Database Theory
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2. Introduction to Datalog

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Motivation

- SQL, relational algebra, relational calculus (both tuple and domain relational calculus) are “relational complete”, i.e., they have the full expressive power of relational algebra.
- But: many interesting queries cannot be formulated in these languages
- Example: no recursive queries (SQL now offers a recursive construct)
Example

- Relation \text{parent}(\text{PARENT}, \text{CHILD})$, gives information on the parent-child relationship of a certain group of people.

- Problem: look for all ancestors of a certain person.

- Solution: define relation \text{ANCESTOR}(X, Y): X is ancestor of Y by generating one generation after the other (one join and one projection each) and finally merge all generations (union):

  \begin{align*}
  \text{grandparent}(\text{GRANDPARENT}, \text{GRANDCHIL}\text{D}) & := \pi_{1,4}(\text{parent}[\text{CHILD} = \text{PARENT}]\text{parent}) \\
  \text{grandgrandparent}(\text{GRANDGRANDPARENT}, \text{GRANDGRANDCHIL}\text{D}) & := \pi_{1,4}(\text{parent}[\text{CHILD} = \text{GRANDPARENT}]\text{grandparent}) \\
  \end{align*}

  \ldots

  \text{ancestor}(\text{ANCESTOR}, \text{NAME}) := \text{parent} \cup \text{grandparent} \cup \text{grandgrandparent} \cup \ldots
Possible Solution

- Use of a programming language with an embedded relational complete query language:

```plaintext
begin
  result := \{\};
  newtuples := parent;
  while newtuples \not\subseteq result do
    begin
      result := result \cup newtuples;
      newtuples := \pi_{1,4}(newtuples[2 = 1]parent);
    end;
  ancestor := result
end.
```

- procedural, needs knowledge of a programming language, leaves little room for query optimization.
Better Solution: Datalog

- Prolog-like logical query language,
- allows recursive queries in a **declarative** way
- Example:
  - compute all ancestors on the basis of the relation `parent`
    ```
    ancestor(X,Y) :- parent(X,Y).
    ancestor(X,Z) :- parent(X,Y), ancestor(Y,Z).
    ```
  - use the ancestor predicate to compute the ancestors of a certain person (Hans):
    ```
    hans_ancestor(X) :- ancestor(X,hans).
    ```
  - compute the ancestors of a certain person (Hans) directly:
    ```
    hans_ancestor(X) :- parent(X,hans).
    hans_ancestor(X) :- hans_ancestor(Y), parent(X,Y).
    ```
Datalog - Syntax

<datalog_program> ::= <datalog_rule> | <datalog_program><datalog_rule>

<datalog_rule> ::= <head> :- <body>

<head> ::= <literal>
<body> ::= <literal> | <body>, <literal>

<literal> ::= <relation_id>(<list_of_args>)

<list_of_args> ::= <term> | <list_of_args>, <term>

<term> ::= <symb_const> | <symb_var>

<symb_const> ::= <number> | <lcc> | <lcc><string>
<symb_var> ::= <ucc> | <ucc><string>

(lcc = lower_case_character; ucc = upper_case_character)
Restrictions on the Datalog Syntax

<relation_id>:

- name of an existing relation of the database (parent) - can be used only in rule bodies
- name of a new relation defined by the datalog program (ancestor)
- has always the same number of arguments.

comparison predicates:

=, <>, <, > are treated like known database relations.

variables:

- each variable that appears in the head of a rule has to be bound in the body
- variables that appear as arguments of comparison predicates must appear in the same body in literals without comparison predicates

A datalog query is also called datalog program
Logical Semantics of Datalog

We consider

\[ R \ldots \text{ datalog rule of the form } L_0 :\! = \! L_1, L_2, \ldots, L_n, \]
\[ L_i \ldots \text{ literal of the form } p_i(t_1, \ldots, t_{n_i}) \]
\[ x_1, x_2, \ldots, x_\ell \text{ variables in } R \]
\[ P \ldots \text{ datalog program with the rules } R_1, R_2, \ldots, R_m \]

We construct

\[ R^* = \forall x_1 \forall x_2 \ldots \forall x_\ell ( (L_1 \land L_2 \land \cdots \land L_n) \Rightarrow L_0 ). \]

We assign to each datalog program \( P \) the (semantically) well-defined formula \( P^* \) as follows:

\[ P^* = R_1^* \land R_2^* \land \cdots \land R_m^* \]
We consider now

\[ \text{REL} \ldots \text{a relation of the database.} \]
\[ \langle t_1, \ldots, t_n \rangle \ldots \text{a tuple of the relation REL.} \]
\[ \text{rel}(t_1, \ldots, t_n) \ldots \text{a fact} \]

\[ \text{DB} \ldots \text{database with relations REL}_1, \text{REL}_2, \ldots, \text{REL}_k \]

We assign to each database relation REL the formula

\[ \text{REL}^* = \text{conjunction of all facts} \]

- a relation is an unordered set of tuples
- the assignment REL \( \mapsto \text{REL}^* \) is therefore not uniquely defined.
- take an arbitrary order (e.g. lexicographical order) since conjunction is associative and commutative.

We assign to each database DB the (semantically) well-defined formula \( \text{DB}^* \) as follows:

\[ \text{DB}^* = \text{REL}_1^* \land \text{REL}_2^* \land \cdots \land \text{REL}_k^*. \]
We have:

\[ DB^* \] is a conjunction of ground atoms (i.e., the facts) and 
\[ P^* \] is a conjunction of formulas with implication

Let \( G \) be a conjunction of facts and formulas with implication. Then the set \( \text{cons}(G) \) of facts that follow from \( G \) is uniquely defined. In other words, we have \( \text{cons}(G) = \{ A \mid A \text{ is a fact with } G \models A \} \).

**Definition**

The semantics of a datalog program \( P \) is defined as the function \( M[P] \), that assigns to each database \( DB \) the set of all facts that follow from the formula "\( P^* \land DB^* \)"

\[
M[P] : DB \rightarrow \text{cons}(P^* \land DB^*)
\]
Example

Consider the database DB with relations `woman(NAME)`, `man(NAME)`, `parent(PARENT, CHILD)` and the datalog program:

\[
\text{grandpa}(X,Y) :\text{ man}(X), \text{ parent}(X,Z), \text{ parent}(Z,Y).
\]

<table>
<thead>
<tr>
<th>woman (NAME)</th>
<th>man (NAME)</th>
<th>parent (PARENT CHILD)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grete</td>
<td>Hans</td>
<td>Hans Linda</td>
</tr>
<tr>
<td>Linda</td>
<td>Karl</td>
<td>Grete Linda</td>
</tr>
<tr>
<td>Gerti</td>
<td>Michael</td>
<td>Karl Michael</td>
</tr>
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<td>Linda Michael</td>
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<td></td>
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<td>Karl Gerti</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Linda Gerti</td>
</tr>
</tbody>
</table>
Let us compute \( DB^* \), \( P^* \) and \( \text{cons}(P^* \land DB^*) \):

\[
DB^* = REL^*_1 \land \cdots \land REL^*_k \text{ with } REL^*_1 = \text{conjunction of all facts}
\]

\[
DB^* = \text{woman(grete)} \land \text{woman(linda)} \land \text{woman(gerti)} \land \\
\text{man(hans)} \land \text{man(karl)} \land \text{man(michael)} \land \\
\text{parent(hans, linda)} \land \text{parent(grete, linda)} \land \\
\text{parent(karl, michael)} \land \text{parent(linda, michael)} \land \\
\text{parent(karl, gerti)} \land \text{parent(linda, gerti)}.
\]

\[
P^* = R^*_1 \land \cdots \land R^*_m \text{ with } R^*_i = \forall x_1 \forall x_2 \cdots \forall x_\ell ((L^*_1 \land \cdots \land L^*_n) \Rightarrow L^*_0).
\]

\[
P^* = \forall X \forall Y \forall Z : ((\text{man}(X) \land \text{parent}(X, Z) \land \text{parent}(Z, Y)) \Rightarrow \text{grandpa}(X, Y)).
\]
The new facts in $\textit{cons}(P^* \land DB^*)$:

grandpa(hans,michael), grandpa(hans,gerti).

The datalog program $P$ with

$P = \text{grandpa}(X,Y) :- \text{man}(X), \text{parent}(X,Z), \text{parent}(Z,Y)$

defines a new relation \textit{grandpa} with the following tuples:

\[
\begin{array}{c|c}
\text{grandpa} & (X & Y) \\
\hline
\text{Hans} & \text{Michael} \\
\text{Hans} & \text{Gerti} \\
\end{array}
\]
Operational Semantics of Datalog

- Datalog rules are seen as inference rules,
- a fact that appears in the head of a rule can be deduced, if the facts in the body of the rule can be deduced.

Example:

facts: parent(linda,michael), parent(linda,gerti)
rule: siblings(michael,gerti) :-
    parent(linda,michael), parent(linda,gerti).

the following fact can be deduced:

siblings(michael,gerti)
Rules with variables

- A rule $R$ with variables represents all variable-free rules we get from $R$ by substituting the variables with the constant symbols.
- The constant symbols are taken from the database $DB$ and the program $P$.
- A variable-free rule resulting from such a substitution is called **ground instance** of $R$ with respect to $P$ and $DB$.
- We write $\text{Ground}(R, P, DB)$ to denote the set of all ground instances over $P$ and $DB$ of $R$. 
Example:

Compute all relations between siblings with the following rule:

\[
\text{siblings}(Y,Z) : \neg \text{parent}(X,Y), \text{parent}(X,Z), Y <> Z. 
\]

All \(6^3\) ground instances of this rule with respect to \(P\) and \(DB\) from above are (Note that there are 6 constant symbols: \{grete, linda, gerti, hans, michael, karl\}):

\[
\begin{align*}
\text{siblings}(\text{grete}, \text{grete}) & : \neg \text{parent}(\text{grete}, \text{grete}), \text{parent}(\text{grete}, \text{grete}), \\
& \text{grete} <> \text{grete} \ (X = Y = Z = \text{grete}) \\
\text{siblings}(\text{grete}, \text{linda}) & : \neg \text{parent}(\text{grete}, \text{grete}), \text{parent}(\text{grete}, \text{linda}), \\
& \text{grete} <> \text{linda} \ (X = Y = \text{grete}, Z = \text{linda}) \\
\cdots & \cdots \\
\text{siblings}(\text{karl}, \text{karl}) & : \neg \text{parent}(\text{karl}, \text{karl}), \text{parent}(\text{karl}, \text{karl}), \\
& \text{karl} <> \text{karl} \ (X = Y = Z = \text{karl})
\end{align*}
\]
**Idea:** execution of a datalog program $P$ on a database $DB$: iterative deduction of facts until saturation is reached (fixpoint)

**Formalization:** define a fixpoint operator

- define Operator $T_P(DB)$: augments $DB$ with all facts, that can be deduced in one step by applying the rules from $P$ to $DB$.

$$T_P(DB) = DB \cup \bigcup_{R \in P} \{ L_0 \mid L_0 ::= L_1, \ldots, L_n \in \text{Ground}(R; P, DB), L_1, \ldots, L_n \in DB \}$$

- $T_P$ is called the immediate consequence operator.
- $T_P^i(DB) = T_P(T_P^{i-1}(DB))$ iterated application of $T_P$. 
\[
T_P^0(DB) = DB
\]
\[
T_P^1(DB) = T_P(T_P^0(DB)) = T_P(DB)
\]
\[
= DB \cup \bigcup_{R \in P} \{ L_0 \mid L_0 : -L_1, \ldots, L_n \in \text{Ground}(R; P, DB),
L_1, \ldots, L_n \in DB \}
\]
\[
T_P^2(DB) = T_P(T_P^1(DB)) = T_P(T_P(DB))
\]
\[
\ldots \quad \ldots
\]
\[
T_P^i(DB) = T_P(T_P^{i-1}(DB)) = T_P(\ldots T_P(DB))
\]
\[
\ldots \quad \ldots
\]
Properties of $T_P(DB)$

- The set of facts is monotonically increasing i.e.:
  \[ T_P^i(DB) \subseteq T_P^{i+1}(DB) \]

- The sequence $\langle T_P^i(DB) \rangle$ converges finitely:
  There exists $n$ with $T_P^m(DB) = T_P^n(DB)$ for all $m \geq n$.

- $T_P^\omega(DB)$ . . . set of facts, to which $\langle T_P^i(DB) \rangle$ converges is the result of the application of $P$ to $DB$.

- The operational semantics of a datalog program $P$ assigns to each database $DB$ the set of facts $T_P^\omega(DB)$:
  \[ O[P] : DB \rightarrow T_P^\omega(DB). \]

Theorem (Equivalence of semantics)

Assume a program $P$. Then it holds that $M[P] = O[P]$. In other words, for any database $DB$, we have:  
$\text{cons}(P^* \land DB^*) = T_P^\omega(DB)$.
Proof of Theorem

Let $P$ be a program and $DB$ a database. We show

$$cons(P^* \land DB^*) = T_P^\omega(DB).$$

(1) We first show $T_P^\omega(DB) \subseteq cons(P^* \land DB^*)$. By induction on $i$, we show that $T_P^i(DB) \subseteq cons(P^* \land DB^*)$ for every $i \geq 0$. Note that this includes the case where $i = \omega$.

Base case. Assume $i = 0$. Take a fact $L \in T_P^0(DB)$. Then by definition of $T_P^0(DB)$, $L \in DB$. By definition, $DB^*$ is a conjunction of literals and $L$ occurs in it. Hence, by classical logic, $L \in cons(P^* \land DB^*)$.

The inductive step. Suppose $T_P^i(DB) \subseteq cons(P^* \land DB^*)$ for $i \geq 0$. We show that $T_P^{i+1}(DB) \subseteq cons(P^* \land DB^*)$. Recall that $T_P^{i+1}(DB) = T_P(T_P^i(DB))$. Thus by the definition of $T_P$,

$$T_P^{i+1}(DB) = T_P^i(DB) \cup \bigcup_{R \in P} \{L_0 \mid L_0 \vdash L_1, \ldots, L_n \in Ground(R, P, DB),
\quad L_1, \ldots, L_n \in T_P^i(DB)\}.$$
By the induction hypothesis, $T_P(DB) \subseteq cons(P^* \land DB^*)$. Thus it remains to show that $L_0 \in cons(P^* \land DB^*)$ for any rule $R \in P$ such that there is $L_0 \vdash L_1, \ldots, L_n \in \text{Ground}(R, P, DB)$ with $L_1, \ldots, L_n \in T_P(DB)$.

Assume such a rule $R = L'_0 \vdash L'_1, \ldots, L'_n$ in $P$, and suppose $\pi$ is the substitution of variables with constants such that applying $\pi$ to $R$ results in $L_0 \vdash L_1, \ldots, L_n$, i.e. $\pi(L'_j) = L_j$ for $j \in \{0, \ldots, n\}$.

By construction, in $P^* \land DB^*$ we have the conjunct

$$R^* = \forall x_1 \forall x_2 \ldots \forall x_\ell ((L'_1 \land L'_2 \land \cdots \land L'_n) \Rightarrow L'_0).$$

Thus, by employing the semantics of classical logic, for any variable substitution $\pi'$ such that $\{\pi'(L'_1), \ldots, \pi'(L'_n)\} \subseteq cons(P^* \land DB^*)$ we also have $\pi'(L'_0) \in cons(P^* \land DB^*)$. Since $\pi$ is a substitution such that $\{\pi(L'_1), \ldots, \pi(L'_n)\} = \{L_1, \ldots, L_n\} \subseteq cons(P^* \land DB^*)$ by the induction hypothesis, we get $\pi(L'_0) = L_0 \in cons(P^* \land DB^*)$. 


(2) We show \( \text{cons}(P^* \land DB^*) \subseteq T^\omega_P(DB) \). To this end, we prove that \( L \notin T^\omega_P(DB) \) implies \( L \notin \text{cons}(P^* \land DB^*) \), for any fact \( L \). We thus simply show that \( T^\omega_P(DB) \) is a model of \( P^* \land DB^* \).

This suffices because of the following simple property: if \( M \) is a model of a formula \( F \), then any fact \( L \notin M \) is not a logical consequence of \( F \) (as witnessed by \( M \) itself).
$T^\omega_P(DB)$ is a model of $DB^*$ because $DB = T^0_P(DB) \subseteq T^\omega_P(DB)$ by the definition of $T^\omega_P(DB)$.

It remains to show that $T^\omega_P(DB)$ is also a model of $P^*$. Consider an arbitrary rule $R \in P$. We have to show that $T^\omega_P(DB)$ is a model of $R^*$ with $R^* = \forall x_1 \forall x_2 \ldots \forall x_\ell ((L_1 \land L_2 \land \cdots \land L_n) \Rightarrow L_0)$.

Consider an arbitrary (ground) variable assignment $\pi$ on the variables $x_1, \ldots, x_\ell$. The only non-trivial case is that all facts $\pi(L_1), \ldots, \pi(L_n)$ are true in $T^\omega_P(DB)$, i.e., $\{\pi(L_1), \ldots, \pi(L_n)\} \subseteq T^\omega_P(DB)$.

We have to show that then also $\pi(L_0)$ is true in $T^\omega_P(DB)$, i.e., $\pi(L_0) \in T^\omega_P(DB)$.

We know $\pi(L_0) : - \pi(L_1), \ldots, \pi(L_n) \in \text{Ground}(R, P, DB)$. Thus by the definition of $T_P$, $\pi(L_0) \in T_P(T^\omega_P(DB))$. Since $T_P(T^\omega_P(DB)) = T^\omega_P(DB)$ by the definition of $T^\omega_P(DB)$, we obtain $\pi(L_0) \in T^\omega_P(DB)$. 
Algorithm: INFER

**INPUT**: datalog program $P$, database $DB$

**OUTPUT**: $T_P^\omega(DB)$ (\(= cons(P^* \land DB^*)\))

**STEP 1.** \(GP := \bigcup_{R \in P} \text{Ground}(R; P, DB),\)

\(*\ GP \ldots \text{ set of all ground instances }*\)

**STEP 2.** \(OLD := \{\}; NEW := DB;\)

**STEP 3.** \(\textbf{while } NEW \neq OLD \textbf{ do begin} \)

\(OLD := NEW;\) \(NEW := \text{ComputeTP}(OLD);\)

\(\textbf{end};\)

**STEP 4.** output $OLD$. 
Subroutine ComputeTP

**INPUT**: Set of facts $OLD$

**OUTPUT**: $T_P(OLD)$

**STEP 1.** $F := OLD$;

**STEP 2.**  
for each rule $L_0 :− L_1,\ldots, L_n$ in $GP$ do 
  if $L_1,\ldots, L_n \in OLD$
    then $F := F \cup \{ L_0 \}$;

**STEP 3.** return $F$;
Example

Apply the following program $P$ to calculate all ancestors of the above given database $DB$.

\[
\text{ancestor}(X,Y) :- \text{parent}(X,Y).
\]

\[
\text{ancestor}(X,Z) :- \text{parent}(X,Y), \text{ancestor}(Y,Z).
\]

Step 1. (INFER) build $GP$

\[
GP = \{ \text{ancestor} (\text{grete}, \text{grete}) :- \text{parent} (\text{grete}, \text{grete}),
    \text{ancestor} (\text{grete}, \text{linda}) :- \text{parent} (\text{grete}, \text{linda}),
    \ldots,
    \text{ancestor} (\text{grete}, \text{grete}) :- \text{parent} (\text{grete}, \text{grete}),
    \text{ancestor} (\text{grete}, \text{grete}),
    \text{ancestor} (\text{grete}, \text{grete}) :- \text{parent} (\text{grete}, \text{linda}),
    \text{ancestor} (\text{linda}, \text{grete}),
    \ldots \}\.
\]

(There are $6^2 + 6^3 = 252$ ground instances.)
Step 2.  \( \text{OLD} := \{\}, \text{NEW} := \text{DB}; \)

Step 3.  \( \text{OLD} \neq \text{NEW} \)

Cycle 1:  \( \text{OLD} := \text{DB}, \text{NEW} := \text{TP(OLD)} = \text{TP(DB)} \)
\[
\text{TP(OLD)} = \text{OLD} \cup \{\text{ancestor}(A, B) \mid \text{parent}(A, B) \in \text{DB}\};
\]

Cycle 2:  \( \text{OLD} := \text{TP(DB)}, \text{NEW} := \text{TP(OLD)} = \text{TP(TP(DB))} \)
\[
\text{TP(OLD)} = \text{OLD} \cup \{\text{ancestor}(\text{hans, michael}), \text{ancestor}(\text{hans, gerti}), \text{ancestor}(\text{grete, michael}), \text{ancestor}(\text{grete, gerti})\}.
\]

Cycle 3:  \( \text{TP(OLD)} = \text{OLD} \), there are no new facts

Step 4.  Output of \( \text{OLD} \).

The result corresponds to the extension of \( \text{DB} \) with the new table \( \text{ancestor} \)
<table>
<thead>
<tr>
<th>parent</th>
<th>(PARENT</th>
<th>CHILD)</th>
<th>ancestor</th>
<th>(ANCESTOR</th>
<th>NAME)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hans</td>
<td>Linda</td>
<td></td>
<td>Hans</td>
<td>Linda</td>
<td></td>
</tr>
<tr>
<td>Grete</td>
<td>Linda</td>
<td></td>
<td>Grete</td>
<td>Linda</td>
<td></td>
</tr>
<tr>
<td>Karl</td>
<td>Michael</td>
<td></td>
<td>Karl</td>
<td>Michael</td>
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<td>Linda</td>
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<td>Grete</td>
<td>Gerti</td>
<td></td>
</tr>
</tbody>
</table>
Datalog with negation

- Without negation, datalog is not relational complete because set difference ($R - S$) cannot be expressed.
- We introduce the negation ($\text{not}$) in bodies of rules.
- Restriction on the application of the negation:
  
  \textit{A relation }$R$\textit{ must not be defined on the basis of its negation.}

- Check for this constraint: with graph-theoretic methods.
Graph representation

Let \( P \) be a datalog program with negated literals in the body of rules.

**Definition: dependency graph**

\( DEP(P) \) is defined as the directed graph, with:

- nodes \( \ldots \) predicates (predicate symbols) \( p \) in \( P \),
- edges \( \ldots p \rightarrow q \), if there exists a rule in \( P \) where \( p \) is the head atom and \( q \) appears in the body (meaning: “\( p \) depends on \( q \)”).

Mark an edge \( p \rightarrow q \) of \( DEP(P) \) with a star “*”, if \( q \) in the body is negated.

**Definition**

A datalog program \( P \) with negation is called valid if the graph \( DEP(P) \) has no directed cycle that contains an edge marked with “*”.

Such programs are called **stratified**, since they can be divided into strata with respect to the negation.
Example

The following program $P$ with the rules:

\[
\text{husband}(X) :- \text{man}(X), \text{married}(X).
\]
\[
\text{bachelor}(X) :- \text{man}(X), \text{not } \text{husband}(X).
\]

is stratified.
The program $P$ with the rules:

$$
\text{husband}(X) := \text{man}(X), \text{not bachelor}(X).
\text{bachelor}(X) := \text{man}(X), \text{not husband}(X).
$$

is not stratified.
Stratification

Definition

A stratum is composed by the maximal set of predicates for which the following holds:

1. if a predicate $p$ appears in the head of a rule, that contains a negated predicate $q$ in the body, then $p$ is in a higher stratum than $q$.

2. if a predicate $p$ appears in the head of a rule, that contains an unnegated (positive) predicate $q$ in the body, then $p$ is in a stratum at least as high as $q$. 
Algorithm

**INPUT:** A set of datalog rules.

**OUTPUT:** the decision whether the program is stratified and the classification of the predicates into strata.

**METHOD:**

1. initialize the strata for all predicates with 1.
2. do for all rules $R$ with predicate $p$ in the head:
   - if (i) the body of $R$ contains a negated predicate $q$, (ii) the stratum of $p$ is $i$, and (iii) the stratum of $q$ is $j$ with $i \leq j$, then set $i := j + 1$.
   - if (i) the body of $R$ contains an unnegated predicate $q$, (ii) the stratum of $p$ is $i$, and (iii) the stratum of $q$ is $j$ with $i < j$, then set $i := j$.

until:

- status is stable ⇒ program is stratified.
- stratum $n > \#$ predicates ⇒ not stratified.
Example

Consider \( R, S \) relations of the database \( DB, P \):

\[
\begin{align*}
v(X,Y) & : - r(X,X), r(Y,Y). \\
u(X,Y) & : - s(X,Y), s(Y,Z), \text{not} \ v(X,Y). \\
w(X,Y) & : \text{not} \ u(X,Y), \ v(Y,X).
\end{align*}
\]

```
  *  \\
  / \\
/  \\

s v  r  \\
/  \\
/  \\
u *  \\
/ \\
/  \\

w  \\
/ \\
/  \\

r, s, v  \\
/  \\
/ \\
level 1

u  \\
/ \\
/ \\
level 2

w  \\
/ \\
/ \\
level 3
```

\(r, s, v\)
Semantics of datalog with negation

**Note:** when calculating the strata of a datalog program with negation the following holds:

- **Step 1:** computation of all relations of the first stratum.
- **Step i:** computation of all relations that belong to stratum $i$.
  $\Rightarrow$ the relations computed in step $i - 1$ are known in step $i$.

**Semantics** of datalog with negation is therefore uniquely defined.

Computation of $P$ from the last example above:

- **Step 1:** compute $V$ from $R$
- **Step 2:** compute $U$ from $S$ and $V$
- **Step 3:** compute $W$ from $U$ and $V
Properties of datalog with negation

- Datalog with negation is relational complete:
  - The difference $D = R - S$ of two (e.g. binary) relations $R$ and $S$:
    $$d(X,Y) :\neg r(X,Y), \ not \ s(X,Y).$$

- syntactical restrictions of datalog with negation:
  
  *all variables that appear in the body within a negated literal must also appear in a positive (unnegated) literal*
Example

Let $DB$ be a database that contains information on graphs, with relations $v(X)$, saying $X$ is a node and $e(X,Y)$ saying there is an edge from $X$ to $Y$. Write a datalog program that computes all pairs of nodes $(X,Y)$, where $X$ is a source, $Y$ is a sink and $X$ is connected to $Y$.

$$p(X,Y) :- \text{source}(X), \text{sink}(Y), \text{connection}(X,Y).$$

$$\text{connection}(X,X) :- v(X).$$
$$\text{connection}(X,Y) :- e(X,Z), \text{connection}(Z,Y).$$

$$n_{-}\text{source}(X) :- e(Y,X).$$
$$\text{source}(X) :- v(X), \text{not} \ n_{-}\text{source}(X).$$

$$n_{-}\text{sink}(X) :- e(X,Y).$$
$$\text{sink}(X) :- v(X), \text{not} \ n_{-}\text{sink}(X).$$
n_source: b, c, e, f  
n_sink: a, b, c, d  
connection: (a,a), . . . , (f,f), (a,b), (a,c), (a,e), (a,f), (b,c), (b,e), (c,e), (d,c), (d,e)  
source: a, d  
sink: e, f  
p: (a,e), (a,f), (d,e)
Learning objectives

- Motivation for Datalog (recursive queries)
- Syntax of Datalog
- Semantics of Datalog:
  - logical semantics,
  - operational semantics.
- Datalog with negation:
  - the need for negation,
  - the notions of dependency graph and stratification,
  - semantics of Datalog with negation.