Exercise 1 (5 credits) Recall the following characterizations of the complexity classes \( \Sigma_i^P \) and \( \Pi_i^P \) for \( i \geq 1 \).

**Theorem.**

- Let \( L \) be a language and \( i \geq 1 \). Then \( L \in \Sigma_i^P \) iff there is a polynomially balanced relation \( R \) such that the language \( \{ x \# y \mid (x, y) \in R \} \) is in \( \Pi_{i-1}^P \) and
  \[
  L = \{ x \mid \text{there exists a } y \text{ with } |y| \leq |x|^k \text{ s.t. } (x, y) \in R \}
  \]

- Let \( L \) be a language and \( i \geq 1 \). Then \( L \in \Pi_i^P \) iff there is a polynomially balanced relation \( R \) such that the language \( \{ x \# y \mid (x, y) \in R \} \) is in \( \Sigma_{i-1}^P \) and
  \[
  L = \{ x \mid \text{for all } y \text{ with } |y| \leq |x|^k, (x, y) \in R \}
  \]

**Corollary.**

- Let \( L \) be a language and \( i \geq 1 \). Then \( L \in \Sigma_i^P \) iff there is a polynomially balanced, polynomial-time decidable \((i + 1)\)-ary relation \( R \) such that
  \[
  L = \{ x \mid \exists y_1 \forall y_2 \exists y_3 \cdots Q y_i \text{ such that } (x, y_1, \ldots, y_i) \in R \}
  \]
  where \( Q \) is \( \forall \) if \( i \) is even and \( \exists \) if \( i \) is odd.

- Let \( L \) be a language and \( i \geq 1 \). Then \( L \in \Pi_i^P \) iff there is a polynomially balanced, polynomial-time decidable \((i + 1)\)-ary relation \( R \) such that
  \[
  L = \{ x \mid \forall y_1 \exists y_2 \forall y_3 \cdots Q y_i \text{ such that } (x, y_1, \ldots, y_i) \in R \}
  \]
  where \( Q \) is \( \exists \) if \( i \) is even and \( \forall \) if \( i \) is odd.

Give a rigorous proof of this corollary.
Hint. Use the above theorem and proceed by induction on $i$. It suffices to prove the correctness of the characterization of $\Sigma_i P$. You may use the characterization of $\Pi_i P$ in the induction step.

Exercise 2 (5 credits) Recall the $\Sigma_2 P$-hardness proof of MINIMAL MODEL SAT by reduction from the $\text{QSAT}_2$-problem: Let an arbitrary instance of $\text{QSAT}_2$ be given by the QBF

$$
\psi = (\exists x_1, \ldots, x_k)(\forall y_1, \ldots, y_\ell)\varphi
$$

Now let $\{x'_1, \ldots, x'_k, z\}$ be fresh propositional variables. Then we construct an instance of MINIMAL MODEL SAT by the variable $z$ and the formula

$$
\chi = (\bigwedge_{i=1}^k (\neg x_i \leftrightarrow x'_i)) \land (\neg \varphi \lor (y_1 \land \ldots \land y_\ell \land z))
$$

Recall from the lecture that we have already proved the following implication:

$\psi$ is true (in every interpretation) $\Rightarrow$ $z$ is true in a minimal model of $\chi$.

Give a rigorous proof also of the opposite direction, i.e.:

$z$ is true in a minimal model of $\chi$ $\Rightarrow$ $\psi$ is true (in every interpretation).

Hint. Let $J$ be a minimal model of $\chi$ and let $z$ be true in $J$.

- First show that then $J(y_j) = \text{true}$ for every $j$.
- Second, let $I$ be the truth assignment obtained by restricting $J$ to the variables $\{x_1, \ldots, x_k\}$. Show that (by the minimality of $J$) $I$ is indeed a partial assignment on $\{x_1, \ldots, x_k\}$ s.t. for any values assigned to $\{y_1, \ldots, y_\ell\}$, the formula $\varphi$ is true.