]_{pr(F)}^{cnf}$.

Remark 9.6 (Implicit Conflicts for Naive Semantics in ZF). Recall that in ZF some AFs do not provide *na* extensions. Thus with Definition 7.11 by Theorem 7.14 we can transform the AF F from Example 9.5 into an AF G that has $\{1,3\}$ and $\{2,4\}$ as only naive extensions while $\{1,4\}_G^{ind}$. Apart from this remark however for the remainder of this chapter we abstain from considering models of ZF without AC.

Observe that all semantics under consideration incorporate conflict-freeness which means that syntactic conflicts are always semantic as well. As highlighted in Example 9.5 some semantic conflicts are not syntactic and thus implicit in nature. The following definition gives a desirable connection between syntactic and semantic conflicts.

Definition 9.7 (Awareness)**.** A semantics σ is called

1. *attack aware*, if σ-extension sets reflect syntactic attacks, that is

$$\text{for each AF } F \text{ and } x,y \subseteq \bigcup\sigma(F) \qquad \text{we have} \qquad [x,y]_F^{cnf} \implies [x,y]_{\sigma(F)}^{cnf};$$

2. *conflict aware*, if σ-realizations reflect semantic conflicts, that is

$$\text{for each AF } F \text{ and } S,T \in \sigma(F) \qquad \text{we have} \qquad [S,T]_{\sigma(F)}^{cnf} \implies [S,T]_F^{cnf}.$$

Lemma 9.8 (Awareness)**.** *Any semantics* $\sigma \in \{cf, ad, co, gr, id, eg, na, pr, sg, sm,_, c2, s2\}$ *satisfies attack as well as conflict awareness.*

Proof. For attack awareness simply observe that each of the considered semantics implements conflict-freeness. For conflict awareness first observe that for $\tau \in \{cf, ad\}$, any AF F and sets $S,T \in \tau(F)$, either $[S,T]_F^{cnf}$ or $S \cup T \in \tau(F)$ holds. The claim for *cf* and *ad* immediately follows. For *co* add that each *ad* set is contained in some *co* set. For all other semantics add I-maximality, i.e. the observation that extensions S,T are either equal or in conflict. □

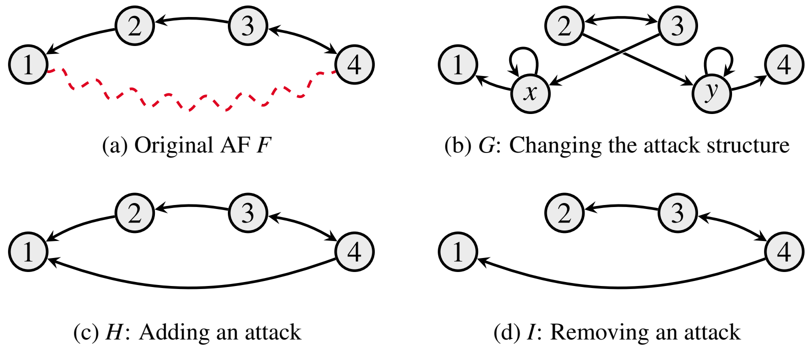


Figure 9.4: AFs illustrating semantic modifications, cf. Example 9.9.

Before approaching the core aspects of this chapter we further develop intuition of semantic modifications (see Definition 5.26) regarding semantic and syntactic conflicts. Observe that awareness (Definition 9.7 and Lemma 9.8) treats the inputs of syntactic and semantic conflicts quite differently. It provides a statement for minimal syntactic conflicts, that is conflicts of arguments and singletons, that naturally extends to any conflicting sets. For semantic conflicts as input it however requires extension sets. The apparent question thus is whether semantic conflicts between arguments give similar insights into syntactic structures. This question is a question of necessary conflicts. We present the following counterexample to this question.

Example 9.9 (Optional and necessary conflicts)**.** Consider the AF F from Example 9.5, depicted in Figure 9.4a with $pr(F) = st(F) = \{\{1,3\},\{2,4\}\}$. We have further semantic modifications (that is modifications with the same pr-semantic evaluation) depicted in Figure 9.4.

In the AF G from Figure 9.4b we have a very reduced attack structure. Here only $[2,3]^{cnf}$ is explicit while the other semantic conflicts are implicit. Observe that $pr(G) = sm(G) = pr(F)$, $sg(G) = s2(G) = c2(G) = \{\{1,2,4\},\{1,3,4\}\}$ and $st(G) = \emptyset$. For symmetry reasons none of the pr-semantic conflicts between arguments from F is necessary. In fact, even $[\{1,3\},4]^{cnf}_{pr(F)}$ is not necessary and thus the minimal necessary conflict for this extension set and preferred semantics is $[\{1,3\},\{2,4\}]^{cnf}$.

In opposition to G, in the AF H from Figure 9.4c all semantic conflicts are made explicit and thus no implicit conflicts remain. We have $pr(H) = st(H) = pr(F)$. This AF thus highlights that none of the semantic conflicts from F is pure, and each is optional.

The AF I from Figure 9.4d represents a further modification of H in that the attack $(2,1)^{att}$ is removed. We have $pr(I) = st(I) = pr(F)$. For symmetry reasons none of the st-semantic conflicts between arguments from F is necessary and all are optional.

Observe that for st semantics AF G does not provide extensions because no conflict-free set attacks arguments x and y simultaneously. For our investigation in necessary conflicts this observation is related to the fact that in F for $S = \{1,3\}$ to be a stable extension we need $2,4 \in S^+$. That is, both $(\{1,3\},2)^{att}$ and $(\{1,3\},4)^{att}$ are st-necessary attacks.

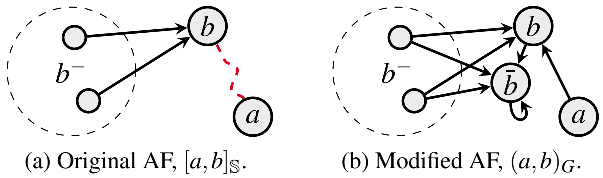


Figure 9.5: Illustration of the Modification $force^{st}$ from Definition 9.10. The dashed cycles refer to a potential collection of arguments. We pick two surrogate arguments from this cloud for illustration purposes.

To summarize insights so far observe that syntactic conflicts can accurately be described as conflicts between arguments. Each syntactic conflict essentially deals with some arguments x,y such that $[x,y]^{cnf}$ and further such syntactic conflict is always semantic. Semantic conflicts also contain conflicting arguments. However as highlighted in Example 9.9 semantic conflicts between arguments might not be reflected by the syntactic structure. It remains to talk about minimal necessity of semantic conflicts and thus influence of argument (sets) $x,y \subseteq \bigcup \mathbb{S}$ with semantic conflict $[x,y]^{cnf}_{\mathbb{S}}$ on σ-realizations of $\mathbb{S}$.

9.2 Modification approach

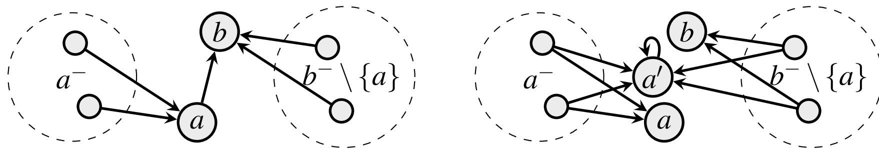
In this section we attempt to provide necessity results by investigating semantic modifications (as facilitated in [DW11, Spa13, DS17]). To be more precise for stable semantics we show that 1-local syntactic modifications are sufficient to transform any given AF F with non-necessary conflict $[x,y]^{cnf}_F$ into some semantically equivalent AF G (with $st(F) = st(G)$) with syntactic independence $\{x,y\}^{ind}_G$. Moreover for the other semantics, and particularly for preferred semantics, we highlight that possible modifications are more complicated.

Since removing attacks might require adding substitute attacks first, the blueprint for getting rid of optional conflicts is a mixture of enforcing and purging attacks. We discuss the according semantic modifications for *st* semantics sequentially.

Definition 9.10 (Stable Attack-Enforcement)**.** Consider some AF F and arguments $a,b \in A_F$. As illustrated in Figure 9.5 we define the (a,b)-*enforcing st-modification* of F as $force^{st}_F(a,b) = (A,R)$ where $A = A_F \cup \{\bar{b}\}$ and $R = R_F \cup \{(\bar{b},\bar{b}),(b,\bar{b}),(a,b)\} \cup \{\{(x,\bar{b}) \mid x \in b^-_F\}$.

Lemma 9.11 (Stable Attack Forcing)**.** *Assume AF F and arguments $a,b \in A_F$ with st-semantic conflict $[a,b]^{cnf}_{st(F)}$ such that $(a,b) \notin R_F$. Then for the (a,b)-enforcing st-modification $G = force^{st}_F(a,b)$ from Definition 9.10 we have $(a,b)^{att}_G$ while $st(F) = st(G)$ and $R_F \subset R_G$.*

Proof. $st(F) \subseteq st(G)$: assume $S \in st(F)$ as given. If $b \in S$, then by assumption $a \notin S$ and thus $S \in cf(G)$. Further $\bar{b} \in b^+_G$ and thus $\bar{b} \in S^+_G$. Since $\bar{b}$ is the only additional argument of G we have $S \in st(G)$. If $b \notin S$, then for some $x \in S$ we have $b \in S^+_F$ and thus by definition $\bar{b} \in S^+_G$. Also then we have $S \in st(G)$.



(a) Original AF, $(a,b) \in R_F$. (b) Modified AF, $(a,b) \notin R_G$.

Figure 9.6: Illustration of the Modification $purge^{st}$ from Definition 9.12.

$st(G) \subseteq st(F)$: Since the only altered attack between acceptable arguments is (a,b) it suffices to assume $a \in S \in st(G)$. Since $\bar{b} \notin a_G^+$ by definition we need some $x \in S$ with $\bar{b} \in x_G^+$ for S to have full range. By definition of $force_F^{st}(a,b)$ then $b \in x_G^+$ and thus $b \in x_F^+$ and subsequently $S \in st(F)$. □

Definition 9.12 (Stable Attack Purging)**.** Assume some AF F and attack $(a,b)_F$. As illustrated in Figure 9.6 we define the (a,b)-*purging st-modification* of F as $purge_F^{st}(a,b) = (A,R)$ where $A = A_F \cup \{a'\}$ and $R = (R_F \setminus \{(a,b)\}) \cup \{(x,a') \mid x \in a^- \cup b^-, x \neq a\}$.

Lemma 9.13 (Stable Attack Purging)**.** *Consider some AF F and arguments $a,b \in A_F$ with $(a,b)_F^{att}$ such that for each $a \in S \in st(F)$ we have $b_F^- \cap S \neq \{a\}$ (i.e. there are alternative attackers of b from S). Then for the (a,b)-purging st-modification of $G = purge_F^{st}(a,b)$ from Definition 9.12 we have $(a,b) \notin R_G$ while $st(F) = st(G)$ and $R_G \cap (A_F \times A_F) \subset R_F$.*

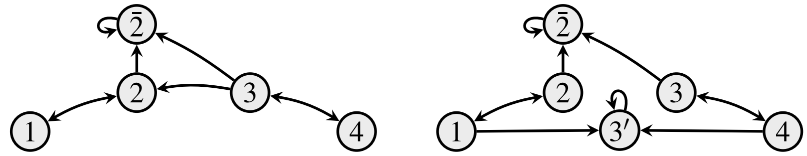
Proof. By the definitions it remains to show that $st(F) = st(G)$.

$st(F) \subseteq st(G)$: assume $S \in st(F)$. In case $a \in S$ by assumption there is $a \neq x \in S$ with $(x,b)_F^{att}$ and thus $(x,b)_G^{att}$ and by definition further $(x,a')_G^{att}$. That is, S still has full range and thus $S \in st(G)$. In case $a \notin S$ by stability we have $a \in S_F^+$ and thus by definition $a' \in S_G^+$, i.e. again $S \in st(G)$.

$st(G) \subseteq st(F)$: assume $S \in st(G)$. Since we did not add any attacks between acceptable arguments the only possibility of S not being a stable extension of F is to have $\{a,b\} \subseteq S$. Observe that by definition of $G = purge_F^{st}(a,b)$ the argument a' is attacked in G only by itself and attackers of a or b from F. Thus for $a' \in S_G^*$ we need $S_G^+ \cap \{a,b\} \neq \emptyset$. But then by conflict-freeness of st semantics $\{a,b\} \not\subseteq S$ and thus $S \in st(F)$. □

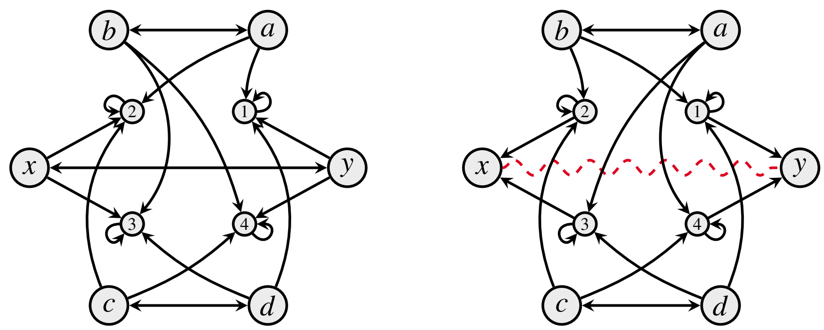
Example 9.14. Consider the AF F from Example 9.5, Figure 9.3. Recall that by Example 9.9 none of the attacks is necessary for st semantics. Observe that for attack $(3,2)_F^{att}$ however Lemma 9.13 does not apply, as for extension $\{1,3\} \in st(F)$ we have $2^- \cap \{1,3\} = \{3\}$, i.e. the only attacker of 2 among S (and indeed in F) is argument 3. However if we first apply $force_F^{st}(1,2) = G$ as illustrated in Figure 9.7a we may then apply $purge_G^{st}(3,2) = H$ to retrieve an AF H with $st(F) = st(H)$ and $(3,2) \notin R_H$.

Theorem 9.15 (Stable-necessary conflicts)**.** *Assume some st-realizable extension set $\mathbb{S}$. A conflict $[x,y]_{\mathbb{S}}^{cnf}$ is necessary syntactic if and only for each $v \in y^0$ there is $S \in \mathbb{S}$ such that $\left[x^0 \cap S, v\right]_{\mathbb{S}}^{cnf}$ and $\left\{S \setminus x^0, v\right\}_{\mathbb{S}}^{ind}$ hold.*



(a) Forcing Attack (a,b) (b) Purging Attack (c,b)

Figure 9.7: Stable attack forcing and purging modifications illustrated.



(a) Stable semantics (b) Preferred Semantics

Figure 9.8: AFs realizing the extension set $\{\{x,a,c\},\{x,b,d\},\{y,a,d\},\{y,b,c\}\}$, cf. Example 9.16.

Proof. Assume for a contradiction that for each $S \in \mathbb{S}$ and each $v \in y^0$ with $\left[x^0 \cap S, v\right]^{cnf}_{\mathbb{S}}$ there is $u \in S \setminus x^0$ such that $[u,v]^{cnf}_{\mathbb{S}}$. We use these outside conflicts and transfinite recursion via repeated application of $force^{st}$ to construct an AF G such that any $v \in y^0$ that is in conflict with x is attacked in any extension that has non-empty intersection with x but by some argument not from x. Then the conditions for Lemma 9.13 are met and we can construct AF H with $st(F) = st(H)$ and $(a,b) \notin R_H$. $\square$

The given construction of Theorem 9.15 is local in regards of the given alternative conflicts. We now proceed to give an example where we first discuss necessities of *st*-conflicts for some particular extension set and then show that for the same extension we have different necessities for *pr* semantics for which such local modifications are necessarily more involved.

Example 9.16. Consider the *st*-realizable extension set $\mathbb{S} = \{\{x,a,c\},\{x,b,d\},\{y,a,d\},\{y,b,c\}\}$. Illustrated in Figure 9.8a we have a *st*-realization F of $\mathbb{S}$. Here the extension sets are enforced by a dimensional approach. We have two times two self-attacking arguments (1 to 4) for which attacks from acceptable arguments are divided into partitions among conflicting arguments. Thus for instance argument set $\{b,c\}$ attacks only arguments $2,3,4$ and for $\{b,c\} \subseteq S \in st(F)$ requires $y \in S$ as well.

Observe that the semantic conflicts of $\mathbb{S}$ are supersets of the argument conflicts $[a,b]^{cnf}_{\mathbb{S}}$, $[x,y]^{cnf}_{\mathbb{S}}$ and $[c,d]^{cnf}_{\mathbb{S}}$. Further for any $\alpha \in \{a,b\}$, $\beta \in \{x,y\}$, $\gamma \in \{c,d\}$ we have pairwise independence $\{\alpha,\beta\}^{ind}_{\mathbb{S}}$, $\{\alpha,\gamma\}^{ind}_{\mathbb{S}}$, $\{\beta,\gamma\}^{ind}_{\mathbb{S}}$. But then, for instance regarding the conflict

$[a,b]^{cnf}_{\mathbb{S}}$, the extension $\{x,a,c\} \in \mathbb{S}$ is required to have $(a,b)^{att}$ for any *st*-realization F. Hence the conflicts $[a,b]^{cnf}$, $[x,y]^{cnf}$ and $[c,d]^{cnf}$ are necessary for *st* semantics.

For *pr* semantics consider the AF G from Figure 9.8b. Now we have admissible enforcing via dimensional approach. That is for instance y can only be defended by some argument set S if both 1 and 4 are attacked. Assuming $c \in S$ we have $4 \in S^+$ and thus require $b \in S$ to also have $1 \in S^+$. It remains to observe that $pr(G) = \mathbb{S}$.

Remarkably now the *st*-necessary conflict $[x,y]^{cnf}_{\mathbb{S}}$ is not *pr*-necessary as illustrated by $\{x,y\}^{ind}_{G}$. Since in $\mathbb{S}$ we have four different extension sets however a pair of syntactically conflicting acceptable arguments does not suffice for *pr*-realizing $\mathbb{S}$. In that case we would have exactly two extension sets. For symmetry reasons none of the semantic conflicts among arguments a,b,c,d,x,y is necessary. However, for *pr*-realizations of $\mathbb{S}$ (as illustrated by G) we need at least two different pairs of syntactically conflicting arguments.

The solution now is that minimal *pr*-necessary conflicts of $\mathbb{S}$ are between pairs of two-sets, for instance $[\{b,x\},\{a,y\}]^{cnf}$ is necessary. Crucially the locality of modifications for *st* semantics as established in Theorem 9.15 is not possible for *pr* semantics as to get rid of the conflict $[a,b]^{cnf}_{G}$ we would need to introduce a syntactic conflict $[x,y]^{cnf}$.

Observe that although the *st*-modifications are local and thus syntactic, the construction from Theorem 9.15 might formally not be. There we require knowledge of alternative conflicts. Given alternative conflicts, we assume similar approaches to work for the other semantics as well. In conclusion, regarding necessity, we believe that the modification approach (specifically for other semantics than *st*) is more obfuscating than enlightening.

9.3 Signature Approach

In this section we give a full characterization of necessary conflicts for the semantics of interest. We do so by distinguishing between maximal conflict-freeness (*na*, *st*, *sg*, *s2*, *c2*) in Subsection 9.3.1, and admissibility (*pr*, *sm*) in Subsection 9.3.2, After presenting rather obvious necessities, we give signatures (as facilitated in [DDLW15, BDL$^+$16, DSLW16]) and realizations making use only of these necessities to show tightness.

9.3.1 Naive-based Semantics

In this subsection we discuss necessary conflicts for semantics $\sigma \in \{na, st, sg, c2, s2\}$, that is the case where extensions are maximal conflict-free. In the following results we consider some conflict, such as $[x,y]^{cnf}$ as given. Observe that no set is in conflict with the empty set and thus we need not consider cases such as $x = \emptyset$, $y = \emptyset$ or $x = y = \emptyset$. We start with a very general result for naive semantics.

Theorem 9.17 (Necessary *cf*/*na*-conflicts)**.** *In ZFC, given AF F, a conflict $[x,y]^{cnf}$ is necessary syntactic for cf/na if and only if there are arguments $u \in x^0$, $v \in y^0$ such that $u^0, v^0 \in cf(F)$ and $[u,v]^{cnf}_{cf(F)}$.*

Proof. First observe that self-attacking arguments are not relevant for cf/na semantic evaluation. Assuming that all arguments $x' \in x^0$ (or $y' \in y^0$) are self-attacking we can easily construct an AF G (with $A_G = A_F$ and $R_G = R_F \setminus \{(z,x'),(x',z) \mid x' \in x^0\}$) such that $[x,y]^{cnf}$ is implicit and not syntactic. Thus, w.l.o.g. assume neither x nor y contain any self-attacking arguments.

By definition of conflict then there are arguments $u \in x$, $v \in y$ with $[u,v]^{cnf}_{cf(F)}$. Assume for a contradiction that $\{u,v\}^{ind}_F$. Using AC (also see Lemma 6.9) we can construct $S \in na(F)$ with $u,v \in S$ and thus $\{u,v\}^{ind}_{na(F)}$. We conclude that necessarily $[u,v]^{cnf}_F$ holds. $\square$

Observation 9.18 (Rejected arguments). Consider as given some AF F, semantics σ, acceptable arguments $X = \bigcup \sigma(F)$ and rejected arguments $Y = A_F \setminus X$. We claim that conflicts involving arguments from Y are not particularly interesting. To this end define $A = A_F \cup \{\bar{y} : y \in Y\}$ and $R_0 = (R_F \cap (X \times X)) \cup \{(x,\bar{y}) \mid (x,y) \in R_F \cap (X \times Y)\} \cup \{(\bar{y},x) \mid (y,x) \in R_F \cap (Y \times X)\} \cup \{(\bar{y},\bar{z}) \mid (y,z) \in R_F \cap (Y \times Y)\}$.

In case $\sigma \notin \{na, cf\}$ and $Y \neq \emptyset$ we have that the AF G with $A_G = A$ and $R_G = R_0 \cup (X \times Y)$ represents a loop-removing modification in that no self-attacks among Y remain and $\sigma(G) = \sigma(F)$ still holds. Similarly, for $\sigma \neq st$ the AF H with $A_H = A$ and $R_H = R_0 \cup \{(y,y) \mid y \in Y\}$ is a conflict-purging modification $\sigma(H) = \sigma(F)$, with the only syntactic conflicts involving arguments from Y being self-attacks. Finally, the AF I with $A_I = A$ and $R_I = R_G \cup R_H \cup (Y \times Y)$ is a conflict-enforcing modification $\sigma(I) = \sigma(F)$ where all conflicts regarding argument (sets) from Y are made explicit.

Consequently in this chapter, unless stated otherwise, we consider only conflicts between acceptable arguments.

In ZFC we have a very straightforward necessity of na-semantic conflicts. That is for na essentially conflicts between arguments and supersets thereof are necessary. With the observation that in Example 9.9 AFs F and I are super-coherent with $st(F) = st(I) = pr(F) = sm(F) = sg(F) = c2(F) = s2(F)$ the same can not be said for any of the other semantics under consideration. A less general observation for naive sets however survives applicability also for other maximal conflict-free semantics.

Lemma 9.19 (Conflicts for Naive Extensions)**.** *Given AF F and naive extension $S \in na(F)$, then for each $x \in A_F \setminus S$ at least one of $[S,x]^{cnf}_F$ or $[x]^{cnf}_F$ holds.*

Proof. Assuming for a contradiction $\{S,x\}^{ind}_F$ and $\{x\}^{ind}_F$ we conclude $\left\{S \cup x^0, x\right\}^{ind}_F$ and by assumption of $S \in cf(F)$ also $\left\{S \cup x^0\right\}^{ind}_F$ and thus, with $S \subset S \cup x^0$, we have $S \notin na(F)$. $\square$

Corollary 9.20. *Given AF F and cf-implicit conflict $[x,y]^{cnf}$ (defined on arguments $x,y \in A_F$), then at least one of $(x)^{att}_F$ or $(y)^{att}_F$ holds.*

For stable semantics we can even do better. The following essentially is a detail of Definition 4.12 regarding stable semantics being range-complete.

Lemma 9.21 (Attacks from Stable Extensions)**.** *Given AF F and stable extension $S \in st(F)$, then for each $x \in A_F \setminus S$ we have $(S,x)^{att}_F$.*

This is not yet a characterization of minimal *st*-necessary conflicts. To see this consider the *st*-realizable extension set $\{\{1,2\},\{1,3\}\}$ (for instance for AF F with $A_F = \{1,2,3\}$ and $R_F = \{(2,3),(3,2)\}$) where the only minimal *st*-necessary conflict apparently is $[2,3]^{cnf}$. Now consider the following observation.

Proposition 9.22 (Necessary Conflicts and Conflict-free Sets)**.** *Consider as given some AF F, semantics $\sigma \in \{na, st, sg, c2, s2\}$, set of acceptable arguments $x \subseteq \bigcup \sigma(F)$, and some conflicting argument $y \in A_F \setminus x$, $[x,y]^{cnf}_{\sigma(F)}$.*

If there is an extension $S \in \sigma(F)$ such that $x \subseteq S$ and $\{S \setminus x, y\}^{ind}_{\sigma(F)}$ hold, then already $[x,y]^{cnf}_F$, i.e. $[x,y]^{cnf}$ is a necessary conflict for σ. In case of $\sigma = st$ we further have $(x,y)^{att}_F$.

Proof. This is a straightforward culmination of Lemma 9.19. Since the conflict $[S,y]^{cnf}$ (for *st* the attack $(S,y)^{att}$, Lemma 9.21) is necessary already, there has to be some argument $u \in S$ with $[u,y]^{cnf}_F$ (for *st* with $(u,y)^{att}_F$). Since arguments not contained in x are not conflicting with y (by assumption $\{S \setminus x, y\}^{ind}_F$ as contraposition to Lemma 9.8), this can only mean $[x,y]^{cnf}_F$ (for *st* even $(x,y)^{att}_F$). □

Clearly supersets of necessary conflicts are still necessary conflicts. The apparent question thus is whether each necessary conflict contains a pair x, y as in Proposition 9.22 for semantics $\sigma \in \{st, sg, c2, s2\}$. In the following we elaborate on this. First we present a result on realizability generalizing [DDLW15, Theorem 1] for *st*/*sg* semantics. Observe that in ZF (with the possible collapse of *na* semantics) application for *na* semantics would be possible as well, however might require a recursive construction since not every *cf* set needs to culminate in some *na* set there.

Theorem 9.23 (Realizability with Maximal Conflict-freeness in ZFC)**.** *Consider incomparable extension set*[2] *$\mathbb{S}$ such that for each $S \in \mathbb{S}$, $x \in \bigcup \mathbb{S} \setminus S$ we have $[S,x]^{cnf}_{\mathbb{S}}$. Then and only then there is some AF F with $\sigma(F) = \mathbb{S}$ for $\sigma \in \{st, sg, c2, s2\}$.*

Proof. For the case of $\mathbb{S} = \emptyset$ we refer to Chapter 7, for the case $\mathbb{S} = \{\emptyset\}$ consider the empty AF $(\emptyset, \emptyset)$. We further on assume $\bigcup \mathbb{S} \neq \emptyset$. First observe that by Chapter 7 we may use σ-bombs B, i.e. AFs B collapsing for semantics σ, $\sigma(B) = \emptyset$. The "only then" part is handled by Lemma 9.19, for realizable extension sets any superset of some extension is syntactically conflicting.

Now consider as given some AF G where for $S \in \mathbb{S}$, $A_G \setminus S$ we have $[S,x]^{cnf}_G$. For $\sigma = st$ we additionally require $(S,x)^{att}_G$, for $\sigma = sg$ we want $\mathbb{S} \subseteq sg(G)$, for $\sigma \in \{c2, s2\}$ we need connected arguments to be already strongly connected. For instance the canonical AF, where each semantic conflict between arguments is realized symmetrically, from [DDLW15] serves this purpose for all considered semantics. Observe that by definition we now have $\mathbb{S} \subseteq \sigma(G)$. The necessity of ZFC (as opposed to ZF) stems from the following need of approachable naive extensions, cf. Theorem 6.15.

[2]For incomparable extension set $\mathbb{S}$ and any $S \neq T \in \mathbb{S}$ we have $S \not\subseteq T$.

Given $\mathbb{T} = na(G) \setminus \mathbb{S}$ and a collection $(A_T, R_T)_{T \in \mathbb{T}}$ of $|\mathbb{T}|$ many disjoint σ-bombs, we use the following construction of AF F:

$$A_F = A_G \cup \bigcup_{T \in \mathbb{T}} A_T \qquad R_F = R_G \cup \bigcup_{T \in \mathbb{T}} R_T \cup \{(x,a) \mid T \in \mathbb{T}, a \in A_T, x \in A_G \setminus T\}.$$

This modification is inspired by [Spa13, DS17, DDLW15] and the bomb constructions from Chapter 7. The intuition is that we identify each unwanted naive extension with some bomb and allow this bomb to be defused only by arguments not contained in that extension. Thus clearly for any $T \in \mathbb{T}$ we have $T \notin \sigma(F)$. Since any $S \in na(G) \setminus \mathbb{T}$ is incomparable to each $T \in \mathbb{T}$ further $(S,a)_F^{att}$ for each $a \in A_T$, that is all other naive extensions fully attack all bombs. Finally, we still have $\mathbb{S} \subseteq \sigma(F)$ which together with $\sigma(F) \subseteq na(F)$ and $\mathbb{T} \cap \mathbb{S} = \emptyset$ leads to $\sigma(F) = \mathbb{S}$. □

We now use this theorem of realizability to give a theorem of necessity.

Theorem 9.24 (Necessity for Maximal Conflict-freeness)**.** *Consider as given some semantics $\sigma \in \{st, sg, c2, s2\}$, AF F with $\mathbb{S} = \sigma(F)$, argument sets $x, y \subseteq A_F$ and conflict $[x,y]_{\mathbb{S}}^{cnf}$. The conflict $[x,y]^{cnf}$ is necessary syntactic if and only if there are $S \in \mathbb{S}$ and $v \in y \setminus S$ (in case of $\sigma \neq st$ we additionally require $v \in \bigcup \mathbb{S}$) such that the following hold:*

$$[S \cap x, v]_{\mathbb{S}}^{cnf}, \qquad\qquad \{S \setminus x, v\}_{\mathbb{S}}^{ind}.$$

Proof. First recall that by Observation 9.18 we do not bother about rejected arguments and thus w.l.o.g. assume $x, y \subseteq \bigcup \mathbb{S}$. The "if" part is covered by Proposition 9.22. Now, assume for a contradiction that no such S, v exist. It remains to show that $\{x,y\}_F^{ind}$ is possible.

By assumption for each $S \in \mathbb{S}$ and $v \in y \setminus S$ with $[S,v]_{\mathbb{S}}^{cnf}$ there is $u_S^v \in S \setminus x$ such that $\left[u_S^v, v\right]_{\mathbb{S}}^{cnf}$. For the construction of some AF G we may thus assume $(u_S^v, v) \in R_G$ for all $v \in y \setminus S$. This way all conflicting arguments among x and y are handled without using an explicit conflict between x and y. For all other conflicts we may choose syntactic realizations arbitrarily. With $\sigma(G) \subseteq st(G)$ always being true we can apply the construction from the proof of Theorem 9.23 and thus receive an AF H with $\sigma(H) = \mathbb{S}$ and $\{x,y\}_H^{ind}$. □

Observe that Theorem 9.24 together with Proposition 9.22 shows that the described necessary conflicts are indeed minimal with regards to the presented sets. Further, Lemma 9.21 tells us something about necessary attacks for *st* semantics. For semantics *sg*, *c2*, *s2* we did not speak about necessary attacks. We handle this question with the following insights.

Lemma 9.25. *For $\sigma \in \{sg, s2, c2\}$ necessary conflicts are without necessary directionality.*

Proof. Consider some σ-realizable extension set $\mathbb{S}$ and AF H such that all necessary conflicts of $\mathbb{S}$ are expressed in H (with arbitrary direction, in general $\sigma(H) \neq \mathbb{S}$). Using the modification G with $A_G = A_H \cup \{o\} \cup \{\bar{x} \mid x \in A_H\}$ and $R_G = R_H \cup \{(o,o)\} \cup \{(x,\bar{x}), (\bar{x},\bar{x}), (\bar{x},o), (o,x) \mid x \in A_H\}$ we have an AF where $\mathbb{S} \subseteq sg(G) = c2(G) = s2(G) = na(G)$. Thus the construction from the proof of Theorem 9.23 applies and we conclude that the considered semantics in general do not provide directionality for necessary conflicts. □

As closing remarks for this subsection observe that our constructions for *sg*, *c2*, *s2* semantics from Theorem 9.23 do not work in the finite case. The necessities from Proposition 9.22 however are still valid.

In the finite case, for *sg* semantics due to its close relationship with stable semantics we still have the same necessary conflicts but Lemma 9.25 does not apply anymore. Thus for restricted AFs (finite, A-realizable, compact or other classes) we do not know about directionality considerations of necessary conflicts for *sg*, *c2*, *s2*. Observe that for the AF I from Example 9.1 even in the finite case semantics *na*, *sg*, *c2*, *s2* do not know of necessary symmetric attacks or directionality though.

9.3.2 Preferred and Semi-stable Semantics

In this subsection we add admissibility to our considerations and present research dedicated to *pr* and *sm* semantics. Observe that awareness (Lemma 9.8) still holds and we can thus present the following first result.

Lemma 9.26 (Attack relations between Admissible Sets)**.** *Consider some AF F and sets x, y such that there are admissible sets $S, T \in ad(F)$ with $x \subseteq S$, $y \subseteq T$, $[S,T]^{cnf}_{ad(F)}$ and $\{S \setminus x, T\}^{ind}_{ad(F)}$ as well as $\{S, T \setminus y\}^{ind}_{ad(F)}$. Then $(x,y)^{att}_F$ and $(y,x)^{att}_F$ hold.*

Proof. This is a rather straightforward observation. S and T are conflicting extensions and by conflict-awareness this conflict needs a syntactic realization. Since S is compatible with all arguments of $T \setminus y$ and T is compatible with all arguments of $S \setminus x$ the conflict has to be among x and y. By admissibility and thus self-defense of S and T finally this conflict has to be realized in a symmetric way. That is, either by some even-cycle, or by an infinite attack chain. □

The following is a construction for a generalization of [DDLW15, Theorem 1] for *pr*/*sm* semantics. In opposition to their result for our construction we use a slightly different approach: dimensional filtering rather than logical formulas.

Definition 9.27 (Realizability with Maximal Admissible Sets)**.** Consider extension set $\mathbb{S}$ and AF G with $A_G = \bigcup \mathbb{S}$ such that for each $S \in \mathbb{S}$ we have $S \in ad(G)$ and for each $S \neq T \in \mathbb{S}$ we have $[S,T]^{cnf}_G$.

For each $x \in A_G$ define $\mathbb{S}_x = (S_i)_{i \in I}$. We make use of a discrete space of $|\mathbb{S}_x|$ (might be an infinite cardinal) dimensions for each x, where coordinates are $x(y_i)_{i \in I}$ such that $y_i \in S_i$. We define the AF F as

$$
\begin{aligned}
A_F &= A_G \cup \{x(y_i)_{i\in I} \mid x \in A_F, \mathbb{S}_x = (S_i)_{i\in I}, y_i \in S_i \textit{ for each } i \in I\},\\
R_F &= R_G \cup \{(x(y_i)_{i\in I}, x(y_i)_{i\in I}), (x(y_i)_{i\in I}, x) \mid x \in A_F, \mathbb{S}_x = (S_i)_{i\in I}, y_i \in S_i \textit{ for each } i \in I\}\\
&\qquad \cup \{(y_j, x(y_i)_{i\in I}) \mid x \in A_F, \mathbb{S}_x = (S_i)_{i\in I}, y_i \in S_i \textit{ for each } i \in I, j \in I\}.
\end{aligned}
$$

Remark 9.28. For an informal discussion of above definition consider the following. For each x we have a separate $|\mathbb{S}_x|$-dimensional space of (self-attacking) arguments attacking x. For

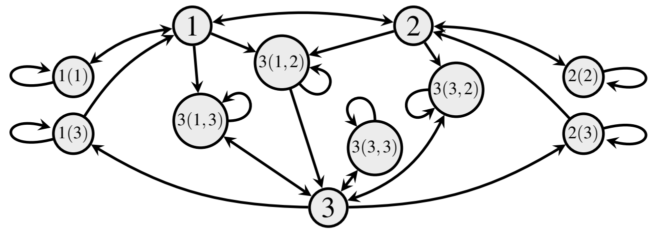


Figure 9.9: AF F pr-realizing the extension set $\mathbb{S} = \{\{1,3\},\{2,3\}\}$ as described in Example 9.29.

finite extension sets and $\mathbb{S}_x = \{S_1, S_2, \ldots, S_n\}$ and argument $\beta \in S_j$ we have that β attacks all new arguments of the form $x(y_1, y_2, \ldots y_{j-1}, \beta, y_{j+1}, \ldots, y_n)$. Similar for infinite AFs arguments $y \in S_j$ attack hyper planes. We use this definition for a characterization of realizability of *pr*/*sm* semantics.[3]

Example 9.29 (Application of *pr*-realizability construction)**.** Consider the extension set $\mathbb{S} = \{\{1,3\},\{2,3\}\}$ and AF G with $A_G = \bigcup \mathbb{S}$ and $R_G = \{(1,2),(2,1)\}$. Since the extensions $\{1,3\},\{2,3\}$ are separated by a symmetric conflict it may serve as input for Definition 9.27. We then have $\mathbb{S}_1 = \{\{1,3\}\}$, $\mathbb{S}_2 = \{\{2,3\}\}$ and $\mathbb{S}_3 = \mathbb{S}$. Thus we get as additional arguments $1(1)$, $1(3)$, $2(2)$, $2(3)$, $3(1,2)$, $3(1,3)$, $3(3,2)$, $3(3,3)$. The resulting AF is depicted in Figure 9.9.

Observe that $\{1,2\}$ does not defend 3 against $3(1,2)$. For 3 to be defended we need some superset of $\{1,3\}$ or $\{2,3\}$.

Theorem 9.30 (Realizability of *pr*/*sm* Semantics in ZFC)**.** *Consider as given an incomparable extension set $\mathbb{S}$ (and for pr also $\mathbb{S} \neq \emptyset$) where for each $S \neq T \in \mathbb{S}$ we have $[S,T]^{cnf}_{\mathbb{S}}$. Then and only then $\mathbb{S}$ is pr/sm-realizable.*

Proof. For the possible collapse of *sm* we refer to Chapter 7, for the case $\mathbb{S} = \{\emptyset\}$ consider the AF $(\{x\},\{(x,x)\})$. For the remainder we consequently assume $\bigcup \mathbb{S} \neq \emptyset$.

The “only then” part is covered by Lemma 9.26, distinct *pr*/*sm* extensions need to be conflicting. Now assume AF G such that $\mathbb{S} \subseteq ad(F)$ and for each $S \neq T \in \mathbb{S}$ we have $[S,T]^{cnf}_G$. Such AF exists since $[S,T]^{cnf}_{\mathbb{S}}$ means there are arguments $s \in S$, $t \in T$ such that $[s,t]^{cnf}_{\mathbb{S}}$ and then for instance (using AC) the assignment $(s,t),(t,s) \in R_G$ serves the purpose. We now apply the construction from Definition 9.27 and show that the resulting AF F has $pr(F) = \mathbb{S}$. For the *sm*-result observe that the modification H with $A_H = A_F \cup \{\bar{x} \mid x \in A_F\}$ and $R_H = R_F \cup \{(\bar{x},\bar{x}),(x,\bar{x}) \mid x \in A_F\}$ ensures $sm(H) = pr(F)$ while conflicts between arguments from F are not altered.

For $\emptyset \neq T \in pr(F)$ and $x \in T$ we have x being attacked by all the coordinates from space $\mathbb{S}_x$. Clearly for sets $S \in \mathbb{S}$ with $S \subseteq T$ we have x being defended against these attacks due to S attacking the whole space of $\mathbb{S}_x$ via hyper planes as discussed in Remark 9.28. This means that

[3] While the result is similar as compared to [DDLW15], our construction is different. Most notably by using the dimensional approach we can directly facilitate DNF formulas whereas their construction requires conversion to CNF formulas.

$\mathbb{S} \subseteq ad(F)$. If there is no $S \in \mathbb{S}_x$ with $S \subseteq T$, w.l.o.g. assume $S \setminus T = \{y(S)\}$ for each $S \in \mathbb{S}_x$. Then x is not defended against the attack from coordinate argument $x(y(S_i))_{i \in I}$ and thus not admissible.

Now for each $x \in T$ there is some $S \in \mathbb{S}_x$ with $S \subseteq T$. But then we are finished already since the case $S_1, S_2 \in \mathbb{S}$, $S_1, S_2 \subseteq T$ means either $S_1 = S_2$ or by construction a syntactic conflict. Thus $\emptyset$ and the sets $S \in \mathbb{S}$ are the only admissible sets in G, which by incomparability of $\mathbb{S}$ subsequently means $pr(F) = \mathbb{S}$. $\square$

Theorem 9.31 (Necessity of *pr*/*sm*-semantic Conflicts in ZFC)**.** *Assume semantics $\sigma \in \{pr, sm\}$ and σ-realizable extension set $\mathbb{S}$. A semantic conflict $[x,y]^{cnf}_{\mathbb{S}}$ is necessary syntactic if and only if there are $S \neq T \in \mathbb{S}$ such that the following hold:*

$$\left\{S \setminus x^0, T\right\}^{ind}_{\mathbb{S}}, \qquad \left\{S, T \setminus y^0\right\}^{ind}_{\mathbb{S}}, \qquad \left[S \cap x^0, T \cap y^0\right]^{cnf}_{\mathbb{S}}.$$

Proof. Assume for a contradiction that for each $\left[S \cap x^0, T \cap y^0\right]^{cnf}_{\mathbb{S}}$ w.l.o.g. there is $u \in S \setminus x^0$ with $\left[u, T \cap y^0\right]^{cnf}_{\mathbb{S}}$ and thus $v \in T \cap y^0$ with $[u,v]^{cnf}_{\mathbb{S}}$. For these arguments facilitate attacks $(u,v), (v,u) \in R_G$ and other necessary conflicts of $\mathbb{S}$ arbitrarily and symmetric (using AC), culminating in an input AF G with $A_G = \bigcup \mathbb{S}$. We can then use the construction from Definition 9.27 and Theorem 9.30 to retrieve an AF F with $\sigma(F) = \mathbb{S}$ and $\left\{S \cap x^0, T \cap y^0\right\}^{ind}_{F}$. Since this holds for each $S, T \in \mathbb{S}$ we get $\{x,y\}^{ind}_F$. $\square$

9.4 Necessary Conflicts Illustrated

In this section we analyse a given AF for necessary conflicts for all semantics of interest. We further consider restricted cases to elaborate on restricted necessities. For instance for *c2*/*s2* semantics in finite AFs we can not make use of collapse and thus the proposed construction from 9.23 is not applicable anymore. We are also particularly interested in $\mathbb{A}$-realizability and thus for instance necessary conflicts for compact AFs. Without further ado we present the example of interest.

Example 9.32. Consider the AF F depicted in Figure 9.10 with $A_F = E_a \cup E_b$ for $E_a = \{u_i, x_j, a_j \mid i \in \{0,1\}, j \in \{0,1,2\}\}$ and $E_b = \{v_i, y_j, b_j \mid i \in \{0,1\}, j \in \{0,1,2\}\}$ with attack set

$$\begin{aligned} R_F = \{&(\alpha_i, \beta_i), (\beta_i, \alpha_i) \mid (\alpha, \beta) \in \{(u,y), (v,x)\}, i \in \{0,1\}\} \\ &\cup \{(\alpha_i, \beta_i) \mid (\alpha, \beta) \in \{(x,b), (y,a)\}, i \in \{0,1,2\}\} \\ &\cup \{(x_2, y_2), (y_2, x_2), (a_2, b_2), (b_2, a_2)\} \\ &\cup \{(b_2, a_0), (a_0, b_1), (b_1, a_2), (a_2, b_0), (b_0, a_1), (a_1, b_2)\}. \end{aligned}$$

Apparently this AF is planar, bipartite and loop-free. We have $st(F) = sg(F) = sm(F) = pr(F) = s2(F) = c2(F) = \mathbb{S}$ with 32 different extensions in $\mathbb{S}$, too many for an explicit listing to provide additional insights. We thus present the following description.

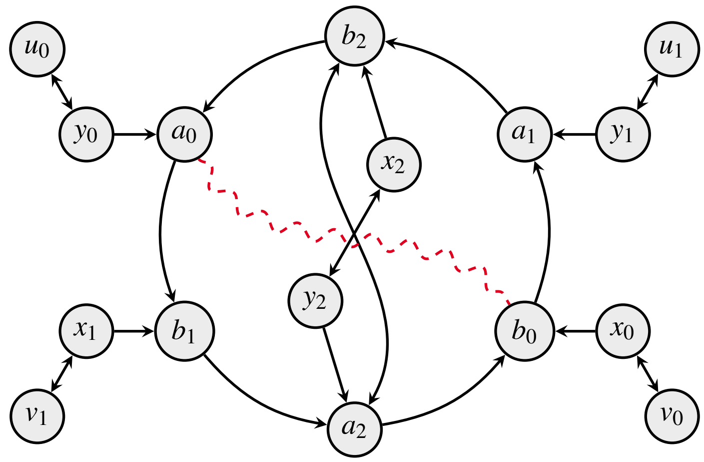


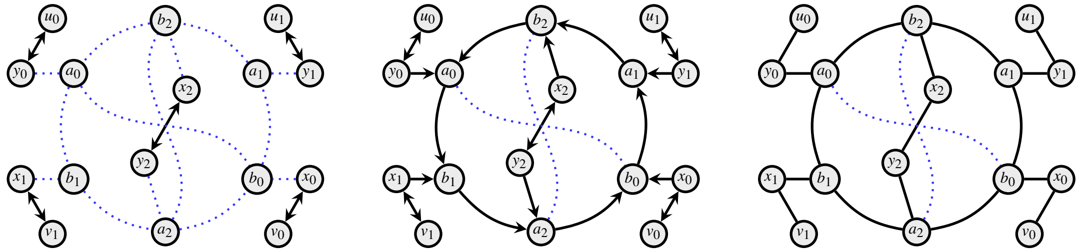
Figure 9.10: Bipartite planar AF from Example 9.32.

Observe that for instance E_a and E_b are stable extensions, hence $st(F) \neq \emptyset$. We now elaborate on the claim that for $\sigma \in \{st, pr, sm, sg, c2, s2\}$ there is no $E \in \sigma(F)$ such that $a_0, b_0 \in E$, and hence a_0 and b_0 are implicitly in conflict. To see this for admissibility based semantics σ (since $st(F) \neq \emptyset$ this includes stage semantics) for a contradiction assume that there is some $E \in \sigma(F)$ with $a_0, b_0 \in E$. Then b_2 needs to be attacked by E, since b_0 attacks a_1 we need $x_2 \in E$. But also a_2 needs to be attacked, however E already attacks all attackers of a_2. The SCCs of F are $\{u_i, y_i\}, \{v_i, x_i\}$ for $i \in \{1,2\}$, $\{x_2, y_2\}$ and the 6-cycle $S_{ab} = \{b_2, a_0, b_1, a_2, b_0, a_1\}$. Since $\{x_2, y_2\}$ is an initial SCC (without attackers from the outside) we have that any *c2*/*s2* extension breaks the 6-cycle and thus $c2(F) = s2(F)$. For *c2*/*s2* then observe that the 6-cycle is the last SCC, assume w.l.o.g. $x_2 \in E$. Again, since we can not defeat a_2 by E we have to choose a_2 over b_0 and thus get $b_0 \notin E$.

Further observe that all other conflicts (than between a_0 and b_0) of F for semantics σ are explicit in F. To see this first observe that for $E \in \sigma(F)$ w.l.o.g. we can guess $\{\alpha_0, \beta_1, \gamma_0, \delta_1, \varepsilon_2 \mid \alpha, \beta \in \{u, y\}; \gamma, \delta \in \{v, x\}; \varepsilon \in \{x, y\}\}$ and then select the uniquely matching arguments from S_{ab}. This uniqueness is either due to admissibility or to SCC-directionality and the fact that at least ε_2 affects S_{ab}. This also means that the AF F is super-coherent and all I-maximal semantics of interest coincide. Thus there are $2^5 = 32$ σ-extensions and no implicit conflicts among α_0, β_1, γ_0, δ_1 and ε_2.

Further for instance for $\alpha_0 = y_0$ and any argument $a \in S_{ab} \setminus \{a_0\}$ we can choose $\beta, \gamma, \delta, \varepsilon$ such that a is defended. The same holds for other combinations between arguments from S_{ab} and $A \setminus S_{ab}$. E_a and E_b are witness to neither the a_i nor the b_i being in conflict. With the given implicit and explicit conflicts it remains to show that a_1 and b_1 are not in conflict, as witnessed by $E = \{y_0, u_1, x_0, v_1, x_2, a_1, b_1\}$.

Definition 9.33 (Extensions of the AF from Example 9.32)**.** We use $S(\alpha\beta\gamma\delta\varepsilon)$ to denote the induced σ extension with $\{\alpha_0, \beta_1, \gamma_0, \delta_1, \varepsilon_2\} \subseteq S(\alpha\beta\gamma\delta\varepsilon)$. For instance we have $S(uuxxx) = \{u_0, u_1, x_0, x_1, x_2, a_0, a_1, a_2\} = E_a$.



(a) *pr*/*sm*-necessity (b) *st*-necessity (c) *sg*/*s2*/*c2*-necessity

Figure 9.11: Necessary conflicts/attacks of the AF from Figure 9.10, cf. Example 9.36.

We now proceed by illustrating necessary conflicts of the given AF.

Lemma 9.34 (Admissibility)**.** *For the AF F from Example 9.32 and semantics* $\sigma \in \{pr, sm, st\}$ *we have as necessary conflicts* $[u_i, y_i]^{cnf}$, $[v_i, x_i]^{cnf}$ *for* $i \in \{0, 1\}$, *and* $[x_2, y_2]^{cnf}$.

Proof. Following the naming scheme from Definition 9.33, for instance $b_0, b_1, b_2 \in S(uuvvy)$ vs. $b_0, b_1, b_2 \in S(yuvvy)$ illustrates the necessary symmetric attack between u_0 and y_0. Further as stated above the choice of $\alpha, \beta, \gamma, \delta, \varepsilon$ in $S(\alpha\beta\gamma\delta\varepsilon)$ determines members of the extension among a_i, b_i. These members can be enforced via attacks between acceptable arguments as in Example 9.32 or via defense constructions as suggested in Theorem 9.31. □

Lemma 9.35 (Maximal Conflictfreeness)**.** *For the AF F from Example 9.32 and semantics* $\sigma \in \{st, sg, s2, c2\}$ *we have as minimal necessary conflicts any arguments* α, β *such that* $\alpha, \beta \notin \{a_2, b_2\}$ *and* $[\alpha, \beta]_F^{cnf}$. *For* $\sigma = st$ *we even get necessary attacks* $(\alpha, \beta)^{att}$ *for all necessary conflicts with* $(\alpha, \beta)_F^{att}$.

Proof. First observe that the conflicts from Lemma 9.34 carry over and in particular for *st* semantics we have symmetric necessary attacks as presented in Figure 9.11a. We derive the remaining necessary conflicts/attacks via Theorem 9.24 and Lemma 9.21 as illustrated in Figure 9.11c.

For the *st*-necessary attacks $(y_i, a_i)^{att}$ and $(x_i, b_i)^{att}$ consider the extensions $a_2 \in S(yyxxx)$ and $b_2 \in (x_i, b_i)^{att}$. For the *st*-necessary attacks $(a_0, b_1)^{att}$ and $(a_2, b_0)^{att}$ consider the extension $a_0, a_1, a_2 \in S(uuvvx)$, while for $(b_0, a_1)^{att}$ and $(b_2, a_0)^{att}$ we have $b_0, b_1, b_2 \in S(uuvvy)$. For the *st*-necessary attacks $(a_1, b_2)^{att}$ and $(b_1, a_2)^{att}$ consider the extensions $a_1, b_1 \in S(yuxvy)$ and $a_1, b_1 \in S(yvxvx)$, respectively. Finally the conflict $[a_2, b_2]^{cnf}$ can be omitted since it is also inherited from $[x_2, y_2]^{cnf}$ via the necessary attacks. □

Example 9.36. We have the necessary conflicts/attacks as presented in Lemmata 9.34 and 9.35 illustrated in Figure 9.11. Observe that necessary conflicts are depicted as lines without arrowheads, while necessary attacks are depicted as arrows. Optional conflicts are depicted as dotted lines.

9.4.1 Compact Necessities

In this subsection we take a step further on our search for necessary conflicts in that we investigate restrictions of AF classes considered. Noteworthy, for *c2*/*s2* semantics and finite AFs there are no collapses which means that our translations from *st* semantics will not work. Also for *pr*/*sm* semantics for instance the AF from Example 9.5 is compact (has only acceptable arguments). Regarding the extension sets $\{1,3\}$ and $\{2,4\}$ it does not suffice to have only one pair of conflicting arguments for compact pr/sm-realizations.

This subsection is not a systematic inspection of AF classes and necessities but rather an investigational excursion into restricted necessities. That is, we take a look at Example 9.32 and try to find further necessities. Since for *st* semantics all necessary conflicts for general AFs already result in an AF with the intended *st* evaluation, there are no further necessities.

Further, even for this AF we do not give a full picture of compact necessities for most semantics. We rather present necessary conflicts we will make use of again later in Section 10.3. That is we essentially show that for each semantics $\sigma \in \{pr, st, sg, sm, s2, c2\}$ and compact realization F of the extension set $\mathbb{S}$ from Example 9.32 the six-cycle $(b_2a_0b_1a_2b_0a_1)$ is necessary.

Lemma 9.37 (Directionality)**.** *Consider the AF F from Example 9.32 and semantics $\sigma \in \{c2, s2\}$. For compact AF G with $A_F = A_G$ and $\mathbb{S} = \sigma(F) = \sigma(G)$ we have $\{(b_2,a_0),(a_0,b_1),(b_1,a_2),(a_2,b_0),(b_0,a_1),(a_1,b_2)\} \subset R_G$ while $(b_1,a_0),(a_1,b_0) \notin R_G$.*

Proof. First observe that the necessary conflicts from Lemma 9.35 are still necessary for compact σ-realizations and further that no $\alpha \in A_G$ can be self-attacking by compactness.

Observe that $(u_i,y_i)^{att}_G$ and $(v_i,x_i)^{att}_G$ necessarily hold for $i \in \{0,1\}$ as otherwise for instance for $(u_0,y_0)^{att}$ the extension $b_2 \in S(uyxxy)$ would be replaced by $\{y_0,y_1,x_0,x_1,y_2,b_2\}$.

We start with $[a_1,b_0]^{cnf}$. Among the extension $b_0,b_1 \in S(yuvvx)$ we have that only b_0 is in conflict with the outside argument a_1. Now if a_1 attacks b_0 in G we end up with an extension $\{y_0,u_1,v_0,v_1,x_2,b_1,a_1\}$. Thus $(a_1,b_0) \notin R_G$ and instead necessarily $(b_0,a_1)^{att}_G$.

For $[a_1,b_2]^{cnf}$ consider the extension set $a_1,b_1 \in S(yuxvy)$. Remarkably this extension set contains neither a_2 nor b_2. However, the only argument of this extension set to be in conflict with b_2 is a_1. Since $\{y_0,u_1,x_0,v_1,y_2,b_1,b_2\}$ is not a member of $\mathbb{S}$ we need $(b_2,a_1) \notin R_G$ and thus by necessity of conflicts $(a_1,b_2)^{att}_G$. For symmetry reasons we also get $(b_1,a_2)^{att}_G$.

Finally, for $[a_2,b_0]^{cnf}_{\mathbb{S}}$ consider the extension set $a_2 \in S(yyvxx)$. Similarly here the only argument in conflict with b_0 is a_2. Consequently (with $\{y_0,y_1,v_0,x_1,x_2,b_0\}$ not being an extension) we cannot have $(b_0,a_2) \in R_G$ and thus need $(a_2,b_0)^{att}_G$ and by symmetry $(b_2,a_0)^{att}_G$.

Hence, the proposed attacks indeed are necessary for compact σ-realizations of $\mathbb{S}$. □

Remark 9.38. Observe that compactness in above proof for Lemma 9.37 is not necessary for *c2* semantics but finiteness suffices. This semantics can make little use of rejected arguments.

We now turn to *pr*/*sm* semantics which due to directionality of *pr* semantics is very similar in technique to above lemma for *c2*/*s2* semantics.

Lemma 9.39 (Admissibility)**.** *Given the AF F from Example 9.32 and semantics $\sigma \in \{pr, sm\}$, we have that the attacks $(\alpha_i, \beta_i)^{att}$ for $\alpha/\beta \in \{y/a, x/b\}$ and $i \in \{0,1,2\}$, as well as the attacks $(b_0,a_1)^{att}$, $(a_0,b_1)^{att}$, $(a_2,b_0)^{att}$, $(b_2,a_0)^{att}$, $(b_1,a_2)^{att}$ are necessary for compact σ-realizations G with $A_G = A_F$, and further $(a_1,b_0) \notin R_G$, $(b_1,a_0) \notin R_G$.*

Proof. For $(b_0,a_1)^{att}$ first consider $a_0,a_1 \in S(uuxxy)$, that is $\{a_0,a_1\}$ defends a_1 against potential attacks from b_2 and $\{a_1,u_1\}$ defends a_1 against potential attacks from y_1. Now consider $a_0,b_0 \in S(uuvvx)$ and observe that neither $\{a_0,a_1,u_0,u_1,v_0,v_1,x_2\}$ nor $\{a_0,a_1,b_0,u_0,u_1,v_0,v_1,x_2\}$ are valid $\mathbb{S}$-extensions. By above defenses that can only mean $(b_0,a_1) \in R_G$ and $(a_1,b_0) \notin R_G$ and by symmetry $(a_0,b_1) \in R_G$ and $(b_1,a_0) \notin R_G$.

Now for $(a_2,b_0)^{att}$ consider $a_1,a_2 \in S(yuvvx)$. Here only a_2 can defend a_1 against the necessary attack $(b_0,a_1)^{att}$. Consequently we need $(a_2,b_0) \in R_G$ and by symmetry $(b_2,a_0) \in R_G$.

For $(x_0,b_0)^{att}$ observe that $b_0,b_1 \in S(yuvvx)$ while $a_1,b_1 \in S(yuxvx)$. Thus x_0 needs to defend a_1 against b_0 and we get $(x_0,b_0) \in R_G$ and by symmetry $(y_0,a_0) \in R_G$. □

Stage semantics does provide neither directionality nor admissibility and is thus more difficult to describe in terms of necessary conflicts for compact AFs. We make use of a backdoor and show a bit more. That is for compact sg-realizations of the proposed extension set we end up with non-empty stable extension set. This results in coincidence of st and sg and thus the compact-necessary conflicts for sg and st are the same. To see that in general this is not the case we refer to Section 10.2, in particular Examples 10.17 and 10.24.

Lemma 9.40 (Range-maximality)**.** *Given the AF F from Example 9.32 with $\mathbb{S} = sg(F) = st(F)$, we have that each sg-compact realization G of $\mathbb{S}$ provides non-empty stable extensions, i.e. $st(G) \neq \emptyset$ and thus $sg(G) = st(G)$ and consequently the sg-compact necessities of $\mathbb{S}$ are the same as for st.*

Proof. First observe that none of the arguments $\alpha \in A_G$ can be self-attacking since sg incorporates conflict-freeness. Now observe that $u_i, v_i \in S(uuvvx)$ while $x_i, y_i \notin S(uuvvx)$ for $i \in \{0,1\}$. However none of the arguments from $S(uuvvx)$ besides u_i may attack y_i while by Lemma 9.35 we need $[u_i,y_i]_G^{cnf}$. If $(u_i,y_i) \notin R_G$ this would mean that $S(uuvvx) \notin sg(G)$. Consequently we get $(u_i,y_i),(v_i,x_i) \in R_G$ for $i \in \{0,1\}$.

In a second step we show $(y_i,u_i),(y_i,a_i),(x_i,v_i),(x_i,b_i) \in R_G$ for $i \in \{0,1\}$. To this end w.l.o.g. for y_0 consider the extensions $a_2 \in S(yyxxx)$. Observe that the conflict $[a_0,y_0]_{\mathbb{S}}^{cnf}$ is necessary syntactic by Theorem 9.24. If $(y_0,a_0) \notin R_G$ or $(y_0,u_0) \notin R_G$ then the proposed extension would be smaller in range than $a_0,a_2 \in S(uyxxx)$. Thus we get $(y_i,u_i)_G^{att}$, $(y_i,a_i)_G^{att}$, $(x_i,v_i)_G^{att}$, $(x_i,b_i)_G^{att}$ for $i \in \{0,1\}$.

We now have $A_G \setminus \{y_0,y_1,x_0,x_1\}_G^+ = \{a_2,b_2,x_2,y_2\} = A_H$. Thus to show that G provides stable extensions it remains to show that for $H = G|_{A_H}$ some $T \in na(H)$ also is $T \in st(H)$. By Theorem 9.24 again we have $[b_2,x_2]^{cnf}$, $[x_2,y_2]^{cnf}$ and $[y_2,a_2]^{cnf}$ as necessary conflicts in G and thus as syntactic conflicts in H.

Now assume w.l.o.g. $(x_2,y_2)_H^{att}$. In case $(x_2,b_2)_H^{att}$ we have $\{x_2,a_2\} \in st(H)$. In case $(b_2,x_2)_H^{att}$ and $(y_2,a_2)_H^{att}$ we have $\{b_2,y_2\} \in st(H)$. In case $(b_2,x_2)_H^{att}$ and $(a_2,y_2)_H^{att}$ we have $\{b_2,a_2\} \in st(H)$. Thus G has stable extensions. Consequently $st(G) = sg(G)$ and the *st*-necessary conflicts from Lemma 9.35 are also necessary for compact *sg*-realizations. □

9.5 Relations between Semantics

Comparison of semantics regarding conflicts is possible in two different settings. First we discuss the case where a given extension set is realizable for different semantics, i.e. the case of a given extension set. After that we assume an AF as given where the different semantics might yield different evaluation. The following is by definition.

Lemma 9.41 ($\mathbb{S}$-necessity)**.** *Assume an extension set $\mathbb{S}$ such that there are AFs F,G with $\sigma(F) = \tau(G) = \mathbb{S}$ and conflict $[x,y]_{\mathbb{S}}^{cnf}$. For choice of semantics σ/τ we have the following:*

- *$\sigma/\tau = pr/sm$: σ-necessity and τ-necessity of $[x,y]_{\mathbb{S}}^{cnf}$ are equivalent;*
- *$\sigma,\tau \in \{st,sg,s2,c2\}$: σ-necessity and τ-necessity of $[x,y]_{\mathbb{S}}^{cnf}$ are equivalent;*
- *$\sigma \in \{pr,sm\}$ and $\tau \in \{st,sg,c2,s2\}$: σ-necessity implies τ-necessity of $[x,y]_{\mathbb{S}}^{cnf}$;*
- *$\sigma \in \{pr,sm\}$ and $\tau = st$: σ-necessity implies τ-necessity of $(x,y)_{\mathbb{S}}^{att}$;*

To see that the implications *pr/sm* to *st/sg/s2/c2* are not equivalences consider Example 9.16. This example is witness that for $\sigma \in \{sm,pr\}$ and $\tau \in \{st,sg,s2,c2\}$ some τ-necessary conflicts are σ-optional.

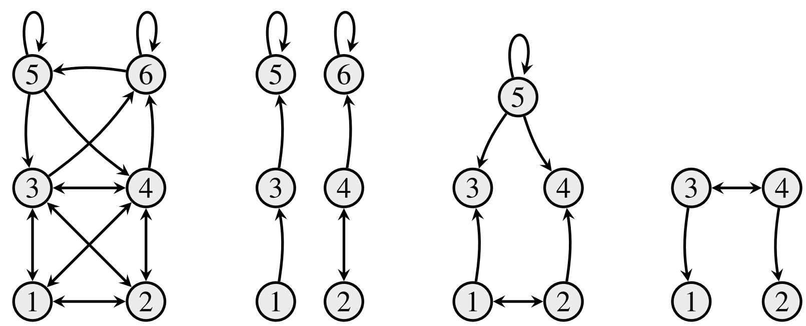
With the straight-forward picture drawn in Lemma 9.41 of relations between semantics σ,τ for a given extension set that is both σ- and τ-realizable, we now ask the same question given some AF and its σ- and τ-evaluation.

Observe that necessary conflicts are given only for acceptable arguments. Hence any example where some σ-acceptable argument is not τ-acceptable is a clear indication that σ-necessity does not imply τ-necessity.

Lemma 9.42. *Given AF F and semantics σ,τ with σ-necessary conflict $[x,y]_F^{cnf}$ such that $y^0 \cap \bigcup \tau(F) = \emptyset$, then the conflict is not τ-necessary.*

Further observe the generality of necessities (Theorems 9.24 and 9.31), given as *pr/sm*-condition or *st/sg/s2/c2/na*-condition. Subset relations of the given semantics consequently yield the following.

Lemma 9.43. *Assume semantics $\sigma,\tau \in \{st,sg,s2,c2,na\}$ or $\sigma,\tau \in \{pr,sm\}$ such that every σ-extension is also a τ-extension. For any given AF F it then holds that σ-necessity implies τ-necessity.*



(a) *sg/s2* vs. *pr/sm* (b) *sg* vs. *s2* (c) *sm/pr* vs. *c2* (d) *sm/pr* vs. *c2*

Figure 9.12: Exemplary AFs as used in the proof of Theorem 9.44.

We now give a characterization of necessary conflicts, given the same input AF for different semantics.

Theorem 9.44. *Consider AF F and syntactic conflict* $[x,y]_F^{cnf}$. *In ZFC the following hold:*

1. *for* $\sigma \in \{pr, sm, st, sg, s2, c2\}$ *we have that* σ*-necessity implies na-necessity but not the other way around;*

2. *for* $\sigma \in \{pr, sm, sg, s2, c2\}$ *we have that* σ*-necessity does not imply st-necessity;*

3. *for* $\sigma \in \{st, sg, s2, c2\}$ *and* $\tau \in \{pr, sm\}$ *we have that* σ*-necessity does not imply* τ*-necessity;*

4. *st-necessity implies sg/s2-necessity, sm-necessity implies pr-necessity, s2-necessity implies c2-necessity;*

5. *pr/sm-necessity and sg/s2-necessity are incomparable;*

6. *sg-necessity and s2-necessity are incomparable;*

7. *we might have c2/pr/na-necessity but not sm/sg/s2/st-necessity;*

8. *pr/sm-necessity do not imply c2-necessity.*

Proof. (1) $\sigma \implies na$: $[x,y]^{cnf}$ being σ-necessary means that there are $u \in x^0$, $v \in y^0$, $u \neq v$ with $[u,v]_F^{cnf}$. Thus $\{u\}_F^{ind}$, $\{v\}_F^{ind}$ and $u,v \in \bigcup na(F)$. But then any *na*-realization G of $na(F)$ requires $[u,v]_G^{cnf}$, for otherwise (Lemma 6.9) we have that $\{u,v\}_{na(G)}^{ind}$ holds.

$na \not\Longrightarrow \sigma$: Consider the AF F with $A_F = \{1,2\}$ and $R_F = \{(1,2)\}$. We have $\sigma(F) = \{\{1\}\}$ while $na(F) = \{\{1\},\{2\}\}$. Thus there are no σ-necessary conflicts in F while $[1,2]^{cnf}$ is *na*-necessary.

(2) $\sigma \not\Longrightarrow st$: Consider the AF F with $A_F = \{1,2,3\}$ and $R_F = \{(1,2),(2,1),(3,3)\}$. We have $\sigma(F) = \{\{1\},\{2\}\}$ and $st(F) = \emptyset$. Here $[1,2]^{cnf}$ is σ-necessary while collapses never provide any necessary arguments or conflicts.

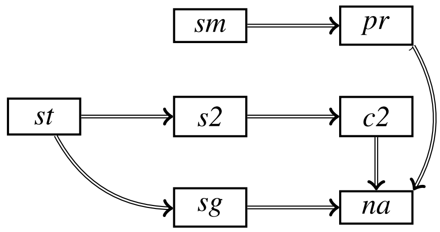


Figure 9.13: Relations between necessities given AF, cf. Theorem 9.44.

(3) $\sigma \not\Rightarrow \tau$: To see this consider Example 9.9 with σ-necessary conflicts that are not τ-necessary.

(4) See Lemma 9.43.

(5) Consider the AF F as depicted in Figure 9.12a, $\sigma \in \{pr, sm\}$ and $\tau \in \{sg, s2\}$. We have $\sigma(F) = \{\{1\},\{2\}\}$ and $\tau(F) = \{\{3\},\{4\}\}$. This means that $[1,2]^{cnf}$ is the only σ-necessary conflict while $[3,4]^{cnf}$ is the only τ-necessary conflict.

(6) Consider the AF F as depicted in Figure 9.12b. We have $sg(F) = \{\{1,4\},\{3,4\}\}$ and $s2(F) = c2(F) = pr(F) = \{\{1,2\},\{1,4\}\}$. That is, the only sg-necessary conflict is $[1,3]^{cnf}$, while the only *s2*/*c2*/*pr*-necessary conflict is $[2,4]^{cnf}$.

(7) Assume $\sigma \in \{c2, pr, na\}$ and $\tau \in \{sm, sg, s2, st\}$ and the AF F with $A_F = \{1,2,3\}$ and $R_F = \{(1,2),(2,1),(2,3),(3,2),(3,3)\}$. We have $\sigma(F) = \{\{1\},\{2\}\}$ and $\tau(F) = \{\{2\}\}$. That is, the σ-necessary conflict $[1,2]^{cnf}$ is not necessary for τ.

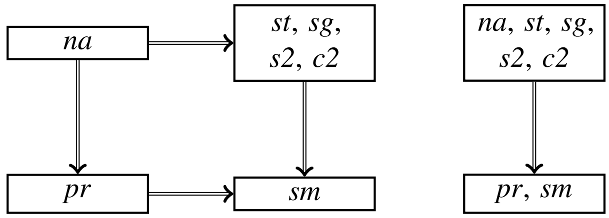
(8) Consider the AFs F from Figure 9.12c and G from Figure 9.12d. We have $c2(F) = c2(G) = pr(G) = sm(G) = \{\{1,4\},\{2,3\}\}$ and $pr(F) = sm(F) = \{\{1\},\{2\}\}$. That is $[1,2]^{cnf}_F$ is *pr*/*sm*-necessary while it is not necessary for *c2* semantics. □

Remark 9.45. Depicted in Figure 9.13 we find the relations between semantics and their necessary conflicts given some input AF F. Here a path via directed arrows from box σ to box τ indicates that any σ-necessary conflict is as well τ-necessary, while absence of arrows means that there are counterexamples.

9.6 Conclusions

In this section we conclude our investigations into necessary conflicts. With Theorems 9.24 and 9.31 we have characterizations of necessary conflicts for all semantics of interest. For *pr*/*sm* it suffices and is necessary to have one conflicting pair of arguments for any two different extension sets. For *st*/*sg*/*s2*/*c2*/*na* it suffices and is necessary to have each extension set being in conflict with each outside argument. One interesting issue for future research is that of counting (see e.g. [BS13]) the number of extension sets. For instance, is any *pr*/*sm*-realizable extension set $\mathbb{S}$ of cardinality 2^n realizable with only n many conflicting pairs of acceptable arguments?

Remark 9.46 (Realizability). As depicted in Figure 9.14 Theorems 9.23 and 9.30 allow for very simple realizability relations between the semantics of interest in the case of arbitrary AF



(a) ZFC (b) ZF, $\neg$AC

Figure 9.14: Realizability/Intertranslatability relations, cf. Remark 9.46.

cardinalities. In models of ZFC we can not realize the empty extension set for *pr*/*na* semantics, resulting in the relations depicted in Figure 9.14a. In models of ZF (see Figure 9.14b) where AC does not hold we can make use of the collapse of *na* and *pr* semantics, and thus have only two blocks of substantially different realizable extension sets. The figures can of course also be interpreted as exact intertranslatability, where an arrow from one box to another means that semantics in the first box are translatable to the semantics in the other box, while containment in the same box indicates bidirectional intertranslatability.

Remark 9.47 (ZF and ZFC). Observe that given some extension set $\mathbb{S}$ the construction of the canonical AF F with $A_F = \bigcup \mathbb{S}$, $R_F = \{(x,y) \mid x,y \in A_F, [x,y]^{cnf}_{\mathbb{S}}\}$ does not require AC. This means that Theorem 9.30 on realizability of *pr*/*sm* semantics would work already in ZF. However, for Theorems 9.23, 9.24 and 9.31 we are in the need of AC.

To some extent with Example 9.16 we might declare the modification approach from Section 9.2 to investigations of necessary attacks as unsuccessful. For one, anyway even the necessary *st*-modifications were not local regarding the input AF F. For another, for *pr* semantics the minimal modifications might not even be local regarding the output AF H. Most importantly discussing necessary conflicts via semantic modifications requires very distinct and elaborate constructions for each specific semantics. For the purpose of initially presenting necessities in abstract argumentation we thus give credit to the clarity of the signature approach from Section 9.3. The signature approach however is very general. On the bright side this allows for our general Theorems 9.24 and 9.31. But this generality also means that for subtleties and for instance syntactic properties of necessary conflicts these theorems leave us widely in the dark. We propose the modification approach from Section 9.2 as an alternative to investigate when looking for such subtleties. Aside from stable semantics (Theorem 9.15) due to space and time restrictions we did not discuss such modifications in this thesis. Syntactic and similar modifications altering the given syntactic conflicts and independences are thus left for future work.

Research on necessary conflicts is particularly important for realizability questions. The approach from the literature [DDLW15, BDL$^+$16, DSLW16] is to consider canonical AFs with all conflicts explicit. As highlighted in Subsection 10.3.2 AFs realizing several extension sets for several semantics (also see [DSLW16]) might result in pure conflicts. In Section 9.3, Theorems 9.23 and 9.30, we point out that the necessary conflicts in general suffice for arbitrary realizations (given one semantics and realizable extension set).

From a practical perspective any form of manipulation with abstract argumentation frameworks [Dyr14], i.e the attempt of refactoring a given knowledge base such that the acceptable sets of arguments change in some desired way, requires in depth knowledge of semantical relations. As far as syntactic investigations with semantic impact are concerned we further relate to [OW11, BB15] for questions of semantic equivalence, [CKMM15, DHK^{+}16] for questions of extension enforcement and [Dun16] for questions of unacceptable sets of arguments.

Chapter 10

Purity

Imagine my surprise, nay, my consternation, when without moving from his privacy, Bartleby, in a singular mild, firm voice, replied,
"I would prefer not to."

Herman Melville, Bartleby the Scrivener [Mel04]

In this chapter we discuss the case of pure conflicts. Pure semantic conflicts (Definition 5.35) can be described as relations between arguments that are formally present (i.e. on the semantic level) yet prefer not to be of structural (syntactic) nature. That is, we consider as given some σ-realizable extension set $\mathbb{S}$ with $[x,y]^{cnf}_{\mathbb{S}}$ such that no realization $\sigma(F) = \mathbb{S}$ (of some AF class) shows $[x,y]^{cnf}_{F}$. This research is reflected by our work on the explicit conflict conjecture and as such (regarding Sections 10.1 and 10.2) featured in [BDL$^+$16]. The progression on our work on conflicts is also reflected in the core of subsequent presentations [Spa15d, Spa15a, Spa16a, Spa16c], particularly regarding Example 9.32 and Subsection 10.3.1. Purity for two-dimensional realizations, Example 10.33 as elaborated on in Subsection 10.3.2, has been incorporated as closing theorem of [DSLW16]. In comparison to [DSLW16, BDL$^+$16] we further enhanced the results by adding considerations for infinite AFs and *s2*/*c2* semantics.

In this thesis we take a different approach when compared to the literature and prior research in that we distinguish between syntactic (Definition 3.7) and semantic (Definition 4.43) instead of explicit and implicit (Definition 9.3) conflicts. We summarize the differences in Table 10.1, where a checkmark in column *c* and line *l* means that *c* conflicts are defined given *l*, whereas a dash means that not. Observe that as opposed to the literature in any case we define conflicts as relations of argument sets or arguments or combinations instead of arguments only.

defined given	implicit	explicit	syntactic	semantic	necessary	pure	optional
AF	-	-	✓	-	-	-	-
AF + semantics	✓	✓	✓	✓	✓	✓	✓
extension set	-	-	-	✓	✓	✓	✓

Table 10.1: Differences between conflict definitions.

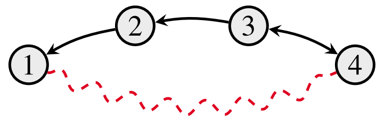


Figure 10.1: AF illustrating different notions of conflict, cf. Example 10.1.

Example 10.1. Given the AF F from Figure 10.1, we have syntactic conflicts $[1,2]_F^{cnf}$, $[2,3]_F^{cnf}$, $[3,4]_F^{cnf}$ and any supersets, i.e. for $x, y \subseteq A_F$ we have $[x,y]_F^{cnf}$ if and only if there is $i \in x$, $j \in y$ with $[i,j]_F^{cnf}$.

For the extension set $\mathbb{S} = \{\{1,3\},\{1,4\},\{2,4\}\}$ we have semantic conflicts $[1,2]_{\mathbb{S}}^{cnf}$, $[2,3]_{\mathbb{S}}^{cnf}$, $[3,4]_{\mathbb{S}}^{cnf}$ and any supersets, i.e. for $x, y \subseteq \bigcup \mathbb{S}$ we have $[x,y]_{\mathbb{S}}^{cnf}$ if and only if there is $i \in x$, $j \in y$ with $[i,j]_{\mathbb{S}}^{cnf}$. Since $\mathbb{S} = na(F)$ all *na*-semantic conflicts of F are syntactic and hence F provides no *na*-implicit conflicts for F.

For the extension set $\mathbb{T} = \{\{1,3\},\{2,4\}\}$ we have semantic conflicts $[1,2]_{\mathbb{T}}^{cnf}$, $[2,3]_{\mathbb{T}}^{cnf}$, $[3,4]_{\mathbb{T}}^{cnf}$, $[1,4]_{\mathbb{T}}^{cnf}$ and any supersets, i.e. for $x, y \subseteq \bigcup \mathbb{T}$ we have $[x,y]_{\mathbb{T}}^{cnf}$ if and only if there is $i \in x$, $j \in y$ with $[i,j]_{\mathbb{T}}^{cnf}$. Since $\mathbb{T} = pr(F)$ the semantic conflict $[1,4]_{\mathbb{T}}^{cnf}$ is not syntactic and hence implicit. Consequently also the semantic conflicts $[1,\{3,4\}]_{\mathbb{T}}^{cnf}$, $[\{1,4\},\{1,4\}]_{\mathbb{T}}^{cnf}$ and similar are not explicit in F and thus implicit for *pr* semantics.

Observe that the constructions used in Theorems 9.23 and 9.30 on realizability are tolerant to syntactic inclusion of any semantic conflicts. While realization of necessary conflicts is sufficient anything between that and the canonical AF is to some extent fine. We reflect this insight with the following lemma.

Lemma 10.2 (Optional Conflicts)**.** *For semantics $\sigma \in \{st, sg, c2, s2, pr, sm\}$ and σ-realizable extension set $\mathbb{S}$ any conflict $[x,y]_{\mathbb{S}}^{cnf}$ that is not necessary is optional.*

Proof. Since we are looking for conflicts we are only interested in cases $|\mathbb{S}| \geq 1$. By realizability (Theorems 9.23 and 9.30) we know that *st*-realizations can be transformed into *sg*/*c2*/*s2*-realizations and vice versa, and *pr*-realizations into *sm*-realizations and vice versa. It hence suffices to consider semantics $\sigma \in \{st, pr\}$.

For stable semantics assume some AF H as given with $[x,y]_{\mathbb{S}}^{cnf}$ and $\{x,y\}_H^{ind}$. Define G arbitrarily by $A_G = A_H$ and $R_G = R_H \cup \{(a,b)\}$ for some $a \in x^0, b \in y^0$. The AF G still satisfies the conditions for realizability of *st* semantics and thus can be used to construct F with $R_F \cap (A_G \times A_G) = R_G$ and $st(F) = \mathbb{S}$.

For preferred semantics similarly assume AF H as given with $[x,y]_{\mathbb{S}}^{cnf}$ and $\{x,y\}_H^{ind}$. We will again apply realizability but this time use $R_G = R_H \cup \{(a,b),(b,a)\}$ for some $a \in x^0, b \in y^0$. This symmetric approach is due to the requirement of extension sets being admissible while as opposed to *st* semantics we do not have full range to start from. □

With this lemma the investigations in this chapter seem to come to a sudden halt, as there are no pure conflicts for the semantics of interest. Instead this insight allows us to shift focus on AF classes and other subtle nuances of pure conflicts. On the one hand in Subsection 10.3.1 we look at AF classes, such as finite or compact (without rejected arguments) that might provide purity. On the other hand we ask purity questions for combinations of semantics in Subsection 10.3.2.

We start with a shift of focus to analytic AFs (see Definition 5.47), i.e. the question whether the reality of optional conflicts, given σ-realizable extension set $\mathbb{S}$, allows construction of AFs that do not show any implicit conflicts. This question is almost but not quite dual to a search for (restricted) pure conflicts. Making some implicit conflicts explicit might yield new implicit conflicts between different argument sets, pure conflicts however require one distinct conflict to remain implicit in any realization. In Section 10.1 we compare analytic AFs for various semantics. In Section 10.2 we discuss the inspiration to our research on conflicts from [BDL$^+$14]: the explicit conflict conjecture, the assumption that for stable (and other) semantics we can always have all-explicit (i.e. analytic) realizations. In Section 10.3 finally we discuss formal purities. In Section 10.4 we conclude and relate to the literature.

10.1 Analytic Argumentation Frameworks

In this section we deal with AFs containing no implicit conflicts, which we call analytic (see Definition 5.47). We differentiate between the concept of an attack (as a syntactic element) and the concept of a (semantic) conflict (with respect to the evaluation under a given semantics).

Definition 10.3. The class of all *analytic argumentation frameworks* for σ is denoted by XAF_σ. We may write $F \in XAF_\sigma$ to denote $[x,y]^{cnf}_F \iff [x,y]^{cnf}_{\sigma(F)}$ for any $x,y \subseteq A_F$, or $XAF_\sigma \subseteq XAF_\tau$ if each σ-analytic AF is also τ-analytic.

Example 10.4. As a simple example consider the AF F from Example 10.1, depicted in Figure 10.1. For $\sigma \in \{st, pr, sm, sg, c2, s2\}$ we have $\sigma(F) = \{\{1,3\},\{2,4\}\}$. Observe that there is an implicit conflict between arguments 1 and 4, denoted by a dashed line in Figure 10.1. Hence F is not σ-analytic, i.e. $F \notin XAF_\sigma$. However we have that $na(F) = \sigma(F) \cup \{\{1,4\}\}$, which means that F is analytic for naive semantics.

As indicated in Example 10.4 the sets of analytic AFs can differ for different semantics. Again, well-known relations between the extensions of certain semantics allow us to obtain $\subseteq$-relations between classes of analytic AFs.

Lemma 10.5. *Consider semantics σ, τ such that for each AF F and every $S \in \sigma(F)$ there is some $T \in \tau(F)$ with $S \subseteq T$, then already $XAF_\sigma \subseteq XAF_\tau$.*

Proof. Let $F \in XAF_\sigma$ and let there be a conflict $[x,y]^{cnf}_{\tau(F)}$. Since for every $S \in \sigma(F)$ there is some $T \in \tau(F)$ with $S \subseteq T$ it follows that $\{u,v\}^{ind}_{\sigma(F)} \implies \{u,v\}^{ind}_{\tau(F)}$. Hence also $[x,y]^{cnf}_{\sigma(F)}$ holds. By the assumption that $F \in XAF_\sigma$ we know that there is an attack $(x,y)^{att}_F$ or $(y,x)^{att}_F$, hence also $F \in XAF_\tau$. $\square$

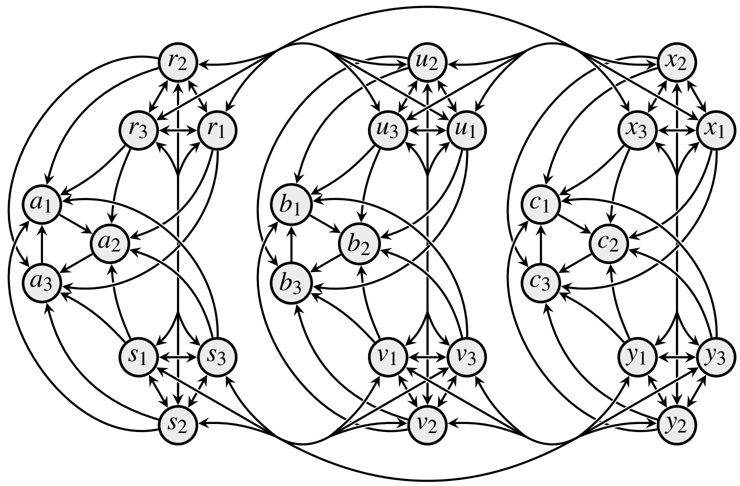


Figure 10.2: AF F with $F \in XAF_\sigma$ for $\sigma \in \{pr, sm, sg, c2, s2\}$ and $F \notin XAF_{st}$.

Above lemma works fine with the common relations between semantics as highlighted in Proposition 6.39 and Figure 6.3. By directionality of *pr* and *c2* alike and the observation that each *pr* extension is contained in some *na* extension we derive the following.

Lemma 10.6 (Relations between *pr* and *c2*)**.** *For finite AF F and $S \in pr(F)$ there is some $T \in c2(F)$ with $S \subseteq T$.*

The next result provides a full picture of the relations between classes of analytic AFs for the semantics we consider (see also Figure 10.6). We will frequently use Lemma 10.5, with either the exact condition or the special case $\sigma(F) \subseteq \tau(F)$.

Theorem 10.7. *The following relations hold:*

1. $XAF_{st} \subset XAF_\sigma \subset XAF_{na}$ *for* $\sigma \in \{pr, sm, sg, c2, s2\}$*;*
2. $XAF_{sm} \subset XAF_{pr}$*;*
3. $XAF_{s2} \subset XAF_{c2}$*;*
4. $XAF_\sigma \not\subseteq XAF_\tau$ *and* $XAF_\tau \not\subseteq XAF_\sigma$ *for* $\sigma \in \{c2, s2\}$ *and* $\tau \in \{sg, sm\}$ *as well as for* $\sigma/\tau = s2/pr$*; and in the finite case* $XAF_{pr} \subset XAF_{c2}$*;*
5. $XAF_{sg} \not\subseteq XAF_\tau$ *and* $XAF_\tau \not\subseteq XAF_{sg}$ *for* $\tau \in \{pr, sm\}$.

Proof. (1) Let $\sigma \in \{pr, sm, sg, c2, s2\}$. The $\subseteq$-relations are due to Lemma 10.5 together with the following facts: (a) in any AF F, $st(F) \subseteq \sigma(F)$; (b) each σ-extension E of an AF F is conflict-free in F, thus there exists a naive extension E' of F with $E \subseteq E'$.

$XAF_\sigma \subset XAF_{na}$: The AF in Figure 10.1 is, as discussed in Example 10.4, *na*-analytic but not σ-analytic.

$XAF_{st} \subset XAF_\sigma$: Consider the AF F from Figure 10.2. It contains several kinds of complete subframeworks, in the sense that each member of such a subframework attacks each other

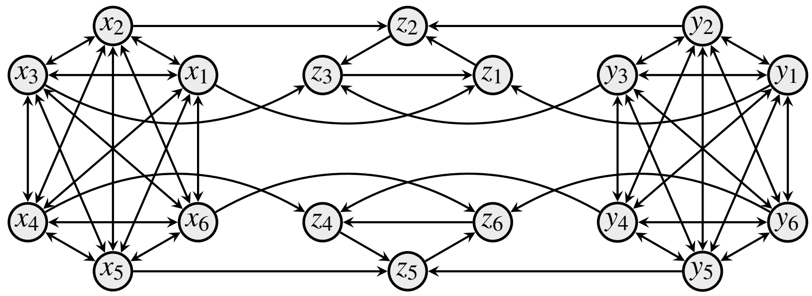


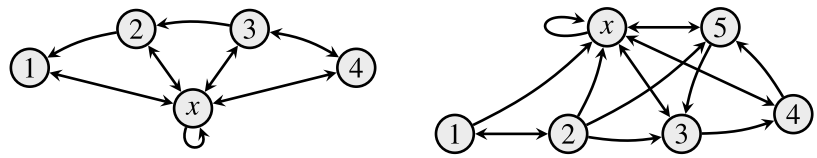
Figure 10.3: AF F with $F \in XAF_{pr}$ and $F \notin XAF_\sigma$ for $\sigma \in \{st, sm, sg\}$.

member. Two complete subframeworks of nine arguments ($\{r_i, u_i, x_i \mid i \in \{1,2,3\}\}$ and $\{s_i, v_i, y_i \mid i \in \{1,2,3\}\}$) and three complete subframeworks of six arguments ($\{r_i, s_i \mid i \in \{1,2,3\}\}$, $\{u_i, v_i \mid i \in \{1,2,3\}\}$ and $\{x_i, y_i \mid i \in \{1,2,3\}\}$). Further there are three directed three-cycles (among $\{a_i \mid i \in \{1,2,3\}\}$, $\{b_i \mid i \in \{1,2,3\}\}$ and $\{c_i \mid i \in \{1,2,3\}\}$), and each argument from the complete subframeworks attacks exactly two arguments from one three-cycle, effectively activating the third one. Now observe that we have $st(F) = \emptyset$, as at least one argument of a_i, b_i, c_i remains out of range due to conflict-freeness, i.e. a conflict-free set in F can have only one argument from each complete nine-component and thus leaves at least one of the three-cycles unattacked. Therefore there is an implicit conflict for st for every pair of non-attacking arguments, hence $F \notin XAF_{st}$. On the other hand we have $pr(F) = sm(F) = \{\{r_i, v_j, a_i, b_j\}, \{s_i, u_j, a_i, b_j\}, \{r_i, y_j, a_i, c_j\}, \{s_i, x_j, a_i, c_j\}, \{u_i, y_j, b_i, c_j\}, \{v_i, x_j, b_i, c_j\} \mid i, j \in \{1,2,3\}\}$ and $sg(F) = c2(F) = s2(F) = \{\{r_i, v_j, a_i, b_j, c_k\}, \{s_i, u_j, a_i, b_j, c_k\}, \{r_i, y_j, a_i, c_j, b_k\}, \{s_i, x_j, a_i, c_j, b_k\}, \{u_i, y_j, b_i, c_j, a_k\}, \{v_i, x_j, b_i, c_j, a_k\} \mid i, j, k \in \{1,2,3\}\}$, which allows to verify that all conflicts for σ are explicit in F, hence $F \in XAF_\sigma$.

(2) By Lemma 10.5 we get $XAF_{sm} \subseteq XAF_{pr}$. In order to obtain properness of this relation consider the AF F from Figure 10.3 and define a cyclic successor function s as $\mathrm{s}(1) = 2, \mathrm{s}(2) = 3, \mathrm{s}(3) = 1$, and $\mathrm{s}(4) = 5, \mathrm{s}(5) = 6, \mathrm{s}(6) = 4$. We have $sm(F) = \{\{x_i, y_j, z_{\mathrm{s}(i)}, z_{\mathrm{s}(j)}\} \mid i \in \{1,2,3\}, j \in \{4,5,6\} \text{ or } i \in \{4,5,6\}, j \in \{1,2,3\}\}$, yielding plenty of implicit conflicts, e.g. between x_i and y_i. Hence F is not analytic for semi-stable semantics. We further define $\mathrm{s}(\{i\}) = \mathrm{s}(i)$ and for $\mathrm{s}(i) = j$ also $\mathrm{s}(\{i, j\}) = \mathrm{s}(j)$. Then on the other hand we have $pr(F) = sm(F) \cup \{\{x_i, y_j, z_{\mathrm{s}(\{i,j\})}\} \mid i, j \in \{1,2,3\} \text{ or } i, j \in \{4,5,6\}\}$, witnessing $F \in XAF_{pr}$.

(3) $XAF_{s2} \subset XAF_{c2}$: By Lemma 10.5 we get $XAF_{s2} \subseteq XAF_{c2}$. Consider the modification G of AF F from Example 10.4, augmenting by a single argument x such that $A_G = \{1,2,3,4,x\}$ and $R_G = \{(2,1), (3,2), (3,4), (4,3), (x,x)\} \cup \{(x,i), (i,x) \mid i \in A_F\}$ as depicted in Figure 10.4a. We have $s2(G) = sg(F) = \{\{1,3\}, \{2,4\}\}$ while $c2(G) = na(F) = sg(F) \cup \{\{1,4\}\}$. Since the new argument x is explicitly in conflict with all available arguments we get $G \in XAF_{c2}$ while $G \notin XAF_{s2}$.

(4) $XAF_\sigma \not\subseteq XAF_\tau$: Consider the AF F depicted in Figure 10.4b. For $\sigma \in \{c2, s2\}$ we have $\sigma(F) = \{\{1,3\}, \{1,4\}, \{1,5\}, \{2,4\}\}$, i.e. all σ-semantic conflicts are explicit. For



(a) $G \in XAF_{c2}$ while $G \notin XAF_{s2}$ (b) $F \in XAF_{c2/s2}$, $F \notin XAF_{pr/sm/sg/st}$

Figure 10.4: AFs illustrating differences of *c2* and *s2* semantics.

$\tau \in \{st, sm, sg\}$ we have $\tau(F) = \{\{2,4\}\}$ and for $\tau = pr$ we have $\tau(F) = \{\{1\},\{2,4\}\}$, in any case arguments 3 and 5 are rejected and thus represent τ-implicit conflicts.

$XAF_{pr} \subset XAF_{c2}$ in the finite case: Consider Lemma 10.6 and above example for properness. Observe that in the infinite case we can run into the trouble of collapse. In essence, we leave this case as open question for future research.

$XAF_{pr} \not\subseteq XAF_{s2}$: Consider the AF F from Figure 10.3 enhanced as G with $A_G = A_F \cup \{o\}$ and $R_G = \{(o,o)\} \cup \{(o,x),(x,o) \mid x \in A_F\}$. By construction we have $pr(F) = pr(G)$ and $s2(G) = sg(F)$ and thus $G \in XAF_{pr}$ while $G \notin XAF_{s2}$.

$XAF_{sg} \not\subseteq XAF_{c2/s2}$: Consider the modification G of the AF F from Figure 10.4b with $A_G = A_F$ and $R_G = R_F \setminus \{(2,1)\}$. Then the first SCC of G is $\{1\}$ only and thus we have $c2(G) = s2(G) = \{\{1,3\},\{1,4\},\{1,5\}\}$ while $sg(G) = s2(G) \cup \{\{2,4\}\}$. That is the AF G has *c2*- and *s2*-implicit conflicts (for instance for rejected argument 2) but is *sg*-analytic.

(5) $XAF_{sg} \not\subseteq XAF_{pr/sm}$: Consider a directed cycle of five arguments F, $A_F = \{x_1,x_2,x_3,x_4,x_5\}$ and $R_F = \{(x_1,x_2),(x_2,x_3),(x_3,x_4),(x_4,x_5),(x_5,x_1)\}$. Here we have $sg(F) = \{\{x_1,x_3\},\{x_1,x_4\},\{x_2,x_4\},\{x_2,x_5\},\{x_3,x_5\}\}$ and thus $F \in XAF_{sg}$. On the other hand $sm(F) = pr(F) = \{\emptyset\}$, marking all pairs of arguments as being in conflict and thus for instance the conflict between x_1 and x_3 is implicit for *pr* and *sm* (and also *st*).

$XAF_{pr} \not\subseteq XAF_{sg}$: The AF F in Figure 10.3 is, as argued in (2), explicit for *pr*, but not for *sm*. However, it holds that $sg(F) = sm(F)$, hence also $F \notin XAF_{sg}$.

$XAF_{sm} \not\subseteq XAF_{sg}$: As witness of $XAF_{sm} \not\subseteq XAF_{sg}$ consider the AF F from Figure 10.5. This AF is composed of two subframeworks, F_X from Figure 10.2 and F_C from [BDL$^+$16, Fig. 6b] (which is *sm*-compact but has *sg*-implicit conflicts), and a connecting interface consisting of argument $\bar{x}$ and its counterpart set $Y = \{\bar{s}_i, \bar{t}_i, \bar{u}_i \mid i \in \{1,2,3\}\}$. There are symmetric attacks between the members $\bar{\alpha}$ of Y and their counterparts α from F_C, between $\bar{x}$ and all members of Y, and between $\bar{x}$ and all arguments from F_X.

A key ingredient to this construction is that both, F_C and F_X, on their own do not provide stable extensions and thus at least one argument remains out of range for any stage or semi-stable extension. In addition observe that F_X is compact for both semi-stable and stage, while F_C is compact only for semi-stable, where a is the argument that does not occur in any $S \in sg(F_C)$.

Considering range-maximal (conflict-free or admissible) sets for F we first distinguish between sets S in relation to the argument $\bar{x}$. In case $\bar{x} \in S$ we have that all arguments from F_X are in range, Y is attacked and thus F_C needs to be evaluated on its own. In case $\bar{x} \notin S$, w.l.o.g.

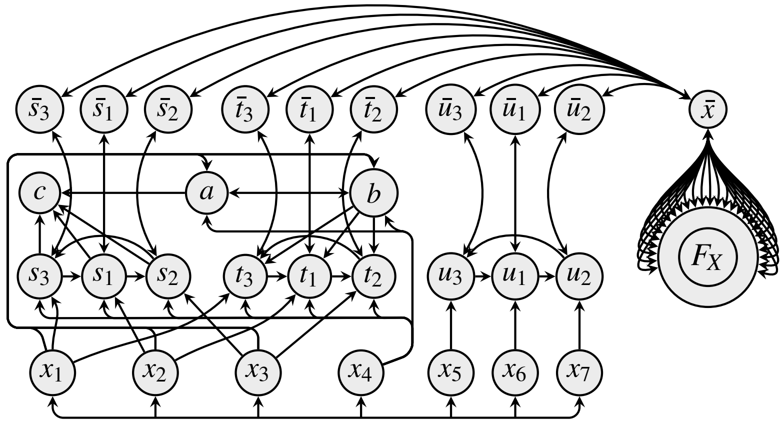


Figure 10.5: AF F with $F \in XAF_{sm}$ for $F \notin XAF_{sg}$. Here F_X refers to the AF from Figure 10.2 and $\bar{x}$ is in a symmetric attack relationship with all arguments from F_X.

assume $Y \subseteq S$ and $a, x_5 \in S$, we have that all of F_C and Y are in range, $\bar{x}$ is attacked and F_X needs to be evaluated on its own. This means that either some argument from F_C or some argument from F_X remains out of range of any *sm* or *sg* extension in F and thus $st(F) = \emptyset$. On a sidenote observe that for very similar reasons F is compact for both, *sm* and *sg* semantics.

Observe that F_C is compact for semi-stable, but not for stage [BDL$^+$16, Theorem 2]. This immediately means that for stage semantics there is an implicit conflict between $\bar{x}$ and F_C (argument a to be precise). This also means that for semi-stable semantics there are no implicit conflicts between $\bar{x}$ and any argument from F_C.

It remains to show that F indeed is analytic for semi-stable semantics. To this end we still need to investigate possible implicit conflicts between F_X and Y, between F_C and Y, as well as between F_X and F_C, and among arguments from F_C, as well as among arguments from Y.

As mentioned before the range of any semi-stable extension covers Y and $\bar{x}$ and either all of F_C or all of F_X. We start with extensions S with $Y \subseteq S$ and thus $\bar{x} \notin S$ and, w.l.o.g. fix the evaluation of F_X and consider some arbitrary $S_X \in sm(F_X)$. First observe that this immediately means that Y does not contain any conflicts and, due to F_X being compact, there are also no conflicts between Y and F_X. As $Y \cup S_X \cup \{c, x_i\} \in sm(F)$ for $i \in \{1,2,3,4\}$, and for $i \in \{5,6,7\}$ also $Y \cup S_X \cup \{a, x_i\} \in sm(F)$ as well as $Y \cup S_X \cup \{b, c, x_i\} \in sm(F)$, there are no conflicts between Y and $a, b, c, x_1 \dots x_7$, between c and $b, x_1 \dots x_7$, or between a, b and x_5, x_6, x_7.

We now investigate extensions $S \in sm(F)$ that contain gradually less arguments from Y. In the following we will omit certain x_i from extensions, due to in F_C explicit conflicts, for instance x_2 as well as x_4 attack s_1 and t_1. For $(Y \setminus \{\bar{s}_1\} \cup \{s_1\}) \subseteq S$ we can have $x_i \in S$ for $i \in \{1,3\}$, and for $i \in \{5,6,7\}$ on the other hand $x_i, a \in S$ or $x_i, b \in S$. For $(Y \setminus \{\bar{t}_1\} \cup \{t_1\}) \subseteq S$ we can have $x_i, c \in S$ for $i \in \{1,3\}$, or for $i \in \{5,6,7\}$ on the other hand $x_i, a \in S$. For $(Y \setminus \{\bar{u}_1\} \cup \{u_1\}) \subseteq S$ we can have $x_i, a \in S$ or $x_i, b, c \in S$ for $i \in \{5,7\}$, or for $i \in \{1,2,3,4\}$ on the other hand $x_i, c \in S$. Hence for symmetry reasons for $i \in \{1,2,3\}$ there are no implicit conflicts between arguments

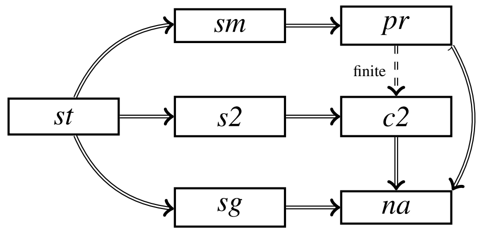


Figure 10.6: Relations between analytic AFs (cf. Theorem 10.7).

s_i, t_i, u_i on the one side and on the other side Y and arguments a, b, c, x_j for $j \in \{1, 2, \dots 7\}$. We conclude that there are no implicit but only explicit conflicts between F_C and Y in F.

For $i, j, k \in \{1, 2, 3\}$ fixed and $S_Y = Y \setminus \{\bar{s}_i, \bar{t}_j, \bar{u}_k\}$ we have that $S_X \cup S_Y \cup \{s_i, t_j, u_k, x_i\} \in sm(F)$. This means that there are no conflicts between s_i, t_j and u_k, and subsequently that the subframework F_C does not have any implicit conflicts in F.

Now finally, as elaborated on, each argument from F_C can appear in semi-stable extensions S of F that do not contain $\bar{x}$ and thus contain some arbitrary F_X-extension S_X. This means that there are no conflicts between F_C and F_X, which closes the gaps and shows that F indeed is analytic for semi-stable semantics. $\square$

Depicted in Figure 10.6 we find the relations from Theorem 10.7 visualized. An arrow from box σ to box τ means that any σ-analytic AF is also τ-analytic. The dashed arrow from *pr* to *c2* with the *finite*-label means that it holds in the finite case, and we do not know for the general case. For all other combinations of σ/τ, if there is no arrow, then there are counterexamples.

Open Question 10.8. *In the general case (with c2-collapses) do we have $XAF_{pr} \subset XAF_{c2}$ or are XAF_{pr} and XAF_{c2} incomparable?*

10.2 Explicit Conflict Conjecture

In this section we take another look at the issue of implicit conflicts and the possibility of making them explicit. In Section 10.1 we identified the classes of AFs where all conflicts are explicit w.r.t. a given semantics. Recall the notion of an analytic AF from Definition 5.47. In [BDL$^+$14] the authors conjectured that, under stable semantics, every AF can be translated to an equivalent analytic AF (having the same set of arguments), i.e. that all implicit conflicts can be made explicit without changing the stable extensions. We will refute this conjecture and show that the claim also does not hold for preferred, semi-stable, stage, cf2 or stage2 semantics.

Definition 10.9. An AF F is called *quasi-analytic* for σ if there is an AF G such that $A_F = A_G$, $\sigma(F) = \sigma(G)$ and G is analytic for σ, i.e., it has only explicit conflicts for σ. On the other hand, F is called *non-analytic* for σ if it is not quasi-analytic for σ.

Example 10.10. Consider again the AF in Figure 10.1. As we have seen in Example 10.4, it is not analytic for $\sigma \in \{st, pr, sm, sg, c2, s2\}$. As highlighted in Example 9.9 and Figure 9.4c the modification H adding the attack $(4,1)^{att}$ is a semantically equivalent AF, where all conflicts are explicit. Thus F is quasi-analytic.

In other words, an AF is quasi-analytic for a given semantics σ if it can be translated to another AF that has the same arguments, has the same extensions under σ, and all conflicts are explicit. The conjecture from [BDL$^+$14] says that every AF containing implicit conflicts for stable semantics is quasi-analytic, in the sense that all implicit conflicts can be made explicit without adding further arguments. We repeat the conjecture from [BDL$^+$14], just parameterized by an arbitrary semantics. In line with the following definition, [BDL$^+$14] claimed that ECC holds for stable semantics.

Definition 10.11. We say that the *Explicit Conflict Conjecture (ECC)* holds for semantics σ if every AF is quasi-analytic for σ.

Lemma 10.12. *In ZFC any AF is quasi-analytic for na semantics.*

Proof. Given AF F, consider the modification G with $A_G = A_F$ and $R_G = \{(x,y) \mid x,y \in A_F, [x,y]^{cnf}_{na(F)}\}$. Observe that in ZFC *na* semantics rejects arguments if and only if they are self-attacking (Proposition 6.40), and further that our definition of AF G ensures self-attacks for arguments in semantic conflict with themselves. The construction further assures that rejected arguments are in syntactic conflict with all other arguments and that any semantic conflict between arguments is availably also syntactically. Now by necessity of *na*-conflicts (Theorem 9.17) and minimality of syntactic conflicts (Lemma 9.2) we have that any *na*-semantic conflict can be traced back to pairs of arguments and is thus reflected in our construction. That is, $na(F) = na(G)$ and $G \in XAF_{na}$. $\square$

Before diving deeper into non-analytic AFs and constructions for other semantics we first present the reason we do not allow additional arguments for the constructions (and thus Definition 10.3). Namely, allowing additional arguments can result in making all implicit conflicts explicit for *st* and *sg* semantics. The following proposition shows that one additional argument allows, together with an appropriate modification of the attack relation, to make any single implicit conflict explicit. Observe that the used construction strongly resembles the construction from Definition 9.10.

Proposition 10.13. *For stable semantics and some AF F, if there is an implicit conflict between a and b, then there is an AF G with $|A_G| = |A_F| + 1$, $R_G \supseteq R_F$, $(a,b) \in R_G$ and $st(G) = st(F)$.*

Proof. Let F be an arbitrary AF with an implicit conflict between two arguments a and b. We define $R' = R_F \cup \{(a,b)\}$. Observe that $F' = (A_F, R')$ has the same and possibly more stable extensions as compared to F. By construction of F', any $S \in st(F') \setminus st(F)$ has $a \in S$ and $b \notin S^+_F$. We collect the arguments of these unwanted extensions in $A_a = \bigcup(st(F') \setminus st(F))$ and observe that $b \notin (A_a)^+_F$. Now define the AF G with $A_G = A_F \cup \{x\}$ and

$$R_G = R' \cup \{(x,x)\} \cup \{(x,v) \mid v \in A_a\} \cup \{(u,x) \mid u \in A_F \setminus A_a\}.$$

First note that obviously $|A_G| = |A_F| + 1$, $R_G \supseteq R_F$, and $(a,b) \in R_G$. Moreover, since the new argument x attacks or is attacked by every other argument, G does not introduce any further implicit conflicts compared to F. It remains to show that $st(G) = st(F)$. Let $S \in st(F)$ and assume that $b \in S$. As by assumption b and a do not occur together in any stable extension of F, we know that $(b,x)^{att}_G$ and thus $S \in st(G)$. On the other hand assume that $b \notin S$. Then we have some $c \in S$ with $(c,b)^{att}_F$. If $S \notin st(G)$, then only because $x \notin S^*_G$, hence $S \subseteq A_a$, a contradiction to $b \notin (A_a)^+_F$. Therefore $S \in st(G)$. Now assume there is some $S \in st(G)$ with $S \notin st(F)$. By the construction of G this S must be among $st(F') \setminus st(F)$. However, we then have $x \notin S^+_G$, a contradiction to $S \in st(G)$, concluding the proof for $st(F) = st(G)$. □

Now we can show that any AF can be transformed into a *st*/*sg*-semantically equivalent one without implicit conflicts.

Theorem 10.14. *For $\sigma \in \{st, sg\}$ it holds that for any given AF F (for sg with $sg(F) \neq \emptyset$) there is an AF G with $\sigma(F) = \sigma(G)$ and for $x, y \subseteq A_G$ we have $[x,y]^{cnf}_G \iff [x,y]^{cnf}_{\sigma(G)}$.*

Proof. For the case $\sigma(F) = \{\emptyset\}$ consider the AF $(\emptyset, \emptyset)$, for the case $st(F) = \emptyset$ consider the AF $(\{x\}, \{(x,x)\})$. We thus assume $\bigcup \sigma(F) \neq \emptyset$. Let $\mathbb{S} = \sigma(F)$. We use transfinite induction over the conflicts of $\mathbb{S}$ with recursive application of Proposition 10.13 to construct AF G. Hence there is an analytic AF G with $\sigma(G) = \mathbb{S}$. For $\sigma = sg$ semantics by Theorem 9.23 we know that there is an AF F' with $st(F') = sg(F)$ and consequently may use F' as input above to construct G with $st(G) = sg(G) = sg(F)$. □

Observe that in above theorem we did not include the collapse of *sg* semantics. This has a simple reason as witnessed by the following theorem.

Theorem 10.15. *Given semantics $\sigma \in \{na, pr, sg, sm, s2, c2\}$, any AF F with $\sigma(F) = \emptyset$ is non-analytic.*

Proof. Observe that σ-collapsing AF F (see Chapter 7 for examples) means first $\sigma(F) = \emptyset$ but further also existence of infinitely many disjoint conflict-free sets (Lemma 7.17) and thus infinitely many arguments that are not self-attacking. We can not have σ-collapse without some arguments not being in syntactic conflict with themselves. Since the extension set $\emptyset$ however says that all arguments are semantically in conflict with themselves any such AF must be non-analytic. □

The question of ECC really is a question restricted to initial argument sets and non-collapsing AFs. In the remainder of this section we will refute ECC for all semantics in $\{st, pr, sm, sg, c2, s2\}$ by providing non-analytic AFs. For a more compact notation of attacks and symmetric attacks for this section we introduce an additional notation.

Definition 10.16 (Symmetric Attacks)**.** Given AF F, we may abbreviate $\langle x,y \rangle = (x,y),(y,x)$, symmetric attacks defined as attacks in both directions.

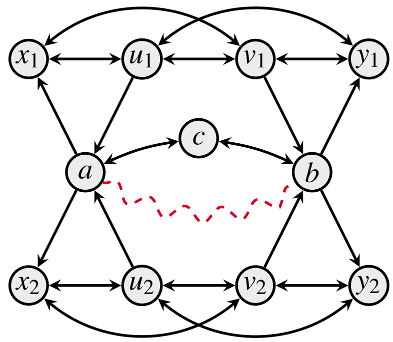


Figure 10.7: Illustration of the AF from Example 10.17.

Example 10.17. Take into account the AF $F = (A,R)$ depicted in Figure 10.7, which features an implicit conflict for stable semantics between a and b:

$$
\begin{aligned}
A =& \{a,b,c\} \cup \{u_i,v_i,x_i,y_i \mid i \in \{1,2\}\} \\
R =& \{\langle a,c\rangle,\langle b,c\rangle\} \cup \{\langle \alpha_i,\beta_i\rangle \mid i \in \{1,2\}, \alpha \in \{x,y\}, \beta \in \{u,v\}\} \\
& \cup \{(u_i,a),(a,x_i),(v_i,b),(b,y_i),\langle u_i,v_i\rangle \mid i \in \{1,2\}\}
\end{aligned}
$$

In the following we refer to $A_1 = \{v_1\}$, $A_2 = \{u_1\}$, $A_3 = \{x_1,y_1\}$, and $B_1 = \{v_2\}$, $B_2 = \{u_2\}$, $B_3 = \{x_2,y_2\}$ The stable extensions of F can be separated into extensions containing c and others. For $i,j \in \{1,2,3\}$ the former are given as:

$$S_{ij} = \{c\} \cup A_i \cup B_j$$

If on the other hand $c \notin S$ one of a,b must be a member of S and thus:

$$
\begin{aligned}
&S_1 = \{a,v_1,v_2\} && S_3 = \{a,v_1,y_2\} && S_5 = \{b,u_1,x_2\} \\
&S_2 = \{b,u_1,u_2\} && S_4 = \{a,y_1,v_2\} && S_6 = \{b,x_1,u_2\}
\end{aligned}
$$

For $S \in st(F)$ and w.l.o.g. $a \in S$ take into account that a is attacked by u_1 and the only possible defenders v_1 and y_1 are explicitly in conflict with b. Thus, clearly a and b share an implicit conflict, as one cannot be defended without the other being attacked. However observe that all the other conflicts implicitly defined by the extension-set $\mathbb{S} = \{S_1,S_2,\ldots,S_6\} \cup \{S_{ij} \mid i,j \in \{1,2,3\}\}$ are already given explicitly in F. Furthermore the remaining maximal conflict-free sets $S_a = \{a,y_1,y_2\}$ and $S_b = \{b,x_1,x_2\}$ do attack, respectively, neither b nor a and thus are not stable extensions of F.

We proceed by showing that the AF depicted in Figure 10.7 and discussed in Example 10.17 serves as a counterexample for ECC for stable semantics.

Theorem 10.18. *There are non-analytic AFs for stable semantics.*

Proof. Consider the AF F from Example 10.17 and recall its set of stable extensions $\mathbb{S}$. We will show that there is no AF G with $A_G = A_F$, $st(G) = \mathbb{S}$ and $(a,b) \in R_G$. Observe that for symmetry reasons we need not consider $(b,a) \in R_G$ and $(a,b) \notin R_G$. For a contradiction take such an AF as given.

The extensions containing c ensure that there is no attack in G between arguments c and α_i for $\alpha \in \{x,u,v,y\}$ and $i \in \{1,2\}$, or between α_1 and α_2. By definition any stable extension $S \in \mathbb{S}$ attacks all outside arguments, $(S,\alpha)^{att}$ for $\alpha \in A_G \setminus S$. That is we can make use of necessity of attacks for stable semantics (Theorem 9.24 and Lemma 9.21). Hence from $S_3 = \{a,v_1,y_2\}$ being a stable extension we conclude $(a,c)^{att}$ and $(\{a,y_2\},\alpha_2)^{att}$ for $\alpha \in \{x,u,v\}$. Similarly due to $S_4 = \{a,y_1,v_2\}$ we conclude that $(\{a,y_1\},\alpha_1)^{att}$ for $\alpha \in \{x,u,v\}$. But now by assumption $(a,b)^{att}$ and thus for $S_a = \{a,y_1,y_2\}$ we acquire full range, $(S_a,\alpha)^{att}$ for any $\alpha \in A_G \setminus S_a$, i.e. S_a becomes an unwanted stable extension. Therefore F is non-analytic. □

We observe that in this counterexample for ECC for stable semantics the stable extensions coincide with semi-stable, preferred, stage and stage2 extensions. With the following lemma this leads to some straightforward generalizations.

Lemma 10.19. *Let F be an AF with $pr(F) = st(F)$ (resp. $sm(F) = st(F)$). If F is quasi-analytic for preferred (resp. semi-stable) semantics, then it is also quasi-analytic for stable semantics.*

Proof. By assumption, for $\sigma \in \{pr, sm\}$, there is a σ-analytic AF G such that $A_G = A_F$ and $\sigma(F) = \sigma(G)$. We want to show that $st(G) = \sigma(G)$. Using the fact that for any AF F, $st(F) \subseteq \sigma(F)$ holds, it remains to show that $\sigma(G) \subseteq st(G)$. To this end observe that any attack of F still represents an explicit conflict in G. Now for $S \in st(F)$ we know that for all $a \in A_F \setminus S$ we have $(S,a)^{att}_F$. Since by assumption also $S \in \sigma(F)$ this immediately implies an explicit conflict between S and a in G. Due to admissibility of σ-extensions this means that actually $(S,a)^{att}_G$ as otherwise S would not defend itself from a in G. Therefore we have $(S,a)^{att}_G$ for all $a \in A_G \setminus S$. Hence $S \in st(G)$, resulting in $\sigma(G) = st(G)$ and thus G being st-analytic and also F being st-quasi-analytic. □

Using the AF F from Example 10.17 and the contraposition of Lemma 10.19 yields the following result, refuting ECC for preferred and semi-stable semantics.

Corollary 10.20. *There are non-analytic AFs for preferred and semi-stable semantics.*

The next example shows that some AFs prove to be non-analytic for preferred semantics while being quasi-analytic for all other semantics under consideration.

Example 10.21. Take into account the AF $F = (A,R)$ as depicted in Figure 10.8 with

$$A = \{a_i, b_i, x_i, u_i \mid i \in \{1,2,3\}\}$$
$$R = \{\langle a_i, b_i \rangle, (b_i, x_i), (x_i, u_i) \mid i \in \{1,2,3\}\} \cup \{(x_1,x_2),(x_2,x_3),(x_3,x_1)\}$$

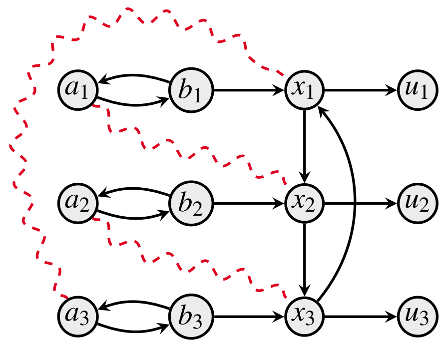


Figure 10.8: A non-analytic AF for *pr* as used in Example 10.21.

We have $pr(F) = \{S_a, S_b, A_1, A_2, A_3, B_1, B_2, B_3\}$ and

$$S_a = \{a_1, a_2, a_3\} \qquad S_b = \{b_1, b_2, b_3, u_1, u_2, u_3\}$$
$$A_1 = \{a_2, a_3, b_1, x_2, u_1, u_3\} \qquad B_1 = \{a_1, b_2, b_3, x_1, u_2, u_3\}$$
$$A_2 = \{a_1, a_3, b_2, x_3, u_1, u_2\} \qquad B_2 = \{a_2, b_1, b_3, x_2, u_1, u_3\}$$
$$A_3 = \{a_1, a_2, b_3, x_1, u_2, u_3\} \qquad B_3 = \{a_3, b_1, b_2, x_3, u_1, u_2\}$$

In the following we show that F is non-analytic for preferred semantics. For a contradiction we assume that there exists an analytic AF G with $A_G = A$ and $pr(F) = pr(G)$. We now investigate this hypothetical AF G. Observe that for $i, j \in \{1,2,3\}$ due to S_b there is no conflict between u_i and b_j, due to A_1, A_2, A_3 there is no conflict between u_i and a_j, and for $i \neq j$ there is no conflict between x_i and u_j; in other words in G the u_i can be attacked only by the x_i. Furthermore we have an implicit conflict between a_1 and x_2. Due to S_a being admissible and G being analytic now $(S_a, x_2)^{att}_G$. But then S_a defends u_2 and thus can not be a preferred extension in G. For symmetry reasons it follows that the implicit conflicts $[a_i, x_j]^{cnf}$ of F cannot be made explicit for preferred semantics.

On the other hand for stable (or stage or semi-stable) semantics we observe that S_a is not an extension. Although the overall conflicts remain the same, this allows us to include conflicts (x_j, a_i) without any harm for the other extensions. Thus for stable, semi-stable and stage semantics this AF is quasi-analytic.

Finally for $\sigma \in \{c2, s2\}$ we have $\sigma(F) = \{A_1, A_2, A_3, B_1, B_2, B_3, S_b\} \cup \{S_a \cup \{x_i, u_j, u_k\} \mid i \neq j \neq k \neq i \in \{1,2,3\}\}$. That is F is already analytic for *c2* and *s2* semantics.

Example 10.22. Consider the AF G depicted in Figure 10.9 which builds upon the AF F from Example 10.21 in that $A_G = A_F \cup \{y_1, y_2, y_3\}$ and $R_G = R_F \cup \{(a_1, y_i), (y_i, y_i) \mid i \in \{1,2,3\}$. Following the extension naming convention from Example 10.21 we even have $pr(G) = sm(G) = \{S_a, S_b, A_1, A_2, A_3, B_1, B_2, B_3\}$.

We now use this Example and in particular the extension set to show that for *pr*/*sm* semantics some AFs can not be realized analytically.

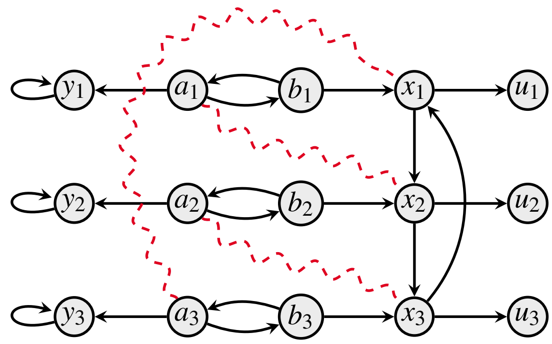


Figure 10.9: A non-analytic AF for *sm* as used in Example 10.22.

Theorem 10.23. *For $\sigma \in \{pr, sm\}$ there are σ-realizable extension sets $\mathbb{S}$ such that no realization F with $\sigma(F) = \mathbb{S}$ is analytic.*

Proof. Consider the extension set from Examples 10.21 and 10.22 and assume some analytic AF F with $\sigma(F) = \mathbb{S}$ as given. We have as semantic conflicts $[a_1,x_2]^{cnf}_{\mathbb{S}}$, $[a_2,x_3]^{cnf}_{\mathbb{S}}$, and $[a_3,x_1]^{cnf}_{\mathbb{S}}$ while there are independencies $\{a_i,x_j\}^{ind}_{\mathbb{S}}$ for all other combinations of indices. Hence with $S_a \in \mathbb{S}$ by analyticity we need $(a_1,x_2)^{att}_F$, $(a_2,x_3)^{att}_F$, and $(a_3,x_1)^{att}_F$. By necessity (Theorem 9.31) we have $(a_i,b_i)^{att}_F$ for $i \in \{1,2,3\}$ as observed for instance for $A_1 = \{a_2,a_3,b_1,x_2,u_1,u_3\}$ vs. $B_2 = \{a_2,b_1,b_3,x_2,u_1,u_3\}$. Further, given any rejected argument $\alpha \in A_F \setminus \bigcup \mathbb{S}$, analyticity requires $[S_a,\alpha]^{cnf}_F$ and admissibility thus $(S_a,\alpha)^{att}_F$. Now for $i, j \in \{1,2,3\}$ we have independencies $\{u_i,u_j\}^{ind}_{\mathbb{S}}$, $\{u_i,a_j\}^{ind}_{\mathbb{S}}$, $\{a_i,a_j\}^{ind}_{\mathbb{S}}$ and thus also for $S_x = \{a_1,a_2,a_3,u_1,u_2,u_3\}$ we get $\{S_x\}^{ind}$. By awareness (Lemma 9.8) then also $S_x \in cf(F)$ holds. But now we already have $S_x^* = A_F$ and thus $S_x \in st(F)$ which by Proposition 6.39 means $S_x \in \sigma(F)$ and thus an unwanted σ-extension. □

We still have not answered the question whether stage semantics possesses non-analytic AFs. A candidate for a non-analytic AF for stage semantics would be the AF F from Example 10.17, but it turns out to be quasi-analytic for stage semantics. In fact, the analytic AF G depicted in Figure 10.10 has the same stage extensions (and by strong connectedness also stage2) as F, $st(F) = sg(F) = s2(F) = sg(G) = s2(G)$.

Example 10.24 (Stage-analytic realization of Example 10.17)**.** Consider the AF F depicted in Figure 10.10. The purpose of this example is to provide a *sg*-analytic interpretation of the same extension set as for the AF from Example 10.17.

When restricting to the sub-AF $G = F|_{\{x_1,u_1,v_1,y_1\}}$ observe that we have as naive sets $na(G) = \{\{x_1,y_1\},\{u_1\},\{v_1\}\}$ where none has full range, i.e. $\{x_1,y_1\}^*_G = \{x_1,v_1,y_1\}$, $\{u_1\}^*_G = \{x_1,u_1,y_1\}$ and $\{v_1\}^*_G = \{u_1,v_1\}$, but all are incomparable. For $H = F|_{\{a,b,c\}}$ on the other hand we have $\{a\}^*_H = \{a,b\}$, $\{b\}^*_H = \{b,c\}$, $\{c\}^*_H = \{a,b,c\}$, i.e. $\{c\}$ has full range while the others do not.

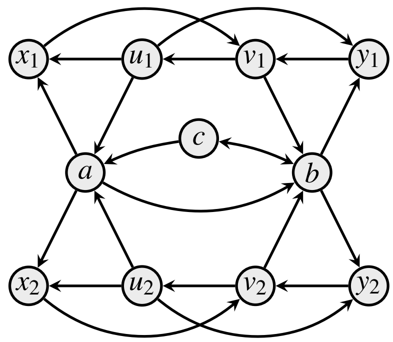


Figure 10.10: Analytic AF for stage semantics, cf. Examples 10.17 and 10.24.

In terms of possible candidates for sg extensions of F, similarly as in Example 10.17 we distinguish between naive sets $S \in na(F)$ containing arguments a, b or c. In case $c \in S$ for $i \in \{1,2\}$ we thus have one of $u_i \notin S_F^*$ (for $x_i, y_i \in S$), $v_i \notin S_F^*$ (for $u_i \in S$) or $x_i, y_i \notin s_F^*$ (for $v_i \in S$). In case $a \in S$ only $\{a, v_i, y_j \mid i, j \in \{1,2\}\}$ are possible due to conflict-freeness, similarly in case $b \in S$ only $\{b, u_i, x_j \mid i, j \in \{1,2\}\}$ are possible. We further on use the same notation of extension sets S_{ij} and S_i as in Example 10.17, and additionally $S_a = \{a, y_1, y_2\}$ and $S_b = \{b, x_1, x_2\}$.

For $a \in S$ for $i \in \{1,2\}$ we have $c, y_i \notin S_F^*$ for $v_i \in S$, or $c, v_i \notin S_F^*$ for $y_i \in S$. Thus S_a is smaller in range than S_{33} with c, v_1, v_2 as opposed to only v_1, v_2 missing. Sets S_1, S_3, S_4 on the other hand are incomparable in range to their S_{11}, S_{13}, S_{31} counterparts.

For $b \in S$ for $i \in \{1,2\}$ we have only $v_i \notin S_F^*$ for $u_i \in S$, or $u_i \notin S_F^*$ for $x_i \in S$ and in case $x_1, x_2 \in S$ additionally $a \notin S_F^*$. That is, S_2 has the same range as S_{22}. The sets S_5, S_6 have the same range as, resp. S_{23}, S_{32}, and S_b is smaller in range than S_{33} since it misses argument a.

That means that this AF provides the same sg extensions as the AF from Example 10.17 while providing all conflicts explicitly. Finally, since the AF consists of only one SCC we have $s2(F) = sg(F)$.

The following slightly more involved example yields a non-analytic AF for stage semantics.

Example 10.25. Take into account the AF $F = (A, R)$ depicted in Figure 10.11 with:

$$
\begin{aligned}
A &= \{a,b,c\} \cup \{u_i, v_i, x_i, y_i, r_i, s_i \mid i \in \{1,2\}\} \\
R &= \{\langle a,c \rangle, \langle b,c \rangle\} \cup \{\langle r_i, x_i \rangle, \langle s_i, y_i \rangle \mid i \in \{1,2\}\} \\
&\quad \cup \{\langle \alpha_i, \beta_i \rangle \mid i \in \{1,2\}, \alpha \in \{x,y\}, \beta \in \{u,v\}\} \\
&\quad \cup \{(u_i, a), (a, x_i), (v_i, b), (b, y_i), \langle u_i, v_i \rangle \mid i \in \{1,2\}\}
\end{aligned}
$$

In the following we will refer to $M_{i1} = \{r_i, v_i, s_i\}, M_{i2} = \{r_i, u_i, s_i\}, M_{i3} = \{r_i, y_i\}, M_{i4} = \{x_i, s_i\}, M_{i5} = \{x_i, y_i\}$. The stable extensions of F can be separated into extensions containing c and others. For $i, j \in \{1 \ldots 5\}$ the former are given as:

$$S_{ij} = \{c\} \cup M_{1i} \cup M_{2j}$$

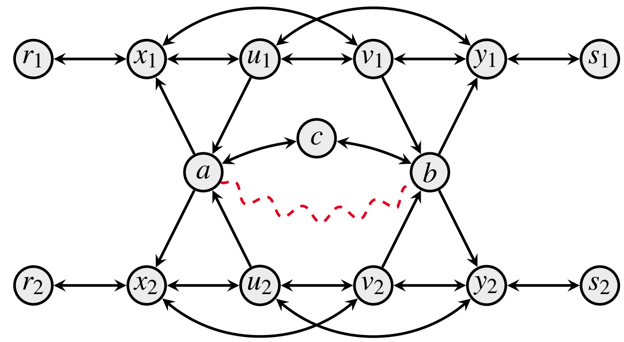


Figure 10.11: Illustration of the AF from Example 10.25.

If, on the other hand, $c \notin S$, one of a, b is a member of S:

$$S_1 = \{a, r_1, r_2, v_1, v_2, s_1, s_2\} \qquad S_4 = \{a, r_1, r_2, y_1, v_2, s_2\}$$
$$S_2 = \{b, r_1, r_2, u_1, u_2, s_1, s_2\} \qquad S_5 = \{b, r_1, u_1, x_2, s_1, s_2\}$$
$$S_3 = \{a, r_1, r_2, v_1, y_2, s_1\} \qquad S_6 = \{b, r_2, x_1, u_2, s_1, s_2\}$$

Similarly to Example 10.17 we have that a and b share an implicit conflict for stable and thus stage semantics, as $st(F) = sg(F) = \mathbb{S} = \{S_1 \dots S_6\} \cup \{S_{ij} \mid i, j \in \{1 \dots 5\}\}$. Again except for the implicit conflict between a and b all conflicts in F already are explicit, and the only other maximal conflict-free sets $S_a = \{a, r_1, r_2, y_1, y_2\}$ and $S_b = \{b, x_1, x_2, s_1, s_2\}$ are not stable extensions here.

Theorem 10.26. *There are non-analytic AFs for stage semantics.*

Proof. Consider the AF $F = (A, R)$ from Example 10.25. We first show that F is non-analytic for stable semantics by assuming a contradicting analytic AF G of the same arguments and extensions. We will then use this observation to proceed similarly for stage semantics. As for any AF G with $st(G) \neq \emptyset$ we have $st(G) = sg(G)$, we will assume some AF G which is analytic for stage semantics where $st(G) = \emptyset$. In fact for both, stable and stage semantics, we show a slightly stronger result; for the given extension-set the conflict between a and b has to be implicit for any compact realization. For symmetry reasons, w.l.o.g. we assume $(a, b) \in R_G$. In what follows, we use the same naming schema for extensions as in Example 10.25.

For stable semantics we need $(a, c)^{att}$, since e.g. S_1 has to be a stable extension. From $S_{33} \in st(G)$, $(a, b)^{att}$ by assumption and as observed $(a, c)^{att}$ we conclude $S_a \in st(G)$, as $c \in S_{33}$ is allowed to attack only a and b. Thus if G is sg-analytic and -compact, then $st(G) = \emptyset$.

We now turn to stage semantics and have the following observations:

- For $i \in \{1, 2\}$, due to maximal conflict-freeness and the given conflicts (Theorem 9.24), we need explicit conflicts between s_i and y_i, r_i and x_i ($r_i, s_i \notin S_{55}$), between c and a, c and b ($a \notin S_{33}$, $b \notin S_{44}$), and between u_i and v_i ($v_i \notin S_{22}$). We will frequently make use of these necessities in the following.

- For the explicit conflict between s_1 and y_1, we need $(s_1,y_1)^{att}$ for otherwise $S^*_{55} \subset S^*_{45}$. Similarly we conclude $(s_2,y_2)^{att}$, $(r_1,x_1)^{att}$ and $(r_2,x_2)^{att}$;

- As the conflict between c and a is explicit, furthermore necessarily $(c,a)^{att}$ for otherwise (in case $c \in a^+$ and $a \notin c^+$) $S^*_{11} \subset S^*_1$;

- Now since u_i and v_i need to be in conflict we need $b \notin c^+$ for otherwise at least one of S_{ij} for $i,j \in \{1,2\}$ becomes a stable extension. By necessity hence $(b,c)^{att}$.

- From $(c,a)^{att}$, $(r_1,x_1)^{att}$ and $(s_1,y_1)^{att}$ we conclude $(u_1,v_1)^{att}$ due to the danger of $S^*_{21} \subset S^*_{11}$. Similarly $(u_2,v_2)^{att}$.

- Since $(c,a)^{att}$ and $(u_i,v_i)^{att}$ furthermore we need $(x_i,r_i)^{att}$, $(x_i,u_i)^{att}$ and $(x_i,v_i)^{att}$, due to range comparison of M_{i4} and M_{i2}.

- By previous range observations we have to assume $a \notin b^+$ and $a \notin u_i^+$, for otherwise S_2 becomes a stable extension.

- But now $S^*_2 \subseteq S^*_b$, i.e. either we gain the unwanted extension S_b or we lose the desired extension S_2. □

With the following result we show that Example 10.25 also serves as non-analytic AF for *s2* semantics.

Theorem 10.27. *There are non-analytic AFs for stage2 semantics.*

Proof. First observe that the AF F from Example 10.25 is strongly connected and thus $s2(F) = sg(F) = st(F) = \mathbb{S}$ with namings as in Example 10.25. We now show that any compact *s2*-realization G of $\mathbb{S}$ with explicit attack $(a,b)^{att}_G$ is strongly connected and thus by Theorem 10.26 we have $s2(G) = sg(G) = st(G)$ and consequently $\{a,b\}^{ind}_G$.

- By Theorem 9.24 aside from $[a,b]^{cnf}_{\mathbb{S}}$ all conflicts among arguments that are explicit in F again need to be explicit in G;

- For $i \in \{1,2\}$ we have $(s_i,y_i)^{att}_G$ and $(r_i,x_i)^{att}_G$ for otherwise w.l.o.g. $S^*_{55} \subset S^*_{45}$ and in particular y_1 would SCC-defeat s_1;

- This means that for $i \in \{1,2\}$ also $(y_i,s_i)^{att}_G$ and $(x_i,r_i)^{att}_G$ hold since otherwise there would be no extension S at all where w.l.o.g. $y_1 \in S$;

- By semantic conflicts and necessity $[a,c]^{cnf}_G$ we conclude $(c,a)^{att}_G$ for otherwise S_1 SCC-defeats S_{11};

- Then we need $(b,c)^{att}_G$ for otherwise $\{c\}$ would need to be member of any *s2*-extension;

- Now observe that S_{11} and S_{22} differentiate only in containing u_1,u_2 or v_1,v_2, which means that for w.l.o.g u_1 not to SCC-defeat v_1 we need attacks $(u_i,v_i)^{att}_G$ and $(v_i,u_i)^{att}_G$ for $i \in \{1,2\}$;

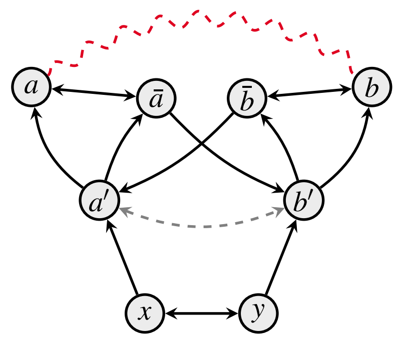


Figure 10.12: Illustration of the $c2$-non-analytic AF from Example 10.28.

- We further need attacks $(\alpha_i, \beta_i)^{att}$ for $i \in \{1,2\}$, $\alpha \in \{x,y\}$ and $\beta \in \{u,v\}$, for otherwise w.l.o.g. S_{11} dominates S_{44};
- Then we also need attacks from $\{u_i, v_i\}$ to $\{x_i, y_i\}$ for otherwise S_a becomes at least as potent as S_1, i.e. $\{r_i, x_i, u_i, v_i, y_i, s_i\}$ are strongly connected;
- Now to prevent SCC-connectedness of G we have to choose one direction for all conflicts $[\alpha, \beta_i]^{cnf}_{\mathbb{S}}$ for $\alpha \in \{a,b\}$ and $\beta \in \{x,u,v,y\}$;
- In either case we can not simultaneously allow for instance S_4 while disallowing S_a. $\square$

Observe that all examples of this section so far were $c2$-analytic. We continue by providing a non-analytic AF for $c2$ semantics.

Example 10.28. Consider the AF F depicted in Figure 10.12 with $A_F = \{x,y,a,a',\bar{a},b,b',\bar{b}\}$ and $R_F = \{\langle x,y\rangle, (\bar{b},a'), (\bar{a},b')\} \cup \{\langle \alpha,\bar{\alpha}\rangle, (\beta,\alpha'), (\alpha',\alpha), (\alpha',\bar{\alpha}) \mid \alpha/\beta \in \{a/x, b/y\}\}$. For $c2$-evaluation observe that as initial component we have $\{x,y\}$. For $S \in c2(F)$ with $x \in S$ we need $S \cap (A_F \setminus S^*_F) \in c2(F|_{A_F \setminus S^*_F})$ and thus consider the next component, which is $\{a,\bar{a}\}$. For $x,a \in S$ we consequently end up with $b' \in S$, while for $x,\bar{a} \in S$ we may choose between b or $\bar{b}$. Thus the syntactic conflict $[x,y]^{cnf}_F$ is transformed into a semantic conflict $[a,b]^{cnf}_{c2(F)}$ and we get

$$\mathbb{S} = c2(F) = \{\{x,a,b'\}, \{x,\bar{a},b\}, \{x,\bar{a},\bar{b}\}, \{y,b,a'\}, \{y,\bar{b},a\}, \{y,\bar{b},\bar{a}\}\}.$$

Similarly we get that $st(F) = pr(F) = \mathbb{S}$ and thus super-coherence. All $\mathbb{S}$-semantic conflicts are reflected in Figure 10.12, either by attacks between arguments, or, for $[a,b]^{cnf}$, by a dashed snake line, or, for $[a',b']^{cnf}$, by a dashed bidirectional arrow. Observe that all attacks of F are necessary conflicts for maximal conflict-free semantics (Theorem 9.24). For instance extension set $\{x,\bar{a},b\}$ has no other means of being in conflict with y than via x. Remarkably, for pr and st semantics addition of $(a',b')^{att}, (b',a')^{att}, (a,b)^{att}, (b,a)^{att}$ does not alter the semantic evaluation.

We now get to the interesting part of this example. We could add the attacks $(a',b'), (b',a')$ without modifying the evaluation. For a potential attack (b,a) however the second SCC (for $x \in S$) in above reasoning becomes $\{a,\bar{a},b',b,\bar{b}\}$ which for $c2$ semantics means we would gain extension set $\{x, a, \bar{b}\}$. While we do not have readable proof of this AF construction being non-analytic, we have computational evidence. Here we give credits to Thomas Linsbichler for computing and evaluating all $3^{13} = 1594323$ possible candidates (by means of 13 semantic conflicts being realized in one or the other direction or both).

The lack of a short proof for above example might be considered a question of missing knowledge. Since for $\mathbb{A}$-realizability, $\mathbb{A}$-necessity and $\mathbb{A}$-purity this thesis anyway did not succeed in solving detailed characterizations we gladly point out that Example 9.32 as discussed in Subsection 10.3.1 provides a readable proof of $\mathbb{A}$-purity and thus non-analyticness for *c2* semantics.

To conclude this section we investigate the question of conditions such that ECC holds. We have mentioned earlier that every AF is quasi-analytic for naive semantics. This insight can be generalized as follows.

Proposition 10.29. *Let $\sigma \in \{sg, st, sm, pr, c2, s2\}$. In ZFC, if for some AF F there exists an AF G such that $\sigma(F) = na(G)$, then F is quasi-analytic for σ.*

Proof. Let F, G be AFs with $\sigma(F) = na(G)$. We define the AF H with $A_H = A_F$ and $R_H = \{(x,y) \mid x, y \in A_F, [x,y]^{cnf}_{\sigma(F)}\}$. As this AF H provides the same conflicts as the AF G for naive semantics, we deduce that also the maximal conflict-free sets are the same, $na(H) = na(G)$. By definition of H, for any $S \in na(H)$ and $a \in A_F \setminus S$ we have $(S,a)^{att}_H$ and hence S is a stable extension of H. Finally observe that $st(H) \subseteq \sigma(H) \subseteq na(H)$ for any AF H, hence the result follows. □

Another property that guarantees that ECC holds relies on the existence of what we call "identifying arguments". We say that an AF F is *determined* for semantics σ if for every $S \in \sigma(F)$ there exists an $a \in S$ such that for $S' \in \sigma(F)$ we have that $a \in S'$ implies $S' = S$. In other words, every σ-extension contains an identifying argument in the sense that it does not occur in any other σ-extension. A simple necessary condition for an AF to be determined for σ is that the number of σ-extensions does not exceed the number of arguments.

Proposition 10.30. *Let $\sigma \in \{st, pr, sm, sg, c2, s2\}$. Then, any AF F determined for σ is quasi-analytic for σ.*

Proof. Consider an AF F determined for σ and for each $S \in \sigma(F)$ let a_S be some fixed identifying argument. Now taking into account the sets $I = \{a_S \mid S \in \sigma(F)\}$ and $R_I = \{\langle a_S, a_{S'}\rangle \mid S, S' \in \sigma(F), S \neq S'\}$, clearly $\sigma((I, R_I)) = \{\{a_S\} \mid S \in \sigma(F)\}$. Furthermore let $O = A_F \setminus I$ be the remaining arguments of F and $R_O = \{\langle a,b\rangle \mid a, b \in O, [a,b]^{cnf}_{\sigma(F)}\}$. We define G as $A_G = A_F = O \cup I$ and $R_G = R_I \cup R_O \cup \{(a_S, b) \mid S \in \sigma(F), b \in (O \setminus S)\}$.

Observe that I forms a clique within G, a clique that is not attacked by arguments from O. Further I is the initial SCC of G. This means that for any σ-extension T we have $T \cap I = \{a_S\}$ for some $S \in \sigma(F)$. For *st* there is no other possibility of having I in range, for *pr*/*s2*/*c2* this

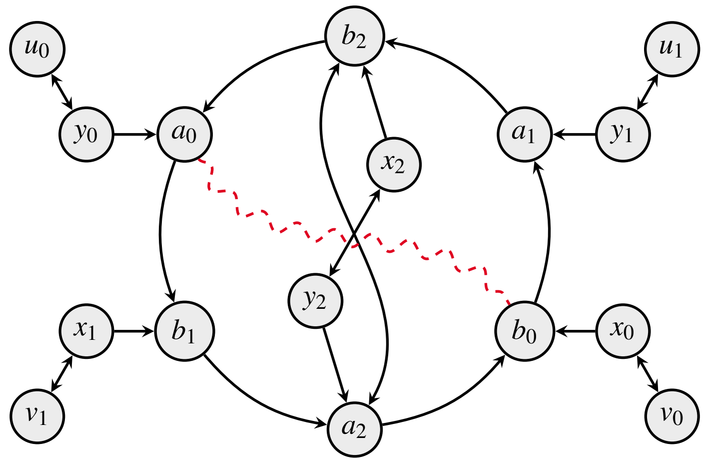


Figure 10.13: AF with compact pure conflict for all semantics of interest, cf. Theorem 10.31.

is by directionality, for *sg*/*sm* we show that $st(G) \neq \emptyset$. Thus, given $a_S \in T$, observe that by construction a_S attacks any $x \in A_G \setminus S$. Consequently we have as AF for further evaluation $G_S = G|_S$. By construction the identity $G_S = (S, \emptyset)$ holds. By I-maximality then $T^*_G = A_G$ and $T = S$ follow. Thus $\sigma(G) = \sigma(F)$.

Finally observe that all conflicts in G for σ (among I, among O or between I and O) are explicit by definition. $\square$

10.3 Restricted Purity

As elaborated on in Lemma 10.2 for general AFs and the semantics of interest there are no pure conflicts. In this section we show cases where purities do play a role. To this end in Subsection 10.3.1 we consider $\mathbb{A}$-realizations, where for input extension set $\mathbb{S}$ aside from arguments $\bigcup \mathbb{S}$ we may make use of arguments from $\mathbb{A}$, that is realizations F with $A_F \subseteq \bigcup \mathbb{S} \cup \mathbb{A}$ (see Definition 5.33). In Subsection 10.3.2 we then discuss multi-dimensional signatures and their possible need for general purities.

10.3.1 Compact Purity

In this subsection we again discuss Example 9.32 as illustrated in Figure 10.13. Before we do that however, we point out that the exemplary AFs from Section 10.2 mostly deal with compact purities. The benefit of Example 9.32 is twofold. Firstly it works for all semantics simultaneously and with rather clear line of argumentation. This clarity is due to its bipartite structure which is also the second benefit, a planar and bipartite AF with compact pure conflict.

Theorem 10.31 (Compact purity)**.** *For semantics $\sigma \in \{pr, sm, st, sg, s2, c2\}$ there are AFs with compact pure conflicts.*

Proof. Consider the AF F from Example 9.32, illustrated in Figure 10.13. Recall that for all semantics σ of interest (see Section 9.4) we have the same extension set $\mathbb{S}$ (we refer to Definition 9.33 for a naming convention of the extensions) and further for any compact σ-realization F of $\mathbb{S}$ we have $(b_2,a_0),(a_0,b_1),(b_1,a_2),(a_2,b_0),(b_0,a_1),(a_1,b_2) \in R_F$, see Subsection 9.4.1 for the why.

Consider AF G with $\mathbb{S} = \sigma(G) = \sigma(F)$ and $A_G = \bigcup \mathbb{S}$ and assume $(a_0,b_0)^{att}_F$. We claim that then also the set $U = \{u_0,u_1,v_0,v_1,y_2,a_0,a_1\}$ becomes an extension while in F this particular set is dominated by $S(uuvvy) = \{u_0,u_1,v_0,v_1,y_2,b_0,b_2\}$.

For *st* and *sg* semantics observe that by Lemmata 9.35 and 9.40. also the attacks $(u_0,y_0)^{att}$, $(u_1,y_1)^{att}$, $(v_0,x_0)^{att}$, $(v_1,x_1)^{att}$, $(y_2,a_2)^{att}$, $(y_2,x_2)^{att}$ are necessary. Consequently we get $U^*_G = A_G$, i.e. U is an unwanted stable extension.

For *pr* and *sm* semantics, as highlighted in Lemma 9.34, we have necessary attacks $\langle x_2,y_2\rangle$ and $\langle u_i,y_i\rangle,\langle x_i,v_i\rangle$ for $i \in \{0,1\}$. With $(a_0,b_0)^{att}$ and the established inner 6-cycle then U becomes admissible with $A_G \setminus U^*_G \subseteq \{a_2\}$. If $a_2 \notin U^+_G$ then U defends a_2 resulting in the stable extension $U \cup \{a\}$. Hence, with $S(uuxvy) \in \sigma(F)$, also $(y_2,a_2)^{att}_G$ holds and thus already U is a stable and thus preferred and semi-stable extension.

For *c2*/*s2* semantics, consider the set $U_0 = \{u_0,u_1,v_0,v_1,y_2\}$. Observe that with $(a_0,b_0)^{att}_G$ we have $A_0 = \{a_0,b_0,a_1,b_2\}$ strongly connected even when cutting out the remains of U_0, i.e. in $G|_{A_F \setminus U^*_0}$. Due to $b_0,b_1,x_2 \in S(yuvvx)$ we have that b_2 can not SCC-precede x_2. Then however since $\{b_0,b_1,b_2\} \subseteq S(uuvvy)$ also U becomes acceptable. □

Remark 10.32. Observe that for *c2* semantics, similarly to Remark 9.38 in above proof of Theorem 10.31, given finite realizations, compactness is not necessary anymore. Thus in opposition to all other semantics under consideration we have that *c2* semantics can provide finite-pure conflicts.

10.3.2 Multi-dimensional Purity

In this subsection we investigate another form of purity, given a combination of semantics. Two-dimensional signatures were first investigated in [DSLW16]. While we did provide results for most combinations there, our constructions were still bound to start from analytic AFs. Opposed to our signature results from Section 9.3 the signature results from the literature rely on so called canonical AFs F, where given extension set $\mathbb{S}$ for arguments $x,y \in \bigcup \mathbb{S}$ and conflict $[x,y]^{cnf}_{\mathbb{S}}$ we have $(x,y)^{att}_F$ and $(y,x)^{att}_F$. Such realizations then are also called *natural realizations*. Further building on Definition 5.30 we say that AF F realizes $\langle \mathbb{S},\mathbb{T}\rangle$ for $\langle \sigma,\tau\rangle$ if $\sigma(F) = \mathbb{S}$ and $\tau(F) = \mathbb{T}$. The following example turns out to block the approach with canonical AFs in that it requires pure conflicts for simultaneous realization of its *pr* and *sm* extension sets.

Example 10.33. Consider the AF $F = (A,R)$ as depicted in Figure 10.14 with arguments b_{xy},c_y,d_y $(x,y \in \{1,2\})$, symmetric 3-cycles over arguments a^i_{xy},u^i and directed 3-cycles over arguments e^i_{xy} $(i \in \{1,2,3\})$. Symmetric arrows between regular arguments (or symmetric 3-cycles) and symmetric 3-cycles indicate symmetric attacks between each of the involved arguments. For instance b_{12} as well as a^3_{12} attack and are attacked by u^1,u^2,u^3. Directed arrows

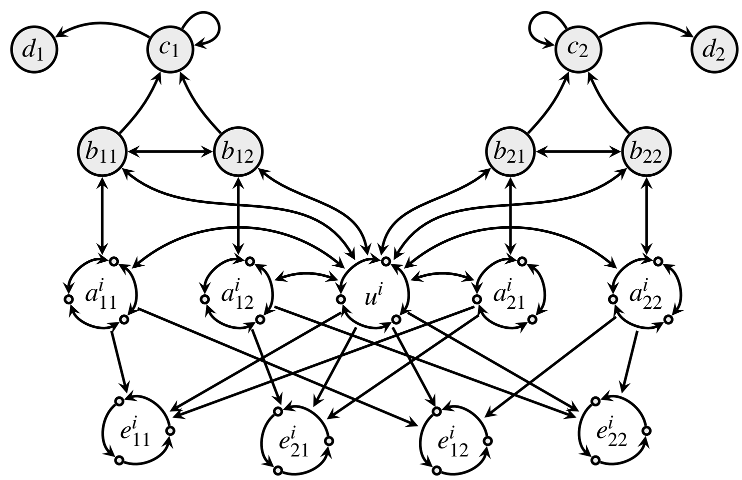


Figure 10.14: F s.t. $\langle sm(F), pr(F)\rangle$ is not naturally realizable.

between symmetric 3-cycles α^i and directed 3-cycles β^i represent attacks (α^i, β^j) for $i \neq j$. For instance a^2_{12} defends e^2_{12} by attacking e^1_{12} and e^3_{12}. Thus a^i_{xy} is in symmetric attack relationship with b_{xy} and with u^j, while e^i_{xy} can be "activated" by a^i_{1x} as well as by a^i_{2y}. To capture this activating relationship, in what follows we denote, for $S \subseteq A$, by $S^{\uparrow}_e$ the union of S and the arguments e^i_{xy} defended by S in F.

The preferred extensions are as follows (with $i, j, k, l \in \{1,2,3\}$ and $x \neq y, x' \neq y' \in \{1,2\}$):

$$E_u = \{u^i, e^i_{11}, e^i_{12}, e^i_{21}, e^i_{22}\}, \qquad E_{ab} = \{a^i_{x1}, a^j_{x2}, a^k_{yx'}, b_{yy'}, d_y\}^{\uparrow}_e,$$
$$E_a = \{a^i_{11}, a^j_{12}, a^k_{21}, a^l_{22}\}^{\uparrow}_e, \qquad E_b = \{a^i_{1x}, a^j_{2x'}, b_{1y}, d_1, b_{2y'}, d_2\}^{\uparrow}_e.$$

Observe that $E^*_u = E^*_a \subset E^*_{ab}$ and E^*_b is incomparable to E^*_{ab}, hence $sm(F)$ only consists of E_{ab} and E_b. Note that E_b misses some e^i_{xy} in range, while E_{ab} misses some d_x in range; thus any realization of $\langle sm(F), pr(F)\rangle$ has some $e^i_{xy} \notin E^*_b$ and $d_x \notin E^*_{ab}$, as members of $pr(F)$ must be conflict-free.

Now observe that the u^i never occur together with d_x in any semi-stable or preferred extension, i.e. we have $[u_i, d_x]^{cnf}$ while $\{u_i, d_x\}^{ind}_F$ holds, an implicit conflict. Canonically realizing $\langle sm(F), pr(F)\rangle$ has to make all implicit conflicts symmetrically explicit, in this case by adding attacks $(u^i, d_x), (d_x, u^i)$. But then we get $d_x, e^i_{xy} \in E^*_u$ for $x, y \in \{1,2\}, i \in \{1,2,3\}$ and thus E^*_u contains or is at least incomparable to E^*_{ab} and E^*_b, which means that E_u cannot be excluded from the semi-stable extensions.

We can now state the following result.

Theorem 10.34. *There are AFs F with pure conflict $[x,y]^{cnf}_{pr(F)}$, $[x,y]^{cnf}_{sm(F)}$, $\{x,y\}^{ind}_F$; that is for any 2-dimensional realization G of $\langle pr(F), sm(F)\rangle$ we still have the implicit conflict $[x,y]^{cnf}_{pr(F)}$, $[x,y]^{cnf}_{sm(F)}$, $\{x,y\}^{ind}_G$.*

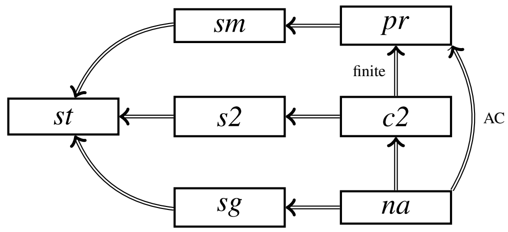


Figure 10.15: Relations between Conflicts for different semantics, cf. Remark 10.35.

10.4 Conclusions

In this chapter we studied implicit conflicts in abstract argumentation. Lemma 10.2 left us with the observation that for any implicit conflict there are semantic modifications getting rid of that conflict. Thus purity in its strongest form is not possible for the semantics under consideration. We then gave an analysis of AFs without implicit conflicts in Section 10.1, an investigation into the nature of conditions under which we might or might not be able to get rid of all implicit conflicts (instead of just one specific) in Section 10.2, and results indicating that for specific cases, such as compact or finite AFs (Subsection 10.3.1) or multi-dimensional realizations (Subsection 10.3.2) purity actually occurs. Our results build on and are inspired by [DW11, DDLW15, DS17] and give a more fine-grained landscape regarding the expressive power of semantics when the shape of AFs is restricted.

Remark 10.35. We have not formally discussed relations between conflicts yet but feel that this fits nicely into this conclusion. Depicted in Figure 10.15 find the conflict relations between semantics of interest. An arrow from semantics σ to semantics τ (with label s) means that, given some AF F (fulfilling s), σ-semantic conflicts of F are always also τ-semantic conflicts.

Most of these relations are straightforward, derived from the subset relations of Proposition 6.39. If every σ-extension is also a τ-extension then τ-semantic conflicts necessarily are σ-semantic. Simple examples built upon acceptance of arguments highlight that the relations are strict. In finite AFs further every *pr* extension is contained in some *c2* extension, in ZFC every *pr* extension is contained in some *na* extension. In infinite AFs *c2* semantics may collapse even with AC. Without AC, cf. Chapter 7, *pr* and *na* semantics may independently collapse.

In Table 10.2 we have illustrated how quasi-analytic AFs relate to established AF classes of interest. A checkmark in line s and column σ means that s AFs are quasi-analytic for semantics σ, a dash means that for this AF class there are non-analytic AFs, a question mark means that we do not know yet. In Table 10.2a we present the restrictions for finite AFs, in Table 10.2b we present possibly infinite AFs in models of ZFC, in Table 10.2c we present models without AC. Observe that *st* semantics is quasi-analytic for collapsing AFs, while all the other semantics are not. The dashes in the tables are either due to the finite, bipartite, planar and compact Example 9.32 or a collapse from Chapter 7. For naive semantics any AF in ZFC is quasi-analytic. Finite circle-free AFs are cycle-free and even well-founded by definition. Well-founded AFs provide exactly one extension for $\sigma \in \{pr, c2, st, sg, sm, s2\}$ and are thus

	na	*pr*	*c2*	*st*	*sg*	*sm*	*s2*
circle-free	✓	✓	✓	✓	✓	✓	✓
cycle-free	✓	✓	✓	✓	✓	✓	✓
well-founded	✓	✓	✓	✓	✓	✓	✓
symmetric	✓	✓	✓	?	?	?	?
symmetric loop-free	✓	✓	✓	✓	✓	✓	✓
bipartite	✓	-	-	-	-	-	-
planar	✓	-	-	-	-	-	-

(a) Finite AFs

	na	*pr*	*c2*	*st*	*sg*	*sm*	*s2*
cifr.	✓	?	?	?	?	?	?
cyfr.	✓	?	-	?	-	-	-
wf.	✓	✓	✓	✓	✓	✓	✓
sym.	✓	✓	✓	?	-	-	-
sylf.	✓	✓	✓	✓	✓	✓	✓
bip.	✓	-	-	-	-	-	-
pln.	✓	-	-	-	-	-	-

(b) Arbitrary cardinalities with AC

	na	*pr*	*c2*	*st*	*sg*	*sm*	*s2*
cifr.	-	-	-	?	-	-	-
cyfr.	-	-	-	?	-	-	-
wf.	✓	✓	✓	✓	✓	✓	✓
sym.	-	-	-	?	-	-	-
sylf.	-	-	-	?	-	-	-
bip.	-	-	-	-	-	-	-
pln.	-	-	-	-	-	-	-

(c) Arbitrary cardinality without AC

Table 10.2: Quasi-analyticity

quasi-analytic. For symmetric AFs *na*/*pr*/*c2* semantics coincide, for symmetric loop-free AFs this applies to all semantics of interest.

In reference to Theorem 10.7 with a detailed listing of relations between analytic AFs for different semantics we remark that we handled the comparison of *pr* and *c2* semantics for finite AFs only. That is finite *pr*-analytic AFs are *c2*-analytic as well. For infinite AFs we might have collapse (see Chapter 7) of *c2* semantics, which might be hindrance in generalizing this to the infinite case. In short, we do not know enough yet about collapse of *c2* semantics to assume one or the other. Thus in particular for *c2* semantics there is still space for improvement.

The explicit conflict conjecture was originally posed in [BDL$^+$14]. Theorem 10.18 on the refutation of the explicit conflict conjecture can be seen as one of the main results of this chapter and this thesis. Particularly Example 9.32 and its describing Theorem 10.31 can be seen as the culmination of our work on implicit conflicts for single semantics so far.

In reference to Theorem 10.34 we may now ask the question whether our result on realizability with necessary conflicts (Theorem 9.31) allows for a characterization of $\langle pr, sm\rangle$-realizability. In [DSLW16] we give an account of two-dimensional signatures for all other admissible cases. Thus this and similar questions for semantics *sg*, *s2*, *c2* remain open. Observe that, given *pr*-realizable extension set $\mathbb{S}$ and compatible *sm*-realizable extension set $\mathbb{T}$, the following holds:

- $\mathbb{T} \subseteq \mathbb{S}$;
- $[x,y]^{cnf}_{\mathbb{S}} \implies [x,y]^{cnf}_{\mathbb{T}}$;
- $\mathbb{T}$-necessary conflicts are $\mathbb{S}$-necessary.

We can thus assume $\langle pr, sm\rangle$-constructions to be bound to *pr*-necessary conflicts. Symmetric attack realizations should be fine though, since admissible sets need to defend themselves anyway. Observe that, given *pr*-realizable extension set $\mathbb{S}$ and AF F incorporating all $\mathbb{S}$-necessary conflicts in a symmetric way, we implicitly define a semilattice on $\mathbb{S}$ via the range in F. That is, F specifies possible *sm* extensions in that local and global maxima need to be worked in.

Finally a few words on the difference between non-analyticity and purity. Clearly pure conflicts result in non-analytic AFs. The reverse however (although it seems to hold for the presented examples) might not be the case. To see this consider some σ-compact AF G with σ-pure conflict $[a,b]^{cnf}$, its isomorphic and disjoint copy $G' = \mathit{rename}(G)$ and their union $H = G \cup G'$. Clearly H now has two σ-pure conflicts $[a,b]^{cnf}$, $[a',b']^{cnf}$ and is still non-analytic. We expand this AF with a single argument x resulting in AF F with $A_F = A_G \cup \{x\}$ and $R_F = R_G \cup \{(x,x), (\alpha, x) \mid \alpha \in A_G\}$. Now however, for all considered semantics except *c2*, the devised techniques allow to facilitate x to make one but only one of the implicit conflicts explicit in a modification F' while still providing the same σ-extensions. This means that F does not provide pure conflicts, however F is still non-analytic.

Chapter 11

Conclusions

Hamlet, Act 4, Scene 5

Ophelia: We know what we are now,
but not what we may become.

William Shakespeare [Sha16]

After a wild journey through set theory, existence of extensions and conflicts, in this Chapter we conclude, connect the dots and point out possible future research directions.

Throughout the thesis we have used various syntactic and semantic modifications and discussed various local and global properties. In this chapter we step away from atomic description and let a selection of results speak for themselves.

To this end we reuse figures and tables mostly collected in previous chapters with a reference to their origin in their captions. In Section 11.1 we give an overview of AF classes and their relations with the presented results. In Section 11.2 we give an overview of relations between semantics regarding some given AF and specific issues of interest. Finally, in Section 11.3 we put focus on what the work presented might lead to in the future.

For a more fine-grained recapitulation, highlighting of important results and possible future work as well as related literature we refer to the conclusions of each chapter in Parts II and III.

11.1 Graph-theoretic classification

As first figure of interest we present Figure 11.1, illustrating relations between AF classes of interest. The importance of this illustration is that it allows us to more directly categorize the subsequent results. Observe that planar AFs as well as compact and analytic AFs are not covered by this figure. Planar AFs are in no relation to any of the other AF classes. Compact and analytic AFs are semantic AF classes, besides compact $\Longrightarrow$ loop-free they do not appear to be in any significant relationship to the other classes either.

Next we have Figure 11.2, illustrating the relationship between AF classes with semantic equivalence. For instance in circle-free AFs *st*, *sm*, *pr*, *sg*, *s2*, and *c2* semantics always produce

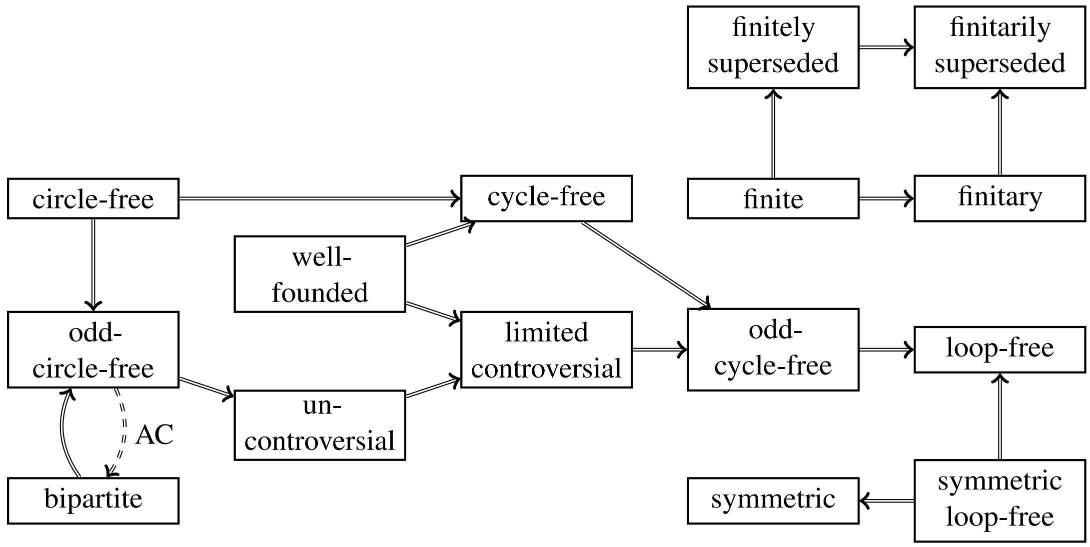


Figure 11.1: Syntactic AF classes put in relation, cf. Figure 3.5.

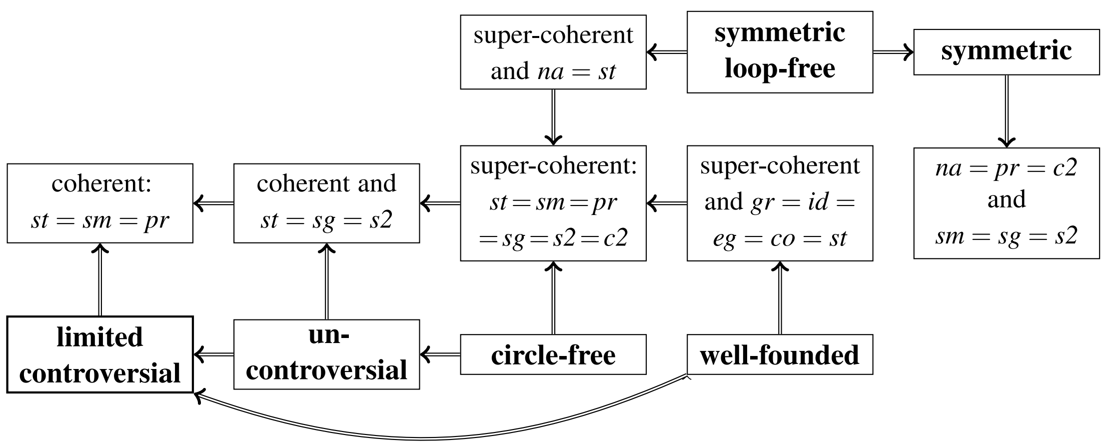


Figure 11.2: Semantic equivalence in light of AF classes, cf. Figure 6.5.

the same extension set. This figure particularly illustrates the tightness of our results in light of Figure 11.1. For instance cycle-free AFs do not guarantee coincidence for any of the semantics of interest. This is largely due to Example 7.21 of a cycle-free AF with collapse for range-based semantics. Also observe that finite and thus finitary, finitely or finitarily superseded AFs may yield different extension sets for any two of the semantics considered. This figure also represents the main results of Chapter 6 and thus the relations between semantics in arbitrarily infinite AFs. Another important aspect of Chapter 6 is our first view on argumentation in respect of ZF as well as ZFC, see Section 6.6.

In Chapter 7 we then present a systematic collection of AFs collapsing for semantics of interest. The regarding AF classification is illustrated by Table 11.1, where ZF refers to possible collapse in models without AC, while a checkmark indicates that an AF class allows a collapse already for models of ZFC. Particularly observe the question marks indicating open questions

	na	*pr*	*id*	*eg*	*st*	*sg*	*sm*	*c2*	*s2*
cycle-free	ZF	ZF	ZF	ZF	✓	✓	✓	✓	✓
circle-free	ZF	ZF	ZF	ZF	ZF	ZF	ZF	ZF	ZF
symmetric	ZF	ZF	ZF	ZF	✓	✓	✓	ZF	✓
symmetric loop-free	ZF	ZF	ZF	ZF	ZF	ZF	ZF	ZF	ZF
finitary	ZF	ZF	ZF	ZF	✓	ZF	ZF	?	?
planar	ZF	ZF	ZF	ZF	✓	?	✓	?	?
finitely superseded	ZF	ZF	ZF	ZF	✓	✓	-	✓	✓
finitarily superseded	ZF	ZF	ZF	ZF	✓	✓	✓	✓	✓

Table 11.1: Collapse in light of AF classes, cf. Table 7.2.

	co	*na*	*pr*	*st*	*sg*	*sm*	*c2*	*s2*	*gr*	*id*	*eg*
well-founded	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
bipartite	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
finite	✓	✓	✓	-	✓	✓	✓	✓	✓	✓	✓
limited controversial	✓	AC	AC	AC	AC	AC	AC	AC	✓	AC	AC
symmetric loop-free	✓	AC	AC	AC	AC	AC	AC	AC	✓	AC	AC
finitary	✓	AC	AC	-	AC	AC	?	?	✓	AC	AC
symmetric	✓	AC	AC	-	-	-	AC	-	✓	AC	AC
planar	✓	AC	AC	-	?	-	?	?	✓	AC	AC
finitely superseded	✓	AC	AC	-	-	-	-	-	✓	AC	AC
finitarily superseded	✓	AC	AC	-	-	-	-	-	✓	AC	AC
arbitrary	✓	AC	AC	-	-	-	-	-	✓	AC	AC

Table 11.2: Perfection in light of AF classes, cf. Table 8.1.

for finitary/planar AFs. Further observe that for the AF classes of interest the only case, where a collapse is not possible, is for finitely superseded AFs.

The dual question of collapse, perfection as elaborated on in Chapter 8, is covered by Table 11.2. Here AC means that perfection is granted for models of ZFC, while a checkmark means that perfection is ensured already in ZF. The question marks again represent open questions and one-to-one correspond to open questions from Table 11.1. Observe that Tables 11.2 and 11.1 complement each other, particularly in light of Figure 3.5. Perfection is defined as the absence of collapse. Thus the collapse of some semantics is possible/impossible for one particular AF class if and only if for this class we have that perfection is violated/ensured. This rough equivalence of course only applies to syntactic AF classes and local properties. For instance by definition compact AFs can not collapse, yet induced sub-AFs thereof might not be compact anymore and might even collapse.

For Part III and relations with AF classes we present Table 11.3 for illustration of AF classes that ensure quasi-analyticity. Observe that most results here are due to the results presented in Figure 11.2 or Table 11.1.

For collapse, perfection and basic properties of abstract argumentation we were able to draw very detailed pictures in light of arbitrary infinities in Part II. For conflicts, in Part III our main achievement is to provide a concise formalism. The results provided there are mostly of

	na	*pr*	*c2*	*st*	*sg*	*sm*	*s2*
circle-free	✓	✓	✓	✓	✓	✓	✓
cycle-free	✓	✓	✓	✓	✓	✓	✓
well-founded	✓	✓	✓	✓	✓	✓	✓
symmetric	✓	✓	✓	?	?	?	?
symmetric loop-free	✓	✓	✓	✓	✓	✓	✓
bipartite	✓	-	-	-	-	-	-
planar	✓	-	-	-	-	-	-

(a) Finite AFs

	na	*pr*	*c2*	*st*	*sg*	*sm*	*s2*
cifr.	✓	?	?	?	?	?	?
cyfr.	✓	?	-	?	-	-	-
wf.	✓	✓	✓	✓	✓	✓	✓
sym.	✓	✓	✓	?	-	-	-
sylf.	✓	✓	✓	✓	✓	✓	✓
bip.	✓	-	-	-	-	-	-
pln.	✓	-	-	-	-	-	-

(b) Arbitrary cardinalities with AC

	na	*pr*	*c2*	*st*	*sg*	*sm*	*s2*
cifr.	-	-	-	?	-	-	-
cyfr.	-	-	-	?	-	-	-
wf.	-	✓	✓	✓	✓	✓	✓
sym.	-	-	-	?	-	-	-
sylf.	-	-	-	?	-	-	-
bip.	-	-	-	-	-	-	-
pln.	-	-	-	-	-	-	-

(c) Arbitrary cardinality without AC

Table 11.3: How analytic AFs relate to other AF classes, cf. Table 10.2.

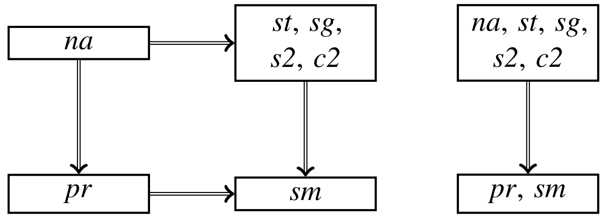
exemplary value. It should be highlighted though that we did succeed in giving characterization theorems for necessary conflicts. To some extent this is muffled by our use of arbitrarily infinite AFs and thus collapsing sub-AFs in our constructions. Said constructions require infinite AFs only for *s2*/*c2* semantics which are for various reasons not very commonly used. For pure conflicts, or quasi-analytic AFs this also leaves ample room for future work.

11.2 Semantic Relations

In Figure 11.3 we present results of realizability in the arbitrarily infinite case. For instance in models of ZF without AC we have that *na*-realizable extension sets are *pr*-realizable, or that *pr*-realizable extension sets are *sm*-realizable but might not be *na*-realizable. By definition the arrows also represent exact intertranslatability. These results make massive use of necessary conflicts and complement work from the literature such as [DW11, Spa13, DS17, DDLW15].

All of the semantic relations established build upon the prime relation of extension containment depicted in Figure 11.4. Given some AF F, for instance any *sm* extension is also a *pr* and *co* extension but might not be a *st* or *s2* extension. These results were partially known before this thesis. Our contribution mainly is to expand the area of applicability to the arbitrarily infinite case and even to models of ZF without AC. Observe that by Example 4.23 the original SCC-recursiveness is not well defined in the infinite case. We hence make use of the alternative Definition 4.24.

In Figure 11.5a we have depicted collapse relations. Given some AF F, then for instance $sg(F) = \emptyset$ implies $st(F) = \emptyset$ while we might still have $sm(F) \neq \emptyset$. This figure directly cor-



(a) ZFC (b) ZF, ¬AC

Figure 11.3: Realizability/Intertranslatability relations, cf. Figure 9.14.

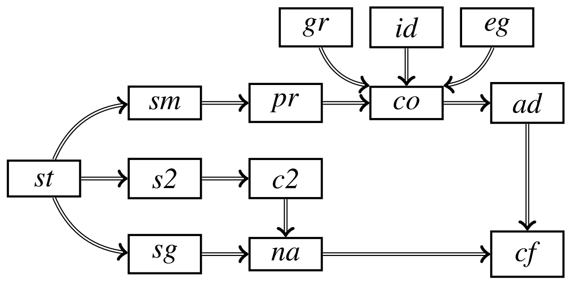
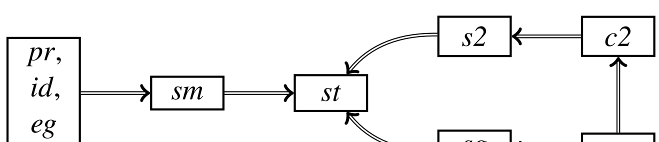


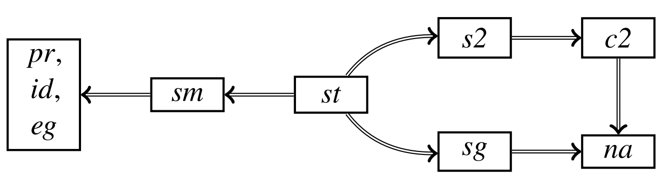
Figure 11.4: Relations between extensions for various semantics, cf. Figure 6.3.

responds to Figure 11.5b, where the perfection relations are illustrated. Both figures draw from Figure 11.4. Perfection means that each induced sub-AF provides extensions, since *sm*-extensions are always also *pr*-extensions thus *sm*-perfection implies *pr*-perfection. Similarly *pr*-collapse needs to imply *sm*-collapse. The main purpose of these two figures is to illustrate that there are no other relations than the ones we expect. This was elaborated by examples mostly in Chapter 7. The attentive reader might observe that Figure 11.5b is not referenced to and indeed not presented at any other place. This is simply due to the close resemblance to Figure 11.5a which makes at least one of the figures somewhat redundant.

Figure 11.6 shows relations between semantics for various forms of conflict. Observe that although these definitions are very closely related, we do find differences in the respective

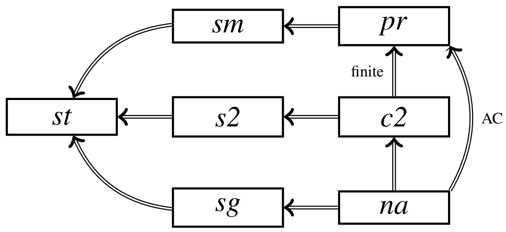


(a) Collapse, cf. Figure 7.14

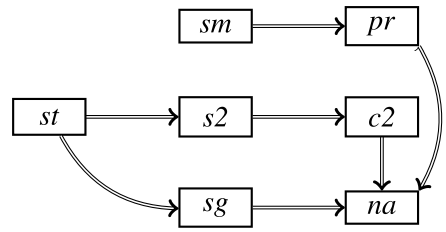


(b) Perfection

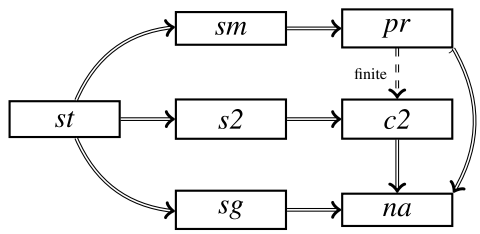
Figure 11.5: Relations between semantic occurrence of global properties.



(a) Semantic conflict, cf. Figure 10.15.



(b) Necessary conflict, cf. Figure 9.13.



(c) Analytic AF, cf. Figure 10.6.

Figure 11.6: Conflict relations for the semantics of interest.

relations. For instance *sm*-necessary conflicts might not be related to *st*-necessary conflicts (Figure 11.6b) while each *sm*-semantic conflict is also a *st*-semantic conflict (Figure 11.6a). In Figure 11.6c we do not relate given conflicts for given AFs, but rather given analytic AFs, i.e. AFs without implicit conflicts. In all three figures observe that we do not strictly follow the relations from Figure 11.4 anymore. For instance now we have direct relations between *pr* and *na* while regarding extensions these semantics can produce disjoint evaluations.

11.3 Future Work

In this thesis we started with the assumption that any describable AF structure should be considered as valuable. This is why we decided to make use of a set theoretic approach and describe AFs as arbitrary sets of arguments combined with an attack relation. A quite different approach to the realm of infinite argumentation was taken in [BCDG13] or [BS17]. In the first the authors consider infinite AFs generated by finite expressions. In the second we assume the availability of arguments to be restricted by some fixed set as universe of arguments.

Generated AFs such as in [BCDG13] in terms of the resulting infinite AFs can be compared to instantiated argumentation [BDKT97, GS04, CA07, BH08, Pra10]. Such formalisms might naturally produce infinite AFs and given the general pictures established in this thesis can be investigated with similar aims. For instantiated infinite structures also [BGR11, Sou08] are valuable resources.

Restricting the universe of available arguments such as in [BS17] on the other hand leads to a radical different setting. We were able to use some results and techniques from this thesis in the referenced paper, but it mostly opens a new way of thinking. In a nutshell, compact AFs gain much more importance. Nonetheless, restricted AFs represent another valid AF class for each set of initial arguments. In light of our constructions for realizability it should be noted that due to the potentially exponential blowup even restrictions in the cardinality of AFs can give intriguing results.

As far as Part II is considered we have further touched the subject of games in abstract argumentation and left a few collapse questions open. Games in abstract argumentation [BMM14, VP00, MP01, MP09, Wal84] are an interesting topic on itself. In Section 6.4 we related games to existence questions. This has also been done for kernels for instance in [DM93]. In our talk at the LABEX CIMI Pluridisciplinary Workshop on Game Theory [Spa15b] we pointed out that semantic conflicts are strongly related to existence of games for certain semantic questions. Thus future research on games seems sensible.

Now for the open collapse/perfection questions. Regarding *c2* semantics we are in essence aware of only one AF structure that collapses. That is all known *c2*-collapsing examples have Example 7.3 as minor (cf. Example 7.44). Future work thus includes search for substantially different collapses for *c2* semantics. Collapse of *s2* semantics is similar to *c2*- or *sg*-collapse and might thus not be considered a research topic on its own. In Section 8.4, Lemma 8.19 and Corollary 8.20, we elaborate on a characterization of *sg*-perfection regarding self-attacking arguments. A more general notion of this characterization (Conjecture 8.23) and the question of range-covering conflict-free sets in *sg*-perfect AFs (Conjecture 8.24) deems us the most pressing matter here. Noteworthy, Conjecture 7.46 would allow for treatment of planar AFs (Conjecture 8.24).

Regarding our work on realizability (Theorems 9.23 and 9.30, also Figure 11.3) it should once more be pointed out that they strongly rely on the possible collapse of semantics. Similarly necessity of conflicts (Theorems 9.24 and 9.31) first might require collapsing AFs as well (for *c2*/*s2* semantics) and further rely on the availability of additional arguments. Section 9.4 gives an example of necessary conflicts while Subsection 9.4.1 elaborates on further techniques showing necessity of specific attacks. In particular there we make use of what we might call *conditional conflicts*. Conditions for necessity of conflicts can for instance be compactness of the AFs of interest, or attack structure requirements such as attack/conflict/independence for selected arguments. Such considerations could be taken into account for manipulation of data [BDD08, CKMM15]. Such considerations are also important for extended questions of realizability [LPS16, Str15, DDLW15]. In particular, knowledge about conditional conflicts allows not only to give realizability results for some semantics, but for each extension set to also derive all possible realizations. In light of [DSLW16] and the two-dimensional purity of Theorem 10.34 such detailed conflict knowledge might allow precise multi-dimensional realization results. Also characterizations of and relations between pure conflicts are prominent questions left open by this thesis. Our work on conflicts operates on the premise that the extension set in question is known. From a practical perspective more often an AF might be known and we might want to make use of semantic conflicts to compute the extension sets. To this end a very promising question is that of syntactic manifestations of semantic conflicts.

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