Given two graphs $G$ and $A$, two players, Red and Green, alternate in coloring the edges of $G$ in their respective color. Aim is to avoid (achieve) to build a monochromatic subgraph isomorphic to $A$. How difficult are these games?
Overview

- In medias res: Let’s play . . .
- Complexity of the graph Ramsey games
- Ultra-strongly solving Sim and Sim$^+$
- About the unlikeliness of solving Sim$_4$ etc.
- Tractable cases
- Provably intractable cases
- The complexity of games: another view
- Open problems and conjectures
- Further remarks
**Sim:** $G = K_{\text{Ramsey}(3,3)} = K_6$, $A = K_3$ on $G_{\text{Avoid-Ramsey}}$
Considering that a hands-on session with an interactive system often is worth more than a thousand images:
... with random permutations between moves:

This Java applet plays Sim and a variant, Sim⁺ (players color one or more edges per move). In case you win, you will be allowed to leave your name in our hall-of-fame!
Sim and $\text{Sim}^+$ can never end in a tie:

$$\text{Ramsey}(3,3)=6$$

(visual proof by courtesy of Ranan Banerji)
A winning strategy for the $G_{\text{Achieve-Ramsey}}$ game $\text{Sim}_A$: 

[Diagram showing a series of graphs connected by arrows labeled 'w.l.o.g.' and 'forced']
No simple winning strategies are known for Sim and Sim$^+$.  

⇒ **Natural question: How “difficult” is a game?**

Translation to complexity theory:

*How does the function bounding the computational resources that are needed in the worst case to determine a winning strategy for the first player grow in relation to the size of the game description?*

Typical results: Generalizations of well-known games such as Chess, Checkers, and Go to boards of size $n \times n$ have been classified as polynomial space and exponential time complete (Fraenkel & Lichtenstein 1981, Fraenkel & al. 1978, Lichtenstein & Sipser 1980).

Note: $P \subseteq NP \subseteq PSPACE \subseteq EXPTIME$ and $P \subset \subseteq EXPTIME$
How to generalize Sim to game boards of arbitrary size?

⇒ **Graph Ramsey theory**

**Definition 1** \( G \rightarrow A \):

*We say that a graph \( G \) arrows a graph \( A \) if every edge-coloring of \( G \) with colors red and green contains a monochromatic subgraph isomorphic to \( A \). \( G \) is called a Ramsey graph of \( A \).*

**Theorem 1** (Chvátal & Harary 1972, Deuber 1975, Erdős & al. 1975, Rödl 1973) *Every graph has Ramsey graphs.*

**Theorem 2** (Burr 1976) *Deciding \( G \notightarrow A \) when \( G \) and \( A \) are part of the input is \( \text{NP} \)-complete.*

**Theorem 3** (M. Schaefer 1999) *Deciding \( G \rightarrow A \) when \( G \) and \( A \) are part of the input is \( \pi_2^P \)-complete.*
Generalizing Sim to graph Ramsey theory leads to:

**Definition 2** \( G_{\text{Avoid-Ramsey}}(G, A, E^r, E^g) \):

Given two graphs \( G = (V, E) \) and \( A \) and two non-intersecting sets \( E^r \cup E^g \subseteq E \) that contain edges initially colored in red and green, respectively. Two players, Red and Green, take turns in selecting at each move one so-far uncolored edge from \( E \) and color it in red for player Red respectively in green for player Green. However, both players are forbidden to choose an edge such that \( A \) becomes isomorphic to a subgraph of the red or the green part of \( G \). It is Red’s turn. The first player who cannot move loses.

Similar definitions of \( G_{\text{Avoid’-Ramsey}} \) (a misère variant) and \( G_{\text{Avoid-Ramsey}^+} \) (one or more edges colored per move).
**Definition 3** \( G_{\text{Achieve-Ramsey}}(G, A, E^r, E^g) \):

Achievement variant: the first player who builds a monochromatic subgraph isomorphic to \( A \) wins.

**Definition 4** A simple strategy-stealing argument tells us that with optimal play on an uncolored board, \( G_{\text{Achieve-Ramsey}} \) must be either a first-player win or a draw, so it is only fair to count a draw as a second-player win. Let us call this variant \( G_{\text{Achieve}'-\text{Ramsey}} \).

**Definition 5** Following the terminology of (Beck & Csirmaz 1982), let us call the variant of \( G_{\text{Achieve-Ramsey}} \) where all the second player does is to try to prevent the first player to build \( A \), without winning by building it himself, the “weak” graph Ramsey achievement game \( G_{\text{Achieve}''-\text{Ramsey}} \).
Main complexity results (Slany 1999)

**Theorem 4**

\[ G_{\text{Avoid-Ramsey}} \text{ and } G_{\text{Avoid'-Ramsey}} \text{ are PSPACE-complete.} \]

**Theorem 5**

\[ G_{\text{Avoid-Ramsey}^+} \text{ is PSPACE-complete.} \]

And, surprisingly,

**Theorem 6**

\[ G_{\text{Achieve''-Ramsey}} \text{ and } G_{\text{Achieve'-Ramsey}} \text{ are PSPACE-complete.} \]

**Theorem 7**

\[ G_{\text{Achieve-Ramsey}} \text{ is PSPACE-complete.} \]

Significance: These games thus are as difficult as other well-known difficult games such as Go, and at least as difficult as any \textbf{NP}-complete problem.
Proof sketch of Theorem 4

- Membership in \textbf{PSPACE}: easy.
- Hardness: via a \textbf{LOGSPACE} reduction from the \textbf{PSPACE}-complete game \(G_{\text{Achieve-POS-CNF}}\) (T. Schaefer 1978):

\textbf{Definition 6} \(G_{\text{Achieve-POS-CNF}}(F)\): We are given a positive CNF formula \(F\). A move consists of choosing some variable of \(F\) which has not yet been chosen. Player I starts the game. The game ends after all variables of \(F\) have been chosen. Player I wins iff \(F\) is true when all variables chosen by player I are set to true and all variables chosen by player II are set to false.

Ex.: On \(F = (x_1 \lor x_4) \land (x_2 \lor x_3) \land (x_2 \lor x_4)\) player I wins. \(F\) is reduced to the following \(G_{\text{Avoid-Ramsey}}\) game . . .
\( F = (x_1 \lor x_4) \land (x_2 \lor x_3) \land (x_2 \lor x_4) \)

Abbreviations:

- \( r_0 \)
- \( r_1 \)
- \( r_2 \)
- \( r_3 \)
- \( r_4 \)
- \( y_1 \)
- \( y_2 \)
- \( y_3 \)
- \( y_4 \)
- \( d_1 \)
- \( d_2 \)
- \( d_3 \)
- \( g_1 \)
- \( g_2 \)
- \( g_3 \)
- \( g_4 \)
- \( u_{1,0} \)
- \( u_{1,1} \)
- \( u_{1,2} \)
- \( u_{1,b} \)
- \( g_{1,t} \)
- \( g_{1,0} \)
- \( g_{1,1} \)
- \( g_{1,2} \)
- \( g_{1,b} \)
Proof sketch of Theorem 5

- A careful analysis of the proof of Theorem 4 reveals that we can reuse the reduction of that proof to show the \textbf{PSPACE}-completeness of $G_{\text{Avoid-Ramsey}^+}$.

- Indeed, all arguments go through even when both players are allowed to color more than one edge per move.

- The difficulty here lies in the analysis of the cases when the opponent plays non-optimally.
Proof sketch of Theorem 6

- Membership in **PSPACE**: easy.
- Hardness: via a **LOGSPACE** reduction from the **PSPACE**-complete game \( G_{\text{Achieve-POS-DNF}} \) (T. Schaefer 1978):

**Definition 7** \( G_{\text{Achieve-POS-DNF}}(F) \): We are given a positive DNF formula \( F \). A move consists of choosing some variable of \( F \) which has not yet been chosen. Player I starts the game. The game ends after all variables of \( F \) have been chosen. Player I wins iff \( F \) is true when all variables chosen by player I are set to true and all variables chosen by player II are set to false.

Ex.: On \( F = (x_1 \land x_2) \lor (x_3 \land x_4 \land x_5) \lor (x_3 \land x_5 \land x_6) \lor (x_3 \land x_4 \land x_7) \) player II wins. The \( G_{\text{Achieve}} \)-Ramsey game ...
\[ F = (x_1 \land x_2) \lor (x_3 \land x_4 \land x_5) \lor (x_3 \land x_5 \land x_6) \lor (x_3 \land x_4 \land x_7) \]

Abbreviation:

\[ \langle \rangle = \boxed{\text{large clause}} \]
Proof sketch of Theorem 7

Similar to the proof for $G_{\text{Achieve}''}$-Ramsey, but some changes (A) and one addition (H) in the reduction are necessary:

$$F = (x_1 \land x_2) \lor (x_3 \land x_4 \land x_5) \lor (x_3 \land x_5 \land x_6) \lor (x_3 \land x_4 \land x_7)$$

Abbreviations:

- $r_1, r_2, r_3, r_4, r_5, \ldots$.
- $n \ldots$ number of variables.
Definition 8 (J. Schaeffer & Lake 1996)

A combinatorial game is . . .

- **ultra-weakly solved** if the game-theoretic value for the initial position has been determined,

- **weakly solved** if it is ultra-weakly solved and if a strategy exists for achieving the game-theoretic value from the opening position, assuming reasonable computing resources,

- **strongly solved** if for all possible positions, a strategy is known for determining the game-theoretic value for both players, assuming reasonable computing resources, and

- **ultra-strongly solved** if for all positions in a strongly solved game, a strategy is known that improves the chances of achieving more than the game-theoretic value against a fallible opponent.
Theoretical size of Sim’s game tree: $15! \approx 1.3 \times 10^{12}$. In case of Sim$^+$: $15! \times 2^{15-1} \approx 2.1 \times 10^{16}$.

Practical size of their directed acyclic game graphs:

- Sim: 2,309 non-isomorphic positions
- Sim$^+$: 13,158 non-isomorphic positions

$\implies$ Strong solutions of Sim and Sim$^+$ are easily feasible.

To ultra-strongly solve Sim, we additionally need a strategy for non-winning positions. In our Java applet, we:

- maximize static chance of opponent to make a mistake
- improve this strategy by probabilistically learning the value of moves through playing over the Internet

$\implies$ Sim and Sim$^+$ are ultra-strongly solved.
Definition 9  \(\text{Sim}_n:\)

\[ G = K_{\text{Ramsey}(n,n)}, \ A = K_n \text{ played on } G_{\text{Avoid-Ramsey}}. \]

Problem: Despite much effort, only \(\text{Ramsey}(4,4) = 18\) is known so far (conjecture (McKay 1998) \(\text{Ramsey}(5,5) \geq 43\) based on 10 cpu-years of computations . . . ).

Let us consider the game \(\text{Sim}_4\) played on a game board \(G\) having \(\binom{18}{2} = 153\) edges, the graph \(A\) to avoid being a tetrahedron. Unfortunately, we found that the number of non-isomorphic game positions in \(\text{Sim}_4\) is around 

\[ 2 \times 10^{54}. \]

\[ \Rightarrow \] There is not much hope to even weakly solve any game \(\text{Sim}_n\) and even less so any game \(\text{Sim}_n^+\) for \(n > 3\).
Tractable cases:

**Theorem 8** (Harary, Slany, Verbitsky 2000)

\[ G_{\text{Avoid-Ramsey}}(K_n, (\{a, b, c\}, \{\{a, b\}, \{b, c\}\}), \{\}, \{\}) \]

for \( n \geq 3 \) is a win for the second player.

Proof sketch:

There is a relatively simple two-phase winning strategy for the second player. The proof uses a counting argument and several lemmas.
Provably intractable cases:

Because of the known exponential lower bounds for classic symmetric binary Ramsey numbers

\[ n2^{n/2} \left( \frac{1}{e\sqrt{2}} \right) + o(1) < \text{Ramsey}(n, n) \]

already computing the size of the game graph of a graph Ramsey game played on \((K_{\text{Ramsey}(n,n)}, K_n, \{\}, \{\})\) given only \(n\) for input will require at least doubly exponential time because of the succinctness of the input (Graham et al. 1990).
The complexity of games: another view

Problem: \textbf{PSPACE}-completeness is a very coarse instrument to measure the difficulty of combinatorial games: no statement about particular instances are possible. For example, how does the complexity of the \textit{real} game Go compare to that of Sim$_4$ or Sim$_5$?

\implies \textbf{time-bounded Kolmogorov complexity of combinatorial game instances:}

What is the “size” $n$ of the “smallest program” that, using at most $n$ “time units”, wins game $G$ whenever a winning strategy exists and plays “optimally” otherwise?

Good upper and lower bounds are most likely difficult \ldots
Open Problem 1  Consider $G_{\text{Avoid-Ramsey}}(K_k, K_n, \emptyset, \emptyset)$ where $k = \text{Ramsey}(n, n)$. Is it always true that the first player has a winning strategy in this game iff $\binom{k}{2}$ is even?

Open Problem 2  Consider $G_{\text{Avoid-Ramsey}}(G, A, E^r, E^g)$, where

\[ c \overset{\text{def}}{=} \min \left\{ r + g \mid (G, E^r, E^g)^{(r, g)} \rightarrow A \right\}, \]

and where $(G, E^r, E^g)^{(r, g)}$ denotes an $(r, g)$ edge-red-green-coloring of the uncolored edges of the precolored graph $(G, E^r, E^g)$. Is it always true that the first player has a winning strategy in this game iff $c$ is even?
Conjecture 1  Graph Ramsey games played on $(G, A, \emptyset, \emptyset)$ are PSPACE-complete.

Conjecture 2  Graph Ramsey achievement games played on $(K_n, A, E^r, E^g)$ are tractable.

Conjecture 3  Graph Ramsey avoidance games played on $(K_k, K_n, E^r, E^g)$ where $k \geq \text{Ramsey}(n, n)$ are PSPACE-complete.

Conjecture 4  The graph Ramsey avoidance games played on $(K_{\text{Ramsey}(n,n)}, K_n, \emptyset, \emptyset)$ are 2-EXPSPACE-complete.
Open Problem 3  Show that $G_{\text{Achieve-Ramsey}}$ remains \textsc{PSPACE}-complete even if the achievement graph $A$ is restricted to a meaningful subclass of graphs such as fixed, bipartite or degree-restricted graphs.

Open Problem 4  Show that Theorems 4–7 hold even if the game graph $G$ is restricted to a meaningful subclass of graphs such as bipartite or degree-restricted graphs.
Further remarks

- Sim and Sim\(^+\): simple enough to analyze perfectly, yet far from trivial.

- Applications: competitive situations where opposing parties try to achieve or to avoid a certain pattern in the structure of their commitments, e.g., analysis of mobile Internet agent warfare (Thomsen & Thomsen 1998).

- Sim and Sim\(^+\) are to be integrated in a role-playing game ⇒ “cheats” will be made very difficult.

- Please try out our applet that plays Sim and Sim\(^+\) on http://www.dbai.tuwien.ac.at/proj/ramsey/

so that it can continue to become even better.