Infinite Argumentation Frameworks*

On the Existence and Uniqueness of Extensions

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Abstract Abstract properties satisfied for finite structures do not necessarily carry over to infinite structures. Two of the most basic properties are *existence* and *uniqueness* of something. In this work we study these properties for acceptable sets of arguments, so-called extensions, in the field of abstract argumentation. We review already known results, present new proofs or explain sketchy old ones in more detail. We also contribute new results and introduce as well as study the question of existence-(in)dependence between argumentation semantics.

1 Introduction

In the past two decades much effort has been spent on abstract argumentation, mainly with finite structures in mind. Be it in the context of non-monotonic reasoning, as an application of modal logic, or as a tool for structural text-analysis and data-mining (see [10] for an excellent summary). From a mathematicians point of view the infinite case has been widely neglected, although one should also highlight efforts of encoding infinite argumentation structures for efficient handling [3] as well as corresponding work in similar areas [1,11] and logical foundations of argumentation [16,9].

Clearly finite or countably infinite structures are an attractive and reasonable restriction, due to their computational nature. But the bigger picture in terms of fulfilled properties (such as existence and uniqueness) tends to hide behind bigger structures or certain subclasses of them. Which is why this work is to be seen as an effort of emphasizing arbitrary infinities for abstract argumentation.

In his seminal paper [14] Phan Minh Dung introduced a formal framework for argumentation, along with notions of acceptance, already including concepts of conflict-freeness, admissibility, completeness and stability (see [2] for an overview of acceptance conditions in argumentation). An argumentation framework (AF) consists of arguments and attacks, where attacks are presented by a directed binary relation on the arguments representing conflict between arguments. Dung and subsequent works use the term semantics to refer to acceptance conditions for sets of arguments. Whether such sets do exist at all is a main property of

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interest. A (dis)proof in case of finite AFs appears to be mostly straightforward, in the general infinite case however conducting such proofs is more intricate. It usually involves the proper use of set theoretic axioms, like the *axiom of choice* or equivalent statements.

Dung already proposed the existence of preferred extensions in the case of infinite argumentation frameworks. It has later on (e.g. [13]) been pointed out that Dung has not been precise with respect to the use of principles. The existence of semi-stable extensions for finitary argumentation frameworks was first shown in [19], with the use of model-theoretic techniques, techniques that could also be extended to stage and other semantics. In this work we provide complete or alternative proofs. Furthermore, beside semi-stable and preferred semantics we consider a bunch of semantics considered in the literature. For instance, as a new result, we show that stage extensions are guaranteed as long as finitary AFs are considered. Finally, we shed light on the question of uniqueness of extensions.

Section 2 gives the necessary background information. We continue warming up with basic observations in Section 3. In Section 4 we present further results for preferred and lesser semantics. We proceed by giving insights into more advanced semantics (e.g. semi-stable) in Section 5. We conclude in Section 6.

2 Background

An *argumentation framework (AF)* $F = (A, R)$ is an ordered pair consisting of a possibly infinite set of arguments A and an attack relation $R \subseteq A \times A$. Instead of $(a, b) \in R$ we might write $a \rightarrowtail b$ and say that a *attacks* b. For sets $E_1, E_2 \subseteq A$ and arguments $a, b \in A$ we write $E_1 \rightarrowtail b$ if some $a \in E_1$ attacks b, $a \rightarrowtail E_2$ if a attacks some $b \in E_2$ and $E_1 \rightarrowtail E_2$ if some $a \in E_1$ attacks some $b \in E_2$. An argument $a \in A$ is *defended* by a set $E \subseteq A$ in F if for each $b \in A$ with $b \rightarrowtail a$, also $E \rightarrowtail b$. An AF $F = (A, R)$ is called *finite* if $|A| \in \mathbb{N}$. Furthermore, we say that F is *finitary* if every argument has only finitely many attackers, i.e. for any $a \in A$, we have $|\{b \in A \mid b \rightarrowtail a\}| \in \mathbb{N}$. The *range* E^+ of a set of arguments E is defined as extension with all the arguments attacked by E, i.e. $E^+ = E \cup \{a \in A \mid E \rightarrowtail a\}$.

A *semantics* σ is a function which assigns to any AF $F = (A, R)$ a set of sets of arguments denoted by $\sigma(F) \subseteq \wp(F)$. Each one of them, a so-called σ-*extension*, is considered to be acceptable with respect to F. For two semantics σ and τ we use $\sigma \subseteq \tau$ to indicate that for any AF F, $\sigma(F) \subseteq \tau(F)$. There is a huge number of commonly established semantics, motivations and intuitions for their use ranging from desired treatment of specific examples to fulfillment of a number of abstract principles. We consider ten prominent semantics, namely admissible, complete, preferred, semi-stable, stable, naive, stage, grounded, ideal and eager semantics (abbreviated by cf, ad, co, pr, ss, stb, na, stg, gr, id and eg respectively). For recent overviews we refer the reader to [4,2].

Definition 1. *Given an AF $F = (A, R)$ and let $E \subseteq A$.*

1. *$E \in cf(F)$ iff for all $a, b \in E$ we have $a \not\rightarrowtail b$,*

2. $E \in ad(F)$ *iff* $E \in cf(F)$ *and for all* $a \rightarrowtail E$ *also* $E \rightarrowtail a$,
3. $E \in co(F)$ *iff* $E \in cf(F)$ *and for any* $a \in A$ *defended by* E *in* F, $a \in E$,
4. $E \in pr(F)$ *iff* $E \in ad(F)$ *and there is no* $E' \in ad(F)$ *s.t.* $E \subsetneq E'$,
5. $E \in ss(F)$ *iff* $E \in ad(F)$ *and there is no* $E' \in ad(F)$ *s.t.* $E^+ \subsetneq E'^+$,
6. $E \in stb(F)$ *iff* $E \in cf(F)$ *and* $E^+ = A$,
7. $E \in na(F)$ *iff* $E \in cf(F)$ *and there is no* $E' \in cf(F)$ *s.t.* $E \subsetneq E'$,
8. $E \in stg(F)$ *iff* $E \in cf(F)$ *and there is no* $E' \in cf(F)$ *s.t.* $E^+ \subsetneq E'^+$,
9. $E \in gr(F)$ *iff* $E \in co(F)$ *and there is no* $E' \in co(F)$ *s.t.* $E' \subsetneq E$,
10. $E \in id(F)$ *iff* $E \in ad(F)$, $E \subseteq \bigcap pr(F)$ *and there is no* $E' \in ad(F)$ *satisfying* $E' \subseteq \bigcap pr(F)$ *s.t.* $E \subsetneq E'$,
11. $E \in eg(F)$ *iff* $E \in ad(F)$, $E \subseteq \bigcap ss(F)$ *and there is no* $E' \in ad(F)$ *satisfying* $E' \subseteq \bigcap ss(F)$ *s.t.* $E \subsetneq E'$.

We recall that the intersection of an empty family of sets does not exist, as it would coincide with the *universal set* leading to the well known *Russel's paradox* (cf. [17] for more details). Consequently, functions like ideal or eager semantics may return *undefined* since their definitions include a subset-check with regard to an intersection.[4] The usual way to avoid undefined intersections is to fix a background set $\mathcal{U}$, a so-called *universe* (which is often explicitly stated or implicitely assumed in argumentation papers), and to define the intersection of a family of subsets $\mathcal{S}$ as $\bigcap \mathcal{S} = \{x \in \mathcal{U} \mid \forall S \in \mathcal{S} : x \in S\}$. Furthermore, in case of ideal and eager semantics one may equivalently replace $\mathcal{U}$ by A since the candidate sets E have to be admissible sets of the considered AF $F = (A, R)$. This means, $\bigcap \sigma(F) = \{x \in A \mid \forall E \in \sigma(F) : x \in E\}$.

The following proposition shows well known relations for the considered semantics.[5] In the interest of readability we present them graphically.

Proposition 1. *For semantics* σ *and* τ, $\sigma \subseteq \tau$ *iff there is a path from* σ *to* τ *in Figure 2, e.g.* $stb \subseteq na$ *for* (stb, stg, na) *is a path from* stb *to* na.

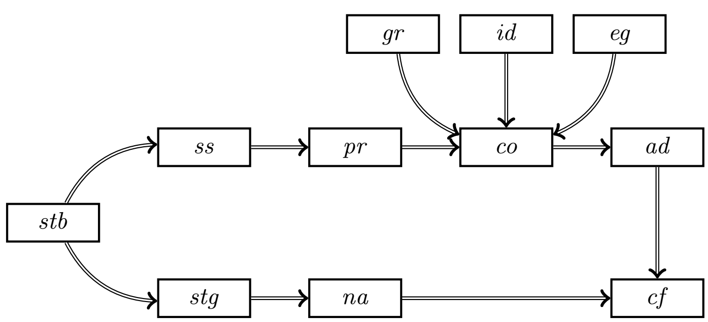

Figure 2. Relations between Semantics

[4] We will see that $ss(F) = \emptyset$ may indeed be the case (Example 4) and thus, these considerations are essential for eager semantics.

[5] Note that the presented relations apply to both finite and infinite AFs. Detailed proofs can be found in [7, Proposition 2.7].

We call a semantics σ *universally defined* if for any AF F, $|\sigma(F)| \geq 1$. Whether a semantics warrants existence of extensions is of high interest. For instance, Dung already showed that AFs can be used to solve well known problems like the stable marriage problem [14]. If the considered problem is modeled correctly and the used semantics provides a positive answer with respect to universal definedness, then solutions of the problem are guaranteed. If a unique solution is guaranteed, i.e. $|\sigma(F)| = 1$ for any F we say that σ follows the *unique status* approach. We will see that existence as well as uniqueness depend on the considered structures. In the following section we start with a preliminary analysis.

3 Warming Up

As we have seen in Figure 2 the general subset relations for the considered semantics are fairly well known. Given two semantics σ, τ such that $\sigma \subseteq \tau$, then (obviously) universal definedness of σ carries over to τ. We start with the investigation of finite AFs.

3.1 Finite AFs

It is well known that stable semantics does not warrant the existence of extensions even in the case of finite AFs. The following minimalistic AFs demonstrate this assertion.

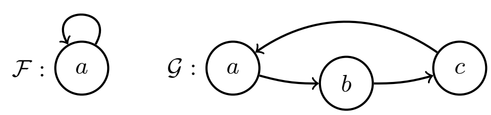

Figure 3. Non-existence of Stable Extension

Both AFs represent odd-cycles and indeed this is a decisive property. It can be shown that being odd-cycle free is sufficient for warranting at least one stable extension.[6] The universal definedness of complete semantics is a well-investigated result from [14].

What about the other semantics considered in this paper? If we take a closer look at Definition 1 we observe that they always possess at least one extension in case of finite AFs.[7] This can be seen as follows: Firstly, the empty set is always admissible and conflict-free. Furthermore, the definitions of the semantics are looking for conflict-free or admissible sets maximal in range or maximal/minimal

[6] This is due to the fact that firstly, *limited controversial* AFs always possess a stable extension [14, Theorem 33] and secondly, in case of finite AFs being odd-cycle free coincides with being limited controversial.

[7] In Sections 4 and 5 we prove this assertion in a rigorous manner for finitary or even arbitrary AFs. The existence of extensions for finite AFs is implied.

with respect to subset relation. Finally, since we are dealing with finite AFs there are only finitely many subsets that have to be considered and thus, the existence of maximal and minimal elements is guaranteed.

3.2 Infinite AFs

It is an important observation that warranting the existence of σ-extensions in case of finite AFs does not necessarily carry over to the infinite case, i.e. the semantics σ does not need to be universally defined. Take for instance semi-stable and stage semantics. To the best of our knowledge the first example showing that semi-stable as well as stage semantics does not guarantee extensions in case of infinite AFs was given in [18, Example 5.8.] and is picked up in the following example.

Example 1. Consider the AF $F = (A \cup B \cup C, R)$ as illustrated in Figure 4 where

- $A = (a_i)_{i \in \mathbb{N}}$, $B = (b_i)_{i \in \mathbb{N}}$, $C = (c_i)_{i \in \mathbb{N}}$ and
- $R = \{a_i \rightarrowtail b_i, b_i \rightarrowtail a_i, b_i \rightarrowtail c_i, c_i \rightarrowtail c_i \mid i \in \mathbb{N}\} \cup$
  $\{b_i \rightarrowtail b_j, b_i \rightarrowtail c_j \mid i, j \in \mathbb{N}, j < i\}$

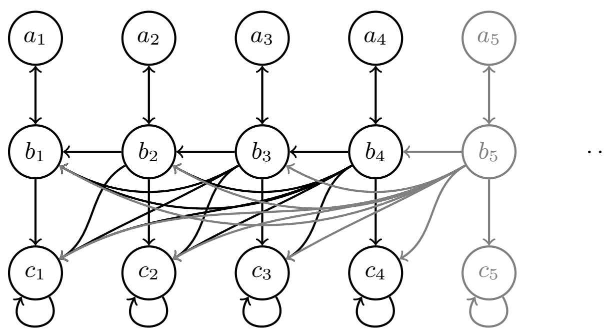

Figure 4. An illustration of the AF from Example 1

The set of preferred and naive extensions coincide, in particular $pr(F) = na(F) = \{A\} \cup \{E_i \mid i \in \mathbb{N}\}$ where $E_i = (A \setminus \{a_i\}) \cup \{b_i\}$. Furthermore, none of these extensions is maximal with respect to range since $A^+ \subsetneq E_i^+ \subsetneq E_{i+1}^+$ for any $i \in \mathbb{N}$. In consideration of $ss \subseteq pr$ and $stg \subseteq na$ (cf. Figure 2) we conclude that this framework does have neither semi-stable nor stage extensions.

There are two questions which arise naturally. Firstly, do stage or semi-stable extensions exist in case of finitary AFs. A positive answer in case of semi-stable semantics was conjectured in [13, Conjecture 1] and firstly proved with substantial effort by Emil Weydert in [19, Theorem 5.1]. Weydert proved his result in a first order logic setup using generalized argumentation frameworks. In this paper we provide an alternative proof using transfinite induction. Moreover, as a new result, we present a proof for the existence of stage semantics in case of finitary AFs.

The second interesting question is whether there is some kind of existence-dependency between semi-stable and stage semantics in case of infinite AFs. The following two examples show that this is not the case. More precisely, it is possible that some AF does have semi-stable but no stage extensions and it is also possible that there are stage but no semi-stable extensions.

Example 2 (No Stage but Semi-stable Extensions). Taking into account the AF $F = (A \cup B \cup C, R)$ from Example 1. Consider a so-called *normal deletion* [6] F' of F as illustrated in Figure 5 where $F' = F|_B$.

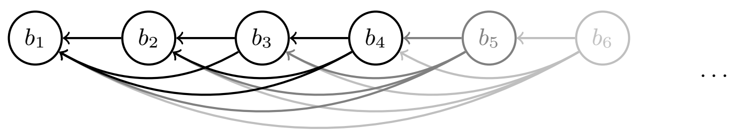

Figure 5. An illustration of the AF from Example 2

We observe that the empty set is the unique admissible extension of F'. Consequently, by definition of semi-stable semantics, $ss(F') = \{\emptyset\}$. On the other hand, $stg(F') = \emptyset$. This can be seen as follows: for any $i \in \mathbb{N}$, $B_i = \{b_i\}$ is a naive extension in F' and there are no other naive extensions. Obviously, there is no range maximal naive set since $B_i^+ \subsetneq B_{i+1}^+$ for any $i \in \mathbb{N}$.

Example 3 (No Semi-Stable but Stage Extensions). Consider again Example 1. We define a so-called *normal expansion* [8] $F' = (A \cup B \cup C \cup D \cup E, R \cup R')$ of F as illustrated in Figure 6, where

- $D = (d_i)_{i \in \mathbb{N}}$, $E = (e_i)_{i \in \mathbb{N}}$ and
- $R' = \{a_i \rightarrowtail d_i, d_i \rightarrowtail a_i, b_i \rightarrowtail d_i, d_i \rightarrowtail b_i, d_i \rightarrowtail c_i, e_i \rightarrowtail d_i, e_i \rightarrowtail e_i \mid i \in \mathbb{N}\}$

In comparison to Example 1 we do not observe any changes as far as preferred and semi-stable semantics are concerned. In particular, $pr(F') = \{A\} \cup \{E_i \mid i \in \mathbb{N}\}$ where $E_i = (A \setminus \{a_i\}) \cup \{b_i\}$ and again, none of these extensions is maximal with respect to range. Hence, $ss(F') = \emptyset$. Observe that we do have additional conflict-free as well as naive sets, especially the set D. Since any $e \in E$ is self-defeating and unattacked and furthermore, $D^+ = A \cup B \cup C \cup D$ we conclude, $stg(F') = \{D\}$.

4 Minor Results

4.1 Universal Definedness of Preferred and Naive Semantics

We start with proving that preferred as well as naive semantics are universally defined. We mention that the case of preferred semantics was already considered

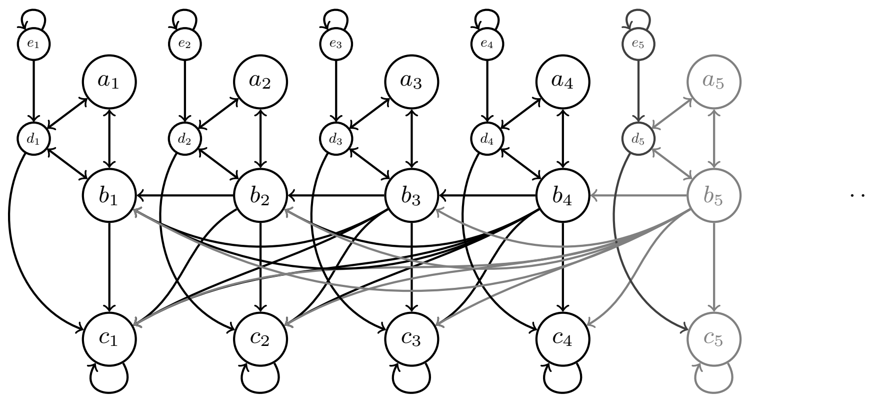

Figure 6. An illustration of the AF from Example 3

in [14, Corollary 12]. The proof is mainly due to *Zorn's lemma*. In order to keep the paper self-contained we recapitulate the famous lemma below.

Lemma 1 ([20]). *Given a partially ordered set* $(P, \leq)$. *If any* $\leq$*-chain possesses an upper bound, then* $(P, \leq)$ *has a maximal element.*

One may easily show that the following "strengthened" version is equivalent to Zorn's lemma.

Lemma 2. *Given a partially ordered set* $(P, \leq)$. *If any* $\leq$*-chain possesses an upper bound, then for any* $p \in P$ *there exists a maximal element* $m \in P$, *s.t.* $p \leq m$.

The following lemma paves the way for showing the universal definedness of naive and preferred semantics.

Lemma 3. *Given* $F = (A, R)$ *and* $E \subseteq A$,

1. *if* $E \in cf(F)$, *then there exists* $E' \in na(F)$ *s.t.* $E \subseteq E'$ *and*
2. *if* $E \in ad(F)$, *then there exists* $E' \in pr(F)$ *s.t.* $E \subseteq E'$.

Proof. For any $F = (A, R)$ we have the associated powerset lattice $(\wp(A), \subseteq)$. Consider now the partially ordered fragments $\mathcal{C} = (cf(F), \subseteq)$ and $\mathcal{A} = (ad(F), \subseteq)$. In accordance with Lemma 2 the existence of naive and preferred supersets is guaranteed if any $\subseteq$-chain possesses an upper bound in $\mathcal{C}$ or $\mathcal{A}$, respectively. Given a $\subseteq$-chain $(E_i)_{i \in I}$ in $\mathcal{C}$ or $\mathcal{A}$, respectively.[8] Consider now $E = \bigcup_{i \in I} E_i$. Obviously, E is an upper bound of $(E_i)_{i \in I}$, i.e. $E_i \subseteq E$ for any $i \in I$. It remains to show that E is conflict-free or admissible, respectively. Conflict-freeness is a finite condition. This means, if there were conflicting arguments $a, b \in E$ there

[8] Remember that any set can be written as an indexed family. This can be done via using the set itself as index set.

would have to be some $i \in I$ with $a, b \in E_i$. Assume now E is not admissible. Consequently, there is some $a \in E$ that is not defended by E. Hence, for some $i \in I$ we have $a \in E_i$ contradicting the admissibility of E_i.

Theorem 7. *For any F, $pr(F) \neq \emptyset$ and $na(F) \neq \emptyset$.*

Proof. Since the empty set is always conflict-free and admissible we may apply Lemma 3 and the assertion is shown.

Since any preferred extension is a complete one (cf. Proposition 1) we deduce that complete semantics is universally defined too. The following proposition shows even more, namely any admissible set is bounded by a complete extension and furthermore, any complete extension is contained in a preferred one.

Proposition 2. *Given $F = (A, R)$ and $E \subseteq A$,*

1. *if $E \in ad(F)$, then there exists $E' \in co(F)$ s.t. $E \subseteq E'$ and*
2. *if $E \in co(F)$, then there exists $E' \in pr(F)$ s.t. $E \subseteq E'$.*

Proof. Given $E \in ad(F)$. Thus, there exists $E' \in pr(F)$ s.t. $E \subseteq E'$ (Lemma 3). Since $pr \subseteq co$ (Proposition 1) the first statement is shown. Consider $E \in co(F)$. Hence, $E \in ad(F)$ (Proposition 1). Consequently, there exists $E' \in pr(F)$ s.t. $E \subseteq E'$ (Lemma 3) and we are done.

4.2 Uniqueness of Grounded and Ideal Semantics

We now turn to grounded as well as the more credulous ideal semantics. The universal definedness in case of grounded semantics was already implicitly given in [14]. Unfortunately, this result was not explicitly stated in the paper. Nevertheless, in [14, Theorem 25] it was shown that firstly, the set of all complete extensions form a complete semi-lattice, i.e. the existence of a greatest lower bound for any non-empty subset S is implied. Secondly, it was proven that the grounded extension is the least complete extension. Consequently, for any AF F we may set $S = co(F)$ and the assertion is shown. The following theorem shows that the same applies to ideal semantics.

Theorem 8. *For any F, $id(F) \neq \emptyset$.*

Proof. Given an arbitrary AF $F = (A, R)$. We define $ad_{\cap pr}(F) = \{E \in ad(F) \mid E \subseteq \bigcap_{P \in pr(F)} P\}$. Now consider $\mathcal{A} = (ad_{\cap pr}(F), \subseteq)$. Obviously, $ad_{\cap pr}(F) \neq \emptyset$ since for any F, $\emptyset \in ad(F)$ and furthermore, $\emptyset \subseteq S$ for any set S. In order to show that $id(F) \neq \emptyset$ it suffices to prove that there is a $\subseteq$-maximal set in $\mathcal{A}$. Again we use Zorn's lemma. Given a $\subseteq$-chain $(E_i)_{i \in I}$ in $\mathcal{A}$. Consider $E = \bigcup_{i \in I} E_i$. Obviously, E is an upper bound of $(E_i)_{i \in I}$ and furthermore, conflict-freeness and even admissibility is given because $E_i \in ad(F)$ for any $i \in I$ (cf. proof of Lemma 3 for more details). Moreover, since $E_i \subseteq \bigcap_{P \in pr(F)} P$ for any $i \in I$ we deduce $E \subseteq \bigcap_{P \in pr(F)} P$ guarenteeing $E \in \mathcal{A}$. Consequently, by Lemma 1 $\mathcal{A} = (ad_{\cap pr}(F), \subseteq)$ possesses $\subseteq$-maximal elements concluding the proof.

The uniqueness of grounded semantics was shown already by Dung [14, Theorem 25, statement 2]. We present a proof for ideal semantics.

Theorem 9. *For any F, $|id(F)| = 1$.*

Proof. $|id(F)| \geq 1$ is already given by Theorem 8. Hence, it suffices to show $|id(F)| \leq 1$. Suppose, to derive a contradiction, that for some $I_1 \neq I_2$ we have $I_1, I_2 \in id(F)$. Consequently, by Definition 1, $I_1, I_2 \in ad(F)$ and $I_1, I_2 \subseteq \bigcap_{P \in pr(F)} P$ as well as neither $I_1 \subseteq I_2$, nor $I_2 \subseteq I_1$. Obviously, $I_1 \cup I_2 \subseteq \bigcap_{P \in pr(F)} P$ and since preferred extensions are conflict-free we obtain $I_1 \cup I_2 \in cf(F)$. Since both sets are assumed to be admissible we derive $I_1 \cup I_2 \in ad(F)$ contradicting the $\subseteq$-maximality of I_1 and I_2.

5 Main Results

When dealing with range-maximal extensions in infinite AFs as seen in the previous examples we might deal with sets of sets of arguments that keep growing in size with respect to their range. For being able to handle constructions of this kind we introduce the following two definitions. The intuition for the first definition is that we want to be able to say something about arguments and sets occuring (un)restricted in collections of extensions. For the second definition we focus on the idea of infinitely range-growing sets of extensions.

Definition 10 (Keepers, Outsiders, Keeping Sets and Compatibility). *Consider some AF F. For $\mathcal{E}$ a set of sets of arguments we call $\mathcal{E}^+ = \bigcup_{E \in \mathcal{E}} E^+$ the* range *of $\mathcal{E}$ and for some argument $a \in \mathcal{E}^+$ we say that:*

- *a is a* keeper *of $\mathcal{E}$ if it occurs range-unbounded in $\mathcal{E}$, i.e. for any $E_1 \in \mathcal{E}$ with $a \notin E_1$ there is some $E_2 \in \mathcal{E}$ such that $a \in E_2$ and $E_1^+ \subseteq E_2^+$;*
- *a is an* outsider *of $\mathcal{E}$ if it is not a keeper of it, i.e. there is some $E_1 \in \mathcal{E}$ with $a \notin E_1$ such that there is no $E_2 \in \mathcal{E}$ with $a \in E_2$ and $E_1^+ \subseteq E_2^+$.*

Furthermore for a set $A \subseteq \mathcal{E}^+$ we say that:

- *A is a* keeping set *of $\mathcal{E}$, or* kept *in $\mathcal{E}$, if it occurs range-unbounded in $\mathcal{E}$, i.e. for every $E_1 \in \mathcal{E}$ with $A \not\subseteq E_1$ there is some $E_2 \in \mathcal{E}$ such that $A \subseteq E_2$ and $E_1^+ \subseteq E_2^+$.*
- *A is called* compatible *with $\mathcal{E}$ if every finite subset of A is kept in $\mathcal{E}$, i.e. for every finite $A_{<\omega} \subseteq^{<\omega} A$ we have that $A_{<\omega}$ is a keeping set of $\mathcal{E}$.*

Definition 11 (Range Chain, Chain Range, Induced AF). *Consider some AF F. A set of sets of arguments $\mathcal{E}$ is called a* range chain *if for any $E_1, E_2 \in \mathcal{E}$ we have $E_1^+ \subseteq E_2^+$ or $E_2^+ \subseteq E_1^+$, again the range of $\mathcal{E}$ (the* chain range *$\mathcal{E}^+$) is defined as $\mathcal{E}^+ = \bigcup_{E \in \mathcal{E}} E^+$.*

Now for a given range chain $\mathcal{E}$ we will consider the by $\mathcal{E}$ induced *AF $F|_{\mathcal{E}}$:*

$$F|_{\mathcal{E}} = (\mathcal{E}^+, \{(a,b) \mid a, b \in \mathcal{E}^+, (a,b) \in R_F\} \cup \{(b,b) \mid b \text{ outsider of } \mathcal{E}\})$$

Observe that naturally finite range chains or chains that have a maximum will not be of interest to us. Also observe the implicit transitivity, i.e. for $E_1, E_2, E_3 \in \mathcal{E}$ from $E_1^+ \subsetneq E_2^+$ and $E_2^+ \subsetneq E_3^+$ it follows that also $E_1^+ \subsetneq E_3^+$. Thus a range chain by definition gives a well-ordering on the equivalence class of elements with equal range. We might need the axiom of choice though, to select one specific extension for every equivalence class.

Lemma 4 (Axiom of Choice). *For every set of non-empty sets $\mathcal{E}$ there is a choice function, i.e. a function f selecting one member of each set, for all $E \in \mathcal{E}$ we have $f(E) \in E$.*

One may show that the axiom of choice is equivalent to Zorn's lemma. It is nowadays widely accepted, but the concept has been shown to be independent from other axioms of set theory. Uses of choice often appear to be implicit, in the following we explicitly mark when the axiom of choice is necessary.

5.1 Semi-stable and Stage Extensions in case of Finitary AFs

In the case of semi-stable and stage extensions we deal with semantics that sometimes are seen as weaker forms of stable semantics. In this sense we think of range chains that range-cover the whole framework, or in other words we will reduce frameworks to arguments being relevant (Definition 11) to some range chain only. The following definition deals with the question whether some argument or sets of arguments might be part of some stable extension. The intuition being that we can recursively try to cover the full range of some AF, the following definition helps in defining the recursion step.

Definition 12 (Unresolved Range). *Given some AF F, a range chain $\mathcal{E}$ such that $F|_{\mathcal{E}} = F$, and a set $A \subseteq \mathcal{E}^+$. We define the* unresolved range *of A as the set A^* that as a next step has to be resolved if A is to be subset of a stable extension. A^* thus consists of arguments endangering A without defense, as well as arguments attacked by A^+ but not by A. Also see Figure 13 for an illustration.*

$$A^* = \left\{ b \notin A^+ \mid b \rightarrowtail A \right\} \cup \left\{ a \notin A^+ \mid A^+ \rightarrowtail a \right\}$$

Lemma 5. *Given some finitary AF F, some range chain $\mathcal{E}$, such that $F|_{\mathcal{E}} = F$, and some with $\mathcal{E}$ compatible set $A \subseteq \mathcal{E}^+$. Then there is some with $\mathcal{E}$ compatible set $B \subseteq \mathcal{E}^+$ such that $A \subseteq B$ and $A^* \subseteq B^+$, we have $A^+ \cup A^* \subseteq B^+$.*

Proof. First observe that for every finite set $A_{<\omega} \subseteq A^+ \cup A^*$ there has to be a finite set $B_{<\omega}$ such that $A_{<\omega} \cap A \subseteq B_{<\omega}$ and $A_{<\omega} \cap A^* \subseteq B^+_{<\omega}$. This is due to the finitary condition and the definitions, for every finite set of arguments there are only a finite number of sets that have at most this range, but since the chain $\mathcal{E}$ is unbounded in F there is at least one. Furthermore if B resolves $A_1 \cup A_2$ then B resolves A_1 and A_2. By transfinite induction on the size of B we can show that there is a set with the desired properties. Observe that the axiom of choice might be necessary though.

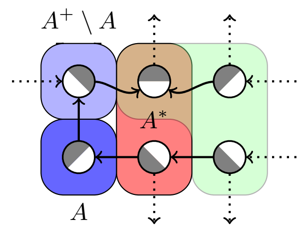

Figure 13. An illustration of unresolved range A^* (Definition 12). Observe that the rightmost area characterizes all arguments that can resolve A^*, when incorporating A.

Theorem 14. *For any finitary F, $|ss(F)| \geq 1$ and $|stg(F)| \geq 1$.*

Proof. Take some finitary AF F, and $\sigma = pr$ or $\sigma = na$, and $\sigma^+ = ss$ or $\sigma^+ = stg$ respectively. We will show that for any range chain $\mathcal{E} \subseteq \sigma(F)$ there is some σ-extension E that covers the full chain range, i.e. $\mathcal{E}^+ \subseteq E^+ \in \sigma(F)$. By then applying Zorn's Lemma it follows that $\mathcal{E}$ also contains at least one range-maximum, i.e. a range-maximal set or in other words a σ^+-extension.

To this end for any range chain $\mathcal{E} \subseteq \sigma(F)$, we proceed with the following steps using transfinite recursion to find an upper bound A with $\mathcal{E}^+ \subseteq A^+$ such that there is some $E \in \sigma(F)$ with $A \subseteq E$.

1. Consider only relevant arguments of F
2. Recursion Start, motivation and intuition
3. Successor Step, augment by resolving keeper sets or compatible keepers
4. Limit Step, collect successor steps
5. Remarks, conflict-freeness and range-completeness

1. Consider only relevant arguments of F: As presented in Definition 11 we will make use of some AF $F|_{\mathcal{E}}$ that contains only arguments from the range of $\mathcal{E}$, plus all outsiders are self-attacking. If we retrieve a conflict-free (admissible) set A such that A contains only keepers of $\mathcal{E}$ and spans the whole range, $A^+ = \mathcal{E}^+$, we can as stated in Lemma 2 retrieve a σ-extension that covers the whole chain range. Clearly every stable extension of $F|_{\mathcal{E}}$ serves this purpose. In the following we will thus construct a stable extension and consider some AF F where $F|_{\mathcal{E}} = F$.

2. Define the recursion start: As recursion start we will use the set $A_0 = \{a\}$ for some keeper a of $\mathcal{E}$. In each step we will augment this set in a clever way, by choosing compatible sets that either cover the unresolved range or some arbitrary compatible keeper.

3. Successor Steps, $\alpha = \beta + 1$: Given some compatible set A_β. If A_β has some unresolved range $A^*_\beta \neq \emptyset$ we choose a compatible set $A_\alpha \supset A_\beta$ such that $A^*_\beta \subset A^+_\alpha$. As stated in Lemma 5 such a set exists, but we might need the axiom of choice to find one. If on the other hand $A^*_\beta = \emptyset$ we pick some compatible keeper $a \notin A_\beta$ such that $A_\alpha = A_\beta \cup \{a\}$ is compatible with $\mathcal{E}$.

4. Limit Steps, α: Given a range chain $\{A_i\}_{i<\alpha}$ where for any $i < j$ we have $A_i \subseteq A_j$ and all A_i are finitely compatible. We define $A_\alpha = \bigcup_{i<\alpha} A_i$, implicitly using the axiom of choice. By definition A_α is compatible with $\mathcal{E}$ for otherwise there would be some finite subset $B \subseteq^{<\omega} A_\alpha$ that is not kept in $\mathcal{E}$, but then due to the construction it follows that already $B \subseteq A_i$ for some $i < \alpha$, in contradiction to the successor step.

5. Conflict-freeness and range-completeness: Conflict-freeness follows from compatibility, range-completeness follows from definition of unresolved range and successor/limit steps resolving this issue. Latest at each limit step, A_α becomes admissible and independent from arguments that are not member of A_α^+, i.e. if $a \rightarrowtail A_\alpha$ then $A_\alpha \rightarrowtail a$, and if $A_\alpha^+ \rightarrowtail a$ then $A_\alpha \rightarrowtail a$, and if $a \rightarrowtail b$ where $b \in A_\alpha^+$ then $A_\alpha \rightarrowtail b$.

Having showed that every range chain of σ-extensions has an upper bound in $\sigma(F)$ using Zorn's lemma we now conclude that there is a range-maximal σ-extension, in other words a σ^+-extension.

5.2 The Special Case of Eager Semantics

One may have wondered why we did not consider eager semantics in Section 4.2. The reason for this simply is that eager semantics does not follow the unique status approach.[9] More precisely, if there are no semi-stable extensions then eager semantics equals preferred semantics. Moreover, in this case we have infinitely many eager extensions. If the set of semi-stable extensions is nonempty then eager semantics is uniquely determined.

Theorem 15. *For any F, we have:*

1. $ss(F) \neq \emptyset \Rightarrow |eg(F)| = 1$,
2. $ss(F) = \emptyset \Rightarrow eg(F) = pr(F)$ *and*
3. $ss(F) = \emptyset \Rightarrow |eg(F)| \geq |\mathbb{N}|$.

Proof. ad 1.) The proof is almost identical with the one presented for Theorem 8. Simply replace preferred by semi-stable semantics.

ad 2.) Let $F = (A, R)$ and assume $ss(F) = \emptyset$. Remember that $\bigcap_{P\in ss(F)} P = \{x \in A \mid \forall P \in ss(F) : x \in P\}$. Given that $ss(F) = \emptyset$ we deduce $\bigcap_{P\in ss(F)} P = A$ since $\forall P \in ss(F) : x \in P$ becomes a vacuous truth. Hence, eager semantics calls for subset-maximal admissible sets. This means, $eg(F) = pr(F)$.

ad 3.) Assume $|eg(F)| = n$ for some finite cardinal $n \in \mathbb{N}$. Due to statement 2 we derive, $|pr(F)| = n$. Remember that $ss \subseteq pr$ (cf. Proposition 1). Consequently, among the finitely many preferred extensions there has to be a range-maximal one. This means, $ss(F) \neq \emptyset$.

Since finitary AFs do always possess semi-stable extensions we state the following corollary.

Corollary 1. *For any finitary F, $|eg(F)| = 1$.*

[9] Observe that our assertion does not contradict the claimed uniqueness in [12] since the author considered the restricted case of finite AFs only.

5.3 A Note on *cf2* and *stg2* Semantics

Two semantics which have defied any attempt of solving w.r.t. the problem of existence in case of finitary AFs are *cf2* and *stg2* semantics [5,15]. Both are defined via a general recursive schema which is based on decomposing AFs along their *strongly connected components (SCCs)*. Roughly speaking,[10] the schema takes a base semantics σ and proceeds along the induced partial ordering and evaluates the SCCs according to σ while propagating relevant results to subsequent SCCs. This procedure defines a $\sigma 2$ semantics.[11]

Given SCC-recursiveness we have to face some difficulties in drawing conclusions with respect to infinite or finitary AFs. If every subframework does have an initial SCC (which is guarenteed for finite AFs), i.e. some strongly connected subframework that is not attacked from the outside, then obviously this AF provides a $\sigma 2$-extension as soon as every single component provides a σ-extension. If on the other hand there is no initial SCC things become more complicated and in particular especially due to the recursive definitions not that easy to handle. So for now we go with the following conjecture.

Conjecture 1. For any finitary F, $|cf2(F)| \geq 1$ and $|stg2(F)| \geq 1$.

A noteworthy observation is that both semantics are not universally defined. Consider therefore the following example.

Example 4 (Example 2 continued). Let $\sigma \in \{cf2, stg2\}$. Consider the AF F' depicted in Figure 5. Here, for a sequence $(b_i)_{i\in\mathbb{N}}$ of arguments we have that $b_i \rightarrowtail b_j$ iff $i > j$. This means, any argument b_i constitutes a SCC $\{b_i\}$ which is evaluated as $\{b_i\}$ by the base semantics of σ. Consequently, $\emptyset$ cannot be a σ-extension. Furthermore, a singleton $\{b_j\}$ cannot be a σ-extension either. The b_i's for $i > j$ are not affected by b_j and thus, the evaluation of $\{b_i\}$ do not return $\emptyset$ as required. Finally, any set containing more than two arguments would rule out at least one of them and thus, cannot be a σ-extension.

5.4 Summary of Results

The following table gives a comprehensive overview over results presented in this paper. The entry $\exists$ ($\exists!$) in row *certain* and column σ indicates that the existence of σ-extension is guaranteed (and uniquely determined) given that *certain* frameworks are considered. The question mark represents an open problem.

[10] Due to the limited space we have to refer the reader to [5] for more details.
[11] Following this terminology we have to rename *cf2* semantics to *na2* semantics since its base semantics is the naive semantics and not conflict-free sets.

	stb	*ss*	*stg*	*cf2*	*stg2*	*pr*	*ad*	*co*	*gr*	*id*	*eg*	*na*	*cf*
finite		∃	∃	∃	∃	∃	∃	∃	∃!	∃!	∃!	∃	∃
finitary		∃	∃	?	?	∃	∃	∃	∃!	∃!	∃!	∃	∃
arbitrary						∃	∃	∃	∃!	∃!	∃	∃	∃

Figure 16. Existence and Uniqueness of Extension

6 Conclusions and Related Work

In this paper we gave an overview on the question whether certain semantics guarantee the existence or even unique determination of extensions. Whereas most of the literature concentrated on finite AFs we stick to the arbitrary infinite case as well as the subclass of finitary AFs. We present full or alternative proofs for already known results like universal definedness of preferred semantics and existence of semi-stable extensions in case of finitary frameworks. Furthermore, we completed the picture for the remaining semantics in case of non-finite structures. To mention two results: Firstly, stage semantics behaves similarly to semi-stable, i.e. extensions are guaranteed as long as finitary AFs are considered. Secondly, eager semantics is universally defined but either there is exactly one or there are infinitely many eager extensions. The former case is ensured for finitary structures. In the latter case eager semantics coincide with preferred semantics. To sum up, eager semantics does not generally follow the unique status approach.

It is a non-trivial problem to decide whether certain abstract properties satisfied for finite AFs carry over to infinite structures. In [2, Section 4.4] the authors wrote “As a matter of fact, we are not aware of any systematic literature analysis of argumentation semantics properties in the infinite case.”. This paper can be seen as a first step in this direction.

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