Characteristics of Multiple Viewpoints in Abstract Argumentation*

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Abstract. The study of extension-based semantics within the seminal abstract argumentation model of Dung has largely focused on definitional, algorithmic and complexity issues. In contrast, matters relating to comparisons of representational limits, in particular, the extent to which given collections of extensions are expressible within the formalism, have been under-developed. As such, little is known concerning conditions under which a candidate set of subsets of arguments are "realistic" in the sense that they correspond to the extensions of some argumentation framework AF for a semantics of interest. In this paper we present a formal basis for examining extension-based semantics in terms of the sets of extensions that these may express within a single AF. We provide a number of characterization theorems which guarantee the existence of AFs whose set of extensions satisfy specific conditions and derive preliminary complexity results for decision problems that require such characterizations.

1 Introduction

The last 15 years have seen an enormous effort to design, compare, and implement different semantics for Dung's abstract argumentation frameworks [13], AFs for short. Not at least this extensive study made argumentation a main topic of current AI research [7, 19]. Surprisingly, a systematic comparison of their capability in terms of multiple extensions, and thus their power in modelling multiple viewpoints with a single AF has been neglected so far. Understanding which extensions can, in principle, go together when a framework is evaluated with respect to a semantics of interest not only clarifies the "strength" of that semantics but also is a crucial issue in several applications.

In this work, we close this gap by studying the *signatures*

$$\Sigma_\sigma = \{\sigma(F) \mid F \text{ is an AF}\},$$

of several important semantics σ namely naive, preferred, semi-stable, stage, and stable semantics [13, 20, 10]. Finding simple criteria to decide whether a set $\mathcal{S}$ is contained in Σ_σ for different semantics σ is essential in many aspects.

First, it adds to the comparison of semantics (see, e.g., [3]) by means of different properties. So far these properties mostly focused on the aspects of a single extension

* This research has been supported by the Austrian Science Fund (FWF) through projects I1102 and P25521.

$S \in \mathcal{S}$ rather than on a set $\mathcal{S}$ thereof. An obvious exception is incomparability (the sets in $\mathcal{S}$ are not proper subsets of each other), but as we will see, all of the standard semantics put additional (yet different) requirements on $\mathcal{S}$ in order to be contained in the signature.

Second, our results are important for constructing AFs. Indeed, knowing whether a set $\mathcal{S}$ is contained in Σ_σ is a necessary condition which should be checked before actually looking for an AF F which *realizes* $\mathcal{S}$ via σ, i.e. $\sigma(F) = \mathcal{S}$. This is of high importance when dynamic aspects of argumentation are considered [18]. As an example, suppose a framework F possesses as its σ-extensions a set $\mathcal{S}$ and one asks for an adaptation of the framework F such that its σ-extensions are given by $\mathcal{S} \cup \{E\}$, i.e. one extension shall be added. Before starting to think how the adapted framework should look like, it is obviously crucial to know whether an appropriate framework exists at all, i.e. whether $S \cup \{E\} \in \Sigma_\sigma$. In a recent paper on revision of AFs [12], the authors circumvent this issue by allowing revision to result in a set of AFs such that the union of their extensions provides the desired outcome. Our results here provide exact conditions under which circumstances their approach can be reduced to single AFs as an outcome of a given revision.

Finally, we note a connection to instantiation-based argumentation [9], where the concept of *rationality postulates* plays an important role as does the underlying principle of evaluating argumentation semantics in *generic* terms. Our results on signatures show that, for a given semantics, certain outcomes (i.e. collections of extensions) are impossible to achieve independent of the concrete way the instantiation process is carried out.

Related work includes studies on enforcing [5, 6] certain outcomes, where the task is to modify AFs in such a way that desired arguments become acceptable. However, the issue of multiple extensions is not covered. In fact, the work which is closest to our investigations are studies of intertranslatability issues [15, 17], where signatures of semantics are put in relation to each other. More precisely, if there is a translation such that θ-extensions of the transformed AF coincide with the σ-extensions of the original AF, then θ is at least as expressive as σ, that is $\Sigma_\sigma \subseteq \Sigma_\theta$ in our terms. These results, however, do not tell us anything about the actual contents of Σ_σ and Σ_θ.

To summarise, the main contributions of our work are:

- We first identify necessary conditions any set of extensions under a given semantics σ satisfies. This not only guides us towards the exact characteristics for the signature of σ, but also determines those sets of extensions that are impossible to be jointly expressed with one AF.
- Then, we provide sufficient conditions for a set of extensions to be realizable under a given semantics σ. For any such realizable set $\mathcal{S}$ of extensions, we moreover provide constructions of canonical frameworks which have $\mathcal{S}$ as their σ-extensions. Together with the already provided necessary conditions, these realizability results yield exact characteristics of the signatures for the considered semantics.
- We also touch upon optimization issues and strengthen the concept of realizability in such a way that we want to find an AF F which is solely built from arguments occurring in $\mathcal{S}$ and delivers $\sigma(F) = \mathcal{S}$ (hence, no additional arguments to express

$\mathcal{S}$ are required). We show that for naive semantics each $\mathcal{S} \in \Sigma_{naive}$ can be strictly realized, while this is not the case for the other semantics.
• One particular application of our results is the problem of recasting, i.e. to decide whether the σ-extensions of a given AF can be expressed via a different semantics θ. We give some preliminary complexity results of the recasting problem, which go up to Π_2^P-completeness.

2 Preliminaries

In what follows, we briefly recall the necessary background on abstract argumentation. For an excellent recent overview, we refer to [1].

Throughout the paper we assume a countably infinite domain $\mathcal{A}$ of arguments. An *argumentation framework* (AF) is a pair $F = (A, R)$ where $A \subseteq \mathcal{A}$ is a non-empty, finite set of arguments and $R \subseteq A \times A$ is the attack relation. The collection of all AFs is given as $AF_{\mathcal{A}}$. We write $a \mapsto_R b$ for $(a, b) \in R$ and $S \mapsto_R a$ (resp. $a \mapsto_R S$) if $\exists s \in S$ such that $s \mapsto_R a$ (resp. $a \mapsto_R s$). We drop subscript R in $\mapsto_R$ if there is no ambiguity. For $S \subseteq A$, the *range* of S (wrt. R), denoted S_R^+, is the set $S \cup \{b \mid S \mapsto_R b\}$.

Given $F = (A, R)$, an argument $a \in A$ is *defended* (in F) by a set $S \subseteq A$ if for each $b \in A$, such that $b \mapsto_R a$, also $S \mapsto_R b$. A set T of arguments is defended (in F) by S if each $a \in T$ is defended by S (in F). The following result is in spirit of Dung's fundamental lemma. We will need it later.

Lemma 1. *Given an* AF $F = (A, R)$ *and two sets of arguments* $S, T \subseteq A$. *If* S *defends itself in* F *and* T *defends itself in* F, *then* $S \cup T$ *defends itself in* F.

Given an AF $F = (A, R)$, a set $S \subseteq A$ is *conflict-free* (in F), if there are no arguments $a, b \in S$, such that $(a, b) \in R$. We denote the set of all conflict-free sets in F as $cf(F)$. $S \in cf(F)$ is called *admissible* (in F) if S defends itself. We denote the set of admissible sets in F as $adm(F)$.

The semantics we focus in this work are the naive, stable, preferred, stage, and semi-stable extensions. Given $F = (A, R)$ they are defined as subsets of $cf(F)$ as follows:

- $S \in naive(F)$, if there is no $T \in cf(F)$ with $T \supset S$
- $S \in stb(F)$, if $S \mapsto a$ for all $a \in A \setminus S$
- $S \in pref(F)$, if $S \in adm(F)$ and $\nexists T \in adm(F)$ s.t. $T \supset S$
- $S \in stage(F)$, if $\nexists T \in cf(F)$ with $T_R^+ \supset S_R^+$
- $S \in sem(F)$, if $S \in adm(F)$ and $\nexists T \in adm(F)$ s.t. $T_R^+ \supset S_R^+$

The objects of our interest are the signatures of a semantics.

Definition 1. *The signature* Σ_σ *of a semantics* σ *is defined as*

$$\Sigma_\sigma = \{\sigma(F) \mid F \in AF_{\mathcal{A}}\}$$

For characterizing the signatures of the semantics of our interest we will make frequent use of the following concepts.

Definition 2. *Given* $\mathcal{S} \subseteq 2^{\mathcal{A}}$, $Args_{\mathcal{S}}$ *denotes* $\bigcup_{S \in \mathcal{S}} S$ *and* $Pairs_{\mathcal{S}}$ *denotes* $\{(a, b) \mid \exists S \in \mathcal{S} : \{a, b\} \subseteq S\}$. $\mathcal{S}$ *is called an* extension-set *(over* $\mathcal{A}$*) if* $Args_{\mathcal{S}}$ *is finite.*

As is easily observed, for all considered semantics σ each element $\mathcal{S} \in \Sigma_\sigma$ is an extension-set.

3 Properties of Argumentation Semantics

Our ultimate goal is to characterize the signatures of the semantics under consideration. In this section, we provide necessary conditions for an extension-set $\mathcal{S}$ to be in the signature. We do so by abstracting away from the syntactical structure of a given AF; instead we provide characterizations for the sets $\sigma(F)$. The first properties, which we define next, enable us to characterize conflict-free sets and naive, stable and stage extensions.

Definition 3. *Let $\mathcal{S} \subseteq 2^{\mathcal{A}}$. The* downward-closure*, $dcl(\mathcal{S})$, of $\mathcal{S}$ is given by $\{S' \subseteq S \mid S \in \mathcal{S}\}$. Further we call $\mathcal{S}$*

- *downward-closed if $\mathcal{S} = dcl(\mathcal{S})$;*
- *tight if for all $S \in \mathcal{S}$ and $a \in Args_{\mathcal{S}}$ it holds that if $S \cup \{a\} \notin \mathcal{S}$ then there exists an $s \in S$ such that $(a, s) \notin Pairs_{\mathcal{S}}$;*
- *incomparable if all elements $S \in \mathcal{S}$ are pairwise incomparable, i.e. for each $S, S' \in \mathcal{S}$, $S \subseteq S'$ implies $S = S'$.*

In words, an extension-set is downward-closed, if for each element of the extension-set, all subsets of this element are in the extension-set too. The notion of being tight, in a way, limits the multitude of incomparable elements of an extension-set.

Proposition 1. *For each* AF $F = (A, R)$*,*

1. *$cf(F)$ is non-empty, downward-closed and tight;*
2. *$naive(F)$ is non-empty, incomparable and its downward-closure is tight;*
3. *$stage(F)$ is non-empty, incomparable and tight;*
4. *$stb(F)$ is incomparable and tight.*

Proof. The properties of being non-empty and incomparable are clear. Likewise, it is easy to see that $cf(F) = dcl(cf(F))$.

To show that $cf(F)$ is tight let $S \in cf(F)$ and $a \in Args_{cf(F)}$, such that $S \cup \{a\} \notin cf(F)$. It follows that $S \neq \emptyset$. Moreover there exists an argument $s \in S$ such that $s \mapsto a$ or $a \mapsto s$. Then $\{a, s\} \notin cf(F)$ and since $cf(F)$ is downward-closed, $\{a, s\} \not\subseteq T$ for any $T \in cf(F)$. It follows that $(a, s) \notin Pairs_{cf(F)}$.

Next, observe that $dcl(naive(F)) = cf(F)$. It follows that $dcl(naive(F))$ is tight. Also note that if a set $\mathcal{S} \subseteq 2^{\mathcal{A}}$ is tight, then the subset-maximal elements in $\mathcal{S}$ form a tight set $\mathcal{S}'$ too (since for each $S \in \mathcal{S}'$ and $a \in Args_{\mathcal{S}'}$, if $S \cup \{a\} \notin \mathcal{S}'$ then $S \cup \{a\} \notin \mathcal{S}$; and moreover, $Pairs_{\mathcal{S}} = Pairs_{\mathcal{S}'}$). In other words, since $dcl(naive(F))$ is tight, it follows that $naive(F)$ is tight. Finally, for each incomparable $\mathcal{S} \subseteq 2^{\mathcal{A}}$ it holds that if $\mathcal{S}$ is tight then $\mathcal{S}'$ is tight for each $\mathcal{S}' \subseteq \mathcal{S}$. Using $stb(F) \subseteq stage(F) \subseteq naive(F)$, the result thus follows. □

Example 1. For the AF F in Figure 1, we have $\mathcal{S} = stb(F) = stage(F) = \{\{a_1, b_2, b_3\}, \{a_2, b_1, b_3\}, \{a_3, b_1, b_2\}\}$. $\mathcal{S}$ is indeed tight. Take, for instance, $E = \{a_1, b_2, b_3\}$; then none of (a_1, b_1), (a_1, a_2), (a_1, a_3) is contained in $Pairs_{\mathcal{S}}$. The other two extensions behave in a symmetric way. However, $dcl(\mathcal{S})$ is not tight. In fact, $\{b_2, b_3\} \in dcl(\mathcal{S})$ and now for b_1, $\{b_1, b_2, b_3\} \notin dcl(\mathcal{S})$ but (b_1, b_2) and (b_1, b_3) are contained in $Pairs_{dcl(\mathcal{S})} = Pairs_{\mathcal{S}}$. By Proposition 1, thus no AF G with $naive(G) = \mathcal{S}$ exists (note that $naive(F) = \mathcal{S} \cup \{\{b_1, b_2, b_3\}\}$).

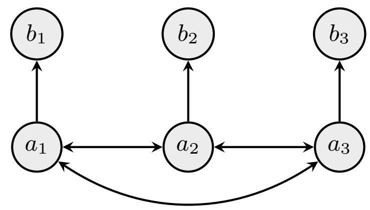


Fig. 1. Argumentation framework F used in Example 1.

We now turn to admissible sets.

Definition 4. *A set $\mathcal{S} \subseteq 2^{\mathcal{A}}$ is called adm-closed if for each $A, B \in \mathcal{S}$ the following holds: if $(a, b) \in \mathit{Pairs}_{\mathcal{S}}$ for each $a, b \in A \cup B$, then also $A \cup B \in \mathcal{S}$.*

The property adm-closed is related to aforementioned properties as follows:

Lemma 2. *For any extension-set $\mathcal{S} \subseteq 2^{\mathcal{A}}$ it holds that if $\mathcal{S}$ is downward-closed and tight, then $\mathcal{S}$ is adm-closed.*

The reverse of Lemma 2 does not hold, i.e. there is an extension-set (e.g. $\{\{a, b\}, \{a, c, e\}, \{b, d, e\}\}$), which is adm-closed, but not tight. The following proposition gives necessary conditions for sets of extensions obtained from the admissible semantics.

Proposition 2. *For each* AF $F = (A, R)$*, $\mathit{adm}(F)$ is an adm-closed extension-set containing $\emptyset$.*

Proof. By definition $\emptyset$ is always admissible. We show that $\mathit{adm}(F)$ is adm-closed. Towards a contradiction, assume $B, C \in \mathit{adm}(F)$ such that for all $b, c \in B \cup C$, $(b, c) \in \mathit{Pairs}_{\mathit{adm}(F)}$, but $B \cup C \notin \mathit{adm}(F)$. From Lemma 1 we know that $B \cup C$ defends itself in F. So for $B \cup C \notin \mathit{adm}(F)$ there must be a conflict in $B \cup C$, i.e. $\exists (b, c) \in R$ such that $\{b, c\} \subseteq B \cup C$. But then, for all $D \in \mathit{adm}(F)$, $\{b, c\} \not\subseteq D$. Hence, $(b, c) \notin \mathit{Pairs}_{\mathit{adm}(F)}$, a contradiction. □

The next property characterizes preferred and semi-stable semantics.

Definition 5. *A set $\mathcal{S} \subseteq 2^{\mathcal{A}}$ is pref-closed if for each $A, B \in \mathcal{S}$, $A \neq B$, there exist $a, b \in A \cup B$ such that $(a, b) \notin \mathit{Pairs}_{\mathcal{S}}$.*

It is easy to verify that each pref-closed extension-set is incomparable. Moreover, for an incomparable set, pref-closed is a stricter notion than tight. Lemma 3 together with Example 2 will show this.

Lemma 3. *For a set $\mathcal{S} \subseteq 2^{\mathcal{A}}$ it holds that if $\mathcal{S}$ is incomparable and tight, then $\mathcal{S}$ is pref-closed.*

Proof. Consider some incomparable and tight extension-set $\mathcal{S} \subseteq 2^{\mathcal{A}}$ and assume that $\mathcal{S}$ is not pref-closed. That means that there are some $A, B \in \mathcal{S}$ with $A \neq B$ such that $\forall a, b \in (A \cup B) : (a, b) \in \mathit{Pairs}_{\mathcal{S}}$. Since $\mathcal{S}$ is incomparable, $B \neq \emptyset$ and $\forall b \in B$: $(A \cup \{b\}) \notin \mathcal{S}$. Considering an arbitrary $b \in B$ we get $\exists a \in A : (a, b) \notin \mathit{Pairs}_{\mathcal{S}}$ by the fact that $\mathcal{S}$ is tight, a contradiction to $\forall a, b \in (A \cup B) : (a, b) \in \mathit{Pairs}_{\mathcal{S}}$. □

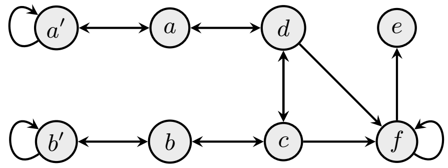


Fig. 2. Argumentation framework F used in Example 2.

We relate the notions of adm- and pref-closed and then show our final characterization result.

Lemma 4. *A set $\mathcal{S} \subseteq 2^{\mathcal{A}}$ is pref-closed iff it is incomparable and adm-closed.*

Proof. Let $\mathcal{S}$ be incomparable and adm-closed and $A, B \in \mathcal{S}$. If $A \neq B$, then $A \cup B \notin \mathcal{S}$ (by incomparability). Since $\mathcal{S}$ is adm-closed, there exist $a, b \in A \cup B$ such that $(a, b) \notin \mathit{Pairs}_{\mathcal{S}}$. It follows that $\mathcal{S}$ is pref-closed. Now consider a set $\mathcal{S} \subseteq 2^{\mathcal{A}}$ not incomparable, i.e. $\exists A, B \in \mathcal{S} : A \subset B$. But then for all $a, b \in A \cup B = B : (a, b) \in \mathit{Pairs}_{\mathcal{S}}$ and thus $\mathcal{S}$ is not pref-closed. Finally consider an incomparable $\mathcal{S}$ which is not adm-closed. Then there are $A, B \in \mathcal{S}$ such that for all $a, b \in A \cup B : (a, b) \in \mathit{Pairs}_{\mathcal{S}}$ and again $\mathcal{S}$ is not pref-closed. □

Proposition 3. *For each* AF $F = (A, R)$, $\sigma \in \{\mathit{pref}, \mathit{sem}\}$, $\sigma(F)$ *is a non-empty, pref-closed extension-set.*

Proof. By definition both semantics $\sigma \in \{\mathit{pref}, \mathit{sem}\}$ always propose at least one extension. Since $\mathit{sem}(F) \subseteq \mathit{pref}(F)$ holds for all AFs F, it is sufficient to show that $\mathit{pref}(F)$ is pref-closed. Towards a contradiction, let $B, C \in \mathit{pref}(F)$ ($B \neq C$), such that for all $a, b \in B \cup C$, $(a, b) \in \mathit{Pairs}_{\mathit{pref}(F)}$. It follows that $B \cup C \in \mathit{cf}(F)$ and by Lemma 1, $B \cup C \in \mathit{adm}(F)$. Since $B \cup C \supset B$, this is a contradiction to $B \in \mathit{pref}(F)$. □

Example 2. Consider the AF F in Figure 2 and let $A = \{a, b\}$, $B = \{a, c, e\}$, $C = \{b, d, e\}$, and $\mathcal{S} = \{A, B, C\}$. We have $\mathit{pref}(F) = \mathit{sem}(F) = \mathcal{S}$ and, indeed, $\mathcal{S}$ is pref-closed: $b, c \in A \cup B$ and $(b, c) \notin \mathit{Pairs}_{\mathcal{S}}$; $a, d \in A \cup C$ and $(a, d) \notin \mathit{Pairs}_{\mathcal{S}}$; $c, d \in B \cup C$ and $(c, d) \notin \mathit{Pairs}_{\mathcal{S}}$. However, we also observe that $\mathcal{S}$ is not tight, since $A \cup \{e\} \notin \mathcal{S}$ but both (a, e) and (b, e) are contained in $\mathit{Pairs}_{\mathcal{S}}$.

4 Realizability and Signatures

In the previous section we have given necessary characteristics for the extension-sets $\mathcal{S} \in \Sigma_\sigma$, where $\sigma \in \{\mathit{cf}, \mathit{adm}, \mathit{naive}, \mathit{stb}, \mathit{stage}, \mathit{pref}, \mathit{sem}\}$ are the semantics of our interest. Now we will show that these characteristics are also sufficient. To this end, we require the concept of *realizability*. In words, an extension-set $\mathcal{S} \subseteq 2^{\mathcal{A}}$ is σ-realizable if there exists an AF $F \in AF_{\mathcal{A}}$, such that $\sigma(F) = \mathcal{S}$. This turns our characteristics into the desired characterizations for Σ_σ.

We start with the following concept of a canonical argumentation framework, which will underlie all subsequent results on realizability.

Definition 6. *Given an extension-set $\mathcal{S}$, we define the canonical argumentation framework for $\mathcal{S}$ as*

$$F^{cf}_{\mathcal{S}} = \big(Arg_{\mathcal{S}}, (Arg_{\mathcal{S}} \times Arg_{\mathcal{S}}) \setminus Pairs_{\mathcal{S}}\big).$$

The underlying idea for the framework is simple. Whenever two arguments occur jointly in a set $S \in \mathcal{S}$, we must not draw a relation between these two arguments; otherwise we do so. Thus, for any $\mathcal{S}$, $F^{cf}_{\mathcal{S}}$ is symmetric and does not contain any self-attacking arguments. For $\mathcal{T} = \{\{a_1, b_2, b_3\}, \{a_2, b_1, b_3\}, \{a_3, b_1, b_2\}, \{b_1, b_2, b_3\}\}$, $F^{cf}_{\mathcal{T}}$ has the same structure as the AF from Figure 1, but with all attacks being symmetric (in fact, orientation does not matter for naive semantics) and $naive(F^{cf}_{\mathcal{T}}) = \mathcal{T}$ holds. When we consider $\mathcal{S} = \{\{a_1, b_2, b_3\}, \{a_2, b_1, b_3\}, \{a_3, b_1, b_2\}\}$, i.e. $\mathcal{S} = \mathcal{T} \setminus \{\{b_1, b_2, b_3\}\}$, we obtain the same framework $F^{cf}_{\mathcal{T}} = F^{cf}_{\mathcal{S}}$. In terms of naive semantics, this is not problematic since $\mathcal{S}$ (as discussed in Example 1) cannot be realized via naive semantics, at all. However, this observation readily suggests that realizing $\mathcal{S}$ with, say, stable semantics, requires additional concepts. We will come back to this issue, but first state some formal results on the canonical framework.

Proposition 4. *For each non-empty, downward-closed and tight extension-set $\mathcal{S}$ it holds that $cf(F^{cf}_{\mathcal{S}}) = \mathcal{S}$.*

Proof. Let $F^{cf}_{\mathcal{S}} = (Args_{\mathcal{S}}, R^{cf}_{\mathcal{S}})$. To show $cf(F^{cf}_{\mathcal{S}}) \subseteq \mathcal{S}$, observe that for each $E \in cf(F^{cf}_{\mathcal{S}})$, $(a, b) \in Pairs_{\mathcal{S}}$ for all $a, b \in E$, by construction of $R^{cf}_{\mathcal{S}}$. Now suppose there exists $E' \in cf(F^{cf}_{\mathcal{S}})$ such that $E' \notin \mathcal{S}$. Wlog. let E' be $\subseteq$-minimal with this property. Then $E' = S \cup \{c\}$ for some $S \in \mathcal{S}$. As $\mathcal{S}$ is tight and $c \in Args_{\mathcal{S}}$ by construction of $F^{cf}_{\mathcal{S}}$ there is an $s \in S$ such that $(s, c) \notin Pairs_{\mathcal{S}}$, a contradiction to the above observation. To show $cf(F^{cf}_{\mathcal{S}}) \supseteq \mathcal{S}$, let $S \in \mathcal{S}$. All $a, b \in S$ are contained as pairs (a, b) in $Pairs_{\mathcal{S}}$, thus by construction, $(a, b) \notin R^{cf}_{\mathcal{S}}$. Hence $S \in cf(F^{cf}_{\mathcal{S}})$. □

We approach the characterization for naive-realizable sets by the following result, which will be useful later.

Lemma 5. *For each incomparable and tight extension-set $\mathcal{S}$ it holds that $\mathcal{S} \subseteq naive(F^{cf}_{\mathcal{S}})$.*

Proof. Assume there is an $S \in \mathcal{S}$ such that $S \notin naive(F^{cf}_{\mathcal{S}})$. If $S \notin cf(F^{cf}_{\mathcal{S}})$, $\exists a, b \in S : (a, b) \notin Pairs_{\mathcal{S}}$, a contradiction to $S \in \mathcal{S}$. Thus $\exists S' \supset S : S' \in cf(F^{cf}_{\mathcal{S}})$. Then by construction of $F^{cf}_{\mathcal{S}}$ $\forall a, b \in S' : (a, b) \in Pairs_{\mathcal{S}}$. Since $\mathcal{S}$ is tight also $S' \in \mathcal{S}$, a contradiction to $\mathcal{S}$ being incomparable. □

We are now ready to give the full characterization.

Proposition 5. *For each incomparable and non-empty extension-set $\mathcal{S}$, where $dcl(\mathcal{S})$ is tight it holds that $naive(F^{cf}_{\mathcal{S}}) = \mathcal{S}$.*

Proof. Since $dcl(\mathcal{S})$ is surely downward-closed, as well as tight and non-empty by definition, we know from Proposition 4 that $cf(F^{cf}_{\mathcal{S}}) = dcl(\mathcal{S})$ (note that $F_{dcl(\mathcal{S})} = F^{cf}_{\mathcal{S}}$). By construction of $dcl(\mathcal{S})$ the $\subseteq$-maximal sets in $dcl(\mathcal{S})$ are the sets $S \in \mathcal{S}$ ($\mathcal{S}$ is incomparable by assumption) and as naive sets are just $\subseteq$-maximal conflict-free, $naive(F^{cf}_{\mathcal{S}}) = \mathcal{S}$. □

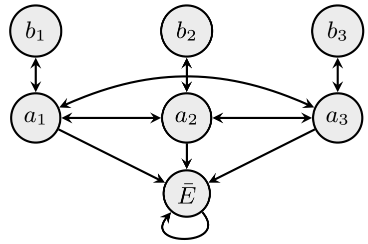


Fig. 3. Excluding the naive extension $\{b_1, b_2, b_3\}$ from $F^{cf}_{\mathcal{S}}$.

So far, in order to realize a set $\mathcal{S}$ we used a framework from $AF_{\mathcal{A}}$ of the form (A, R) with $A = \mathit{Args}_{\mathcal{S}}$. For the subsequent results we require, in general, frameworks with $\mathit{Args}_{\mathcal{S}} \subset A$. In the next section, we will show that this cannot be avoided. For the moment, we recall that $\mathcal{A}$ is infinite, hence there are always enough arguments available in $\mathcal{A}$.

Let us proceed with stable and stage semantics. Stable semantics are the only semantics that can realize $\mathcal{S} = \emptyset$. Note that $\mathcal{S} = \emptyset$ is easily *stb*-realizable, for instance with the framework $(\{a\}, \{(a, a)\})$. In Proposition 1 the only difference between stable and stage semantics was the case $\mathcal{S} = \emptyset$. The next result shows that this indeed is the only difference between the signatures for stable and stage semantics.

The idea of the construction used in the forthcoming proof is to suitably extend the canonical framework from Definition 6 such that undesired stable and stage extensions are excluded[4]. Coming back to our example with $\mathcal{S} = \{\{a_1, b_2, b_3\}, \{a_2, b_1, b_3\}, \{a_3, b_1, b_2\}\}$, recall that $F^{cf}_{\mathcal{S}}$ had one such undesired extension, $E = \{b_1, b_2, b_3\}$. To get rid of it we add a new argument which is attacked by all other sets from $\mathcal{S}$ but not by E, see Figure 3 for illustration.

Proposition 6. *For each non-empty, incomparable and tight extension-set $\mathcal{S}$, there exists an* AF *F such that $\mathit{stb}(F) = \mathit{stage}(F) = \mathcal{S}$.*

Proof. Since $\mathcal{S}$ is non-empty, showing existence of an AF F with $\mathit{stb}(F) = \mathcal{S}$ is sufficient (for each F with $\mathit{stb}(F) \neq \emptyset$, $\mathit{stb}(F) = \mathit{stage}(F)$ holds).

By Lemma 5 we already know that $\mathcal{S} \subseteq \mathit{naive}(F^{cf}_{\mathcal{S}})$. Let $\mathcal{X} = \mathit{naive}(F^{cf}_{\mathcal{S}}) \setminus \mathcal{S}$ and consider the AF extending $F^{cf}_{\mathcal{S}}$ as follows: $F'_{\mathcal{S}} = (\mathit{Args}_{\mathcal{S}} \cup \{\bar{E} \mid E \in \mathcal{X}\}, R'_{\mathcal{S}})$ with $R'_{\mathcal{S}} = \{((\mathit{Args}_{\mathcal{S}} \times \mathit{Args}_{\mathcal{S}}) \setminus \mathit{Pairs}_{\mathcal{S}}) \cup \{(\bar{E}, \bar{E}), (a, \bar{E}) \mid E \in \mathcal{X}, a \in \mathit{Args}_{\mathcal{S}} \setminus E\}\}$ (this construction is borrowed from [15]). We show that $\mathit{stb}(F'_{\mathcal{S}}) = \mathcal{S}$.

$\mathit{stb}(F'_{\mathcal{S}}) \subseteq \mathcal{S}$: Let $E \in \mathit{stb}(F'_{\mathcal{S}})$. As all new arguments $\bar{E}$ are self-attacking, also $E \in \mathit{naive}(F'_{\mathcal{S}}) = \mathit{naive}(F^{cf}_{\mathcal{S}}) = \mathcal{X} \cup \mathcal{S}$. If $E \in \mathcal{X}$, by construction of $F'_{\mathcal{S}}$, $E \not\mapsto \bar{E}$ and also $\bar{E} \notin E$, thus $E \notin \mathit{stb}(F'_{\mathcal{S}})$. Hence it must be that $E \in \mathcal{S}$.

$\mathit{stb}(F'_{\mathcal{S}}) \supseteq \mathcal{S}$: Let $E \in \mathcal{S}$. By Lemma 5, $E \in \mathit{naive}(F^{cf}_{\mathcal{S}})$, and, as $F^{cf}_{\mathcal{S}}$ is symmetric, $E \in \mathit{stb}(F^{cf}_{\mathcal{S}})$. Now consider $F'_{\mathcal{S}}$. As we do not change attacks between the arguments $\mathit{Args}_{\mathcal{S}}$, $E \in \mathit{naive}(F'_{\mathcal{S}})$ and E attacks all arguments in $\mathit{Args}_{\mathcal{S}} \setminus E$. Now consider an arbitrary argument $\bar{E}'$ for $E' \in \mathcal{X}$. $\bar{E}'$ is attacked by all arguments $a \in \mathit{Args}_{\mathcal{S}} \setminus E'$ and

[4] Recall that in every symmetric AF F it holds that $\mathit{naive}(F) = \mathit{stb}(F) = \mathit{stage}(F)$.

as E, E' are both naive sets (and thus incomparable) at least one of these arguments must be contained in E. Hence $E \in stb(F'_{\mathcal{S}})$ follows. □

Towards a suitable canonical AF for admissibility-based semantics we introduce the following technical concept.

Definition 7. *Given an extension-set $\mathcal{S}$, the defense-formula $D_a^{\mathcal{S}}$ of an argument $a \in Args_{\mathcal{S}}$ is $\top$ if $\{a\} \in \mathcal{S}$ and*

$$\bigvee_{S \in \mathcal{S} \text{ s.t. } a \in S} \quad \bigwedge_{s \in S \setminus \{a\}} s$$

otherwise. $D_a^{\mathcal{S}}$ converted to (a logical equivalent) conjunctive normal form in clause-form is then called CNF-defense-formula $C_a^{\mathcal{S}}$ of a (in $\mathcal{S}$).

Intuitively, $D_a^{\mathcal{S}}$ describes the conditions for the argument a being in an extension. The variables coincide with the arguments. Each disjunct represents a set of arguments which jointly allows a to "join" an extension, i.e. represents a collection of arguments defending a.

Example 3. Consider $\mathcal{T} = \{\{a\}, \{b, c\}, \{a, c, d\}\}$. Then $D_a^{\mathcal{T}} = \top$, $D_b^{\mathcal{T}} = c$, $D_c^{\mathcal{T}} = b \vee (a \wedge d)$ and $D_d^{\mathcal{T}} = a \wedge c$. The corresponding CNF-defense-formulas are given as $C_a^{\mathcal{T}} = \{\}$, $C_b^{\mathcal{T}} = \{\{c\}\}$, $C_c^{\mathcal{T}} = \{\{a, b\}, \{b, d\}\}$, and $C_d^{\mathcal{T}} = \{\{a, c\}\}$.

The following lemma shows that the (CNF-)defense-formula for any argument a really captures the intuition of describing which arguments it takes for a in order to join an element of the given extension-set.

Lemma 6. *Given an extension-set $\mathcal{S}$ and an argument $a \in Args_{\mathcal{S}}$, for each $S \subseteq Args_{\mathcal{S}}$ with $a \in S$ the following holds: $(S \setminus \{a\})$ is a model of $D_a^{\mathcal{S}}$ (resp. $C_a^{\mathcal{S}}$) iff there exists an $S' \subseteq S$ with $a \in S'$ such that $S' \in \mathcal{S}$.*

Proof. The if-direction follows straight by definition of $D_a^{\mathcal{S}}$.

To show the only-if-direction consider some $S \subseteq Args_{\mathcal{S}}$ with $a \in S$ where $S \setminus \{a\}$ is a model of $D_a^{\mathcal{S}}$. If $D_a^{\mathcal{S}} = \top$ then by Definition 7 it holds that $\{a\} \in \mathcal{S}$.For $S \setminus \{a\}$ to be a model of $D_a^{\mathcal{S}} \neq \top$, there must be some disjunct $\delta \in D_a^{\mathcal{S}}$, whose elements form a subset of $S \setminus \{a\}$. Consider such a term $\delta \in D_a^{\mathcal{S}}$. Then by construction of $D_a^{\mathcal{S}}$ there is some $S' \in \mathcal{S}$ with $a \in S'$, where $S' \setminus \{a\}$ coincides with the elements of δ. So $S' \subseteq S$.

Since $D_a^{\mathcal{S}} \equiv C_a^{\mathcal{S}}$, these formulas can be used interchangeably in this context. □

Having at hand a formula for each argument, where its models coincide with the sets of arguments that defend this original argument, we can give the construction of our canonical defense-argumentation-framework.

Definition 8. *Given an extension-set $\mathcal{S}$, the* canonical defense-argumentation-framework $F_{\mathcal{S}}^{def} = (A_{\mathcal{S}}^{def}, R_{\mathcal{S}}^{def})$ *extends the canonical* AF $F_{\mathcal{S}}^{cf} = (Args_{\mathcal{S}}, R_{\mathcal{S}}^{cf})$ *as follows:*

$$A_{\mathcal{S}}^{def} = Args_{\mathcal{S}} \cup \bigcup_{a \in Args_{\mathcal{S}}} \{\alpha_{a,\gamma} \mid \gamma \in C_a^{\mathcal{S}}\}, \text{ and}$$

$$R_{\mathcal{S}}^{def} = R_{\mathcal{S}}^{cf} \cup \bigcup_{a \in Args_{\mathcal{S}}} \{(b, \alpha_{a,\gamma}), (\alpha_{a,\gamma}, \alpha_{a,\gamma}), (\alpha_{a,\gamma}, a) \mid \gamma \in C_a^{\mathcal{S}}, b \in \gamma\}.$$

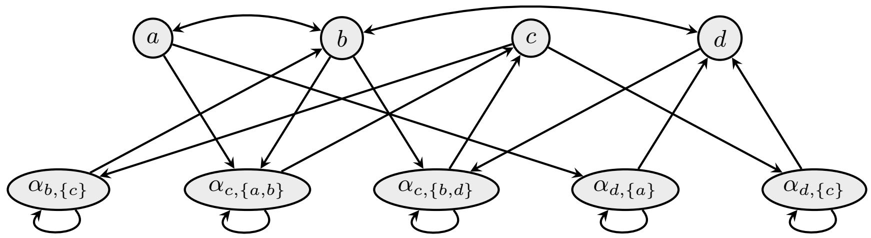


Fig. 4. AF $F^{def}_{\mathcal{T}}$ used in Example 4

$F^{def}_{\mathcal{S}}$ consists of all arguments given in the extension-set plus a certain amount of additional arguments, $\alpha_{a,c}$. Each $\alpha_{a,c}$ attacks argument a and is attacked by all arguments occurring as literals in clause c of the CNF-defense-formula of a. So in $F^{def}_{\mathcal{S}}$ for a to be defended from $\alpha_{a,c}$ it takes at least one argument of these occurring as atoms in clause c of $C^{\mathcal{S}}_a$, simulating the intended meaning of the defense-formulas.

The following proposition shows that, given any extension-set $\mathcal{S}$, each element of $\mathcal{S}$ is admissible in the canonical defense-argumentation-framework $F^{def}_{\mathcal{S}}$.

Proposition 7. *For each extension-set $\mathcal{S}$ it holds that $\mathcal{S} \subseteq adm(F^{def}_{\mathcal{S}})$.*

Proof. Let $S \in \mathcal{S}$. If $S = \emptyset$, the assertion trivially holds. If $S = \{a\}$, then $C^{\mathcal{S}}_a$ is the empty set of clauses. By definition of $R^{def}_{\mathcal{S}}$, a is then defended in $F^{def}_{\mathcal{S}}$ and thus $S \in adm(F^{def}_{\mathcal{S}})$. Thus let $S \in \mathcal{S}$ contain at least two arguments. By construction, S is conflict-free in $F^{def}_{\mathcal{S}}$. It remains to show that each $s \in S$ is defended by S in $F^{def}_{\mathcal{S}}$. Let $s \in S$. If $\{s\} \in \mathcal{S}$, we know from the one-element case that $\{s\} \in adm(F^{def}_{\mathcal{S}})$, so s is defended. On the other hand, if $\{s\} \notin \mathcal{S}$, we know from Lemma 6 that $S \setminus \{s\}$ is a model of $D^{\mathcal{S}}_s$ as well as of $C^{\mathcal{S}}_s$. Hence, each clause $\gamma \in C^{\mathcal{S}}_s$ contains at least one variable $t_\gamma \in S \setminus \{s\}$. Thus, by construction of $R^{def}_{\mathcal{S}}$, $S \setminus \{s\} \mapsto \alpha_{s,\gamma}$ for each $\gamma \in C^{\mathcal{S}}_s$, i.e. $S \setminus \{s\}$ defends s in $F^{def}_{\mathcal{S}}$. □

While $\mathcal{S} \subseteq adm(F^{def}_{\mathcal{S}})$ holds for any extension-set, $F^{def}_{\mathcal{S}}$ may contain additional admissible sets which do not occur in $\mathcal{S}$. In order to ensure that $adm(F^{def}_{\mathcal{S}})$ coincides with $\mathcal{S}$ it takes $\mathcal{S}$ to be adm-closed and to contain $\emptyset$. Before showing this we give an example.

Example 4. Again consider the extension-set $\mathcal{T} = \{\{a\}, \{b, c\}, \{a, c, d\}\}$. We have given the CNF-defense-formulas in Example 3. $F^{def}_{\mathcal{T}}$ thus is given by the AF in Figure 4. Considering, for instance, argument c, where $C^{\mathcal{T}}_c = \{\{a, b\}, \{b, d\}\}$, one can see that in $F^{def}_{\mathcal{T}}$ it takes a or b in order to defend c from $\alpha_{c,\{a,b\}}$, and b or d in order to defend c from $\alpha_{c,\{b,d\}}$.

Proposition 8. *For each adm-closed extension-set $\mathcal{S}$ where $\emptyset \in \mathcal{S}$ it holds that $adm(F^{def}_{\mathcal{S}}) = \mathcal{S}$.*

Proof. By Proposition 7, $\mathcal{S} \subseteq adm(F^{def}_{\mathcal{S}})$ holds for every extension-set $\mathcal{S}$.

It remains to show $adm(F^{def}_{\mathcal{S}}) \subseteq \mathcal{S}$. Consider some $S \in adm(F^{def}_{\mathcal{S}})$. First of all, S cannot contain any of the self-attacking arguments $\alpha_{a,\gamma}$. For $S = \emptyset$, $S \in \mathcal{S}$ by definition. If S consists of exactly one argument, i.e. $S = \{a\}$, it must hold that $\forall b \in A\ s.t.\ b \mapsto a : a \mapsto b$. For that, $C^{\mathcal{S}}_a = \{\}$ must hold, therefore $S \in \mathcal{S}$. Now assume S contains at least two arguments. S being conflict-free, by construction of $R^{cf}_{\mathcal{S}}$, guarantees that $\forall a, b \in S : (a, b) \in Pairs_{\mathcal{S}}$. Let $s \in S$ with $\{s\} \notin adm(F^{def}_{\mathcal{S}})$. Then we have $\alpha_{s,\gamma} \mapsto s$ for each $\gamma \in C^{\mathcal{S}}_s$. Since s is defended by S, for each $\gamma \in C^{\mathcal{S}}_s$, $\exists t_\gamma \in (S \setminus \{s\}) : t_\gamma \mapsto \alpha_{s,\gamma}$. By definition of $F^{cf}_{\mathcal{S}}$, thus t_γ occurs in the clause γ. It follows that $T = \{t_\gamma \mid c \in C^{\mathcal{S}}_s\}$ is a model of $C^{\mathcal{S}}_s$ and thus also of the defense-formula $D^{\mathcal{S}}_s$. Then by Lemma 6 there is some $S'_s \subseteq T \cup \{s\}$ (note that also $S'_s \subseteq S$) with $s \in S'_s$ such that $S' \in \mathcal{S}$. Recall also that in case $\{s\} \in adm(F^{def}_{\mathcal{S}})$, we know from above that $\{s\} \in \mathcal{S}$ (say $S'_s = \{s\}$). Knowing that $\forall a, b \in S : (a, b) \in Pairs_{\mathcal{S}}$, since $\mathcal{S}$ is adm-closed we get $S'_{s_1} \cup S'_{s_2} \in \mathcal{S}$ for any $s_1, s_2 \in \mathcal{S}$. Hence $S \in \mathcal{S}$. □

Lemma 7. *For each incomparable extension-set $\mathcal{S}$, it holds that $\mathcal{S}$ is pref-closed iff $\mathcal{S} \cup \{\emptyset\}$ is adm-closed.*

Proof. Follows immediately from Lemma 4, the fact that $Pairs_{\mathcal{S}} = Pairs_{\mathcal{S}\cup\{\emptyset\}}$ for all $\mathcal{S} \subseteq 2^{\mathcal{A}}$, and Definition 4. □

Proposition 9. *For each non-empty, pref-closed extension set $\mathcal{S}$ it holds that $pref(F^{def}_{\mathcal{S}}) = \mathcal{S}$.*

Proof. Let $\mathcal{S}' = \mathcal{S} \cup \{\emptyset\}$. By construction, $F^{def}_{\mathcal{S}'} = F^{def}_{\mathcal{S}}$. From Lemma 7 and Proposition 8 we thus obtain $adm(F^{def}_{\mathcal{S}}) = \mathcal{S}'$. As preferred extensions are $\subseteq$-maximal admissible sets and since $\mathcal{S}$ is incomparable (Lemma 4), $pref(F^{def}_{\mathcal{S}}) = \mathcal{S}$. □

This result together with the fact that for each AF F' there is an AF F such that $pref(F') = sem(F)$ (see [17]), yields the following result.

Proposition 10. *For each non-empty, pref-closed extension set $\mathcal{S}$, there exists an AF F, such that $sem(F) = \mathcal{S}$.*

We now have all results at hand to characterize the signatures for the semantics we deal with in this paper. All relations in the subsequent theorem follow immediatly from results in this section together with the corresponding characterizations given in Proposition 1–3.

Theorem 1. *The signatures for the considered semantics are given by the following collections of extension sets.*

$$\begin{aligned}
\Sigma_{cf} &= \{\mathcal{S} \neq \emptyset \mid \mathcal{S} \text{ is downward-closed and tight}\}\\
\Sigma_{adm} &= \{\mathcal{S} \neq \emptyset \mid \mathcal{S} \text{ is adm-closed and contains } \emptyset\}\\
\Sigma_{naive} &= \{\mathcal{S} \neq \emptyset \mid \mathcal{S} \text{ is incomparable and } dcl(\mathcal{S}) \text{ is tight}\}\\
\Sigma_{stb} &= \{\mathcal{S} \mid \mathcal{S} \text{ is incomparable and tight}\}\\
\Sigma_{stage} &= \{\mathcal{S} \neq \emptyset \mid \mathcal{S} \text{ is incomparable and tight}\}\\
\Sigma_{pref} = \Sigma_{sem} &= \{\mathcal{S} \neq \emptyset \mid \mathcal{S} \text{ is pref-closed}\}
\end{aligned}$$

Theorem 2. *The following relations hold:*

$$\Sigma_{naive} \subset \Sigma_{stage} \subset \Sigma_{sem} = \Sigma_{pref}, \qquad \Sigma_{stb} = \Sigma_{stage} \cup \{\emptyset\}$$
$$\{dcl(\mathcal{S}) \mid \mathcal{S} \in \Sigma_{naive}\} = \Sigma_{cf} \subset \Sigma_{adm} \supset \{\mathcal{S} \cup \{\emptyset\} \mid \mathcal{S} \in \Sigma_{pref}\}$$

Proof. In what follows, we make implicit use of the results from Theorem 1. First, if an incomparable extension-set $\mathcal{S}$ is tight, then also $dcl(\mathcal{S})$ is tight (using $Pairs_{\mathcal{S}} = Pairs_{dcl(\mathcal{S})}$). Thus, $\Sigma_{naive} \subseteq \Sigma_{stage}$; $\Sigma_{naive} \subset \Sigma_{stage}$, i.e. that the relation is proper, can been seen from the AF discussed in Example 1. Relation $\Sigma_{stage} \subseteq \Sigma_{sem}$ follows from Lemma 3. $\Sigma_{stage} \supseteq \Sigma_{sem}$ does not hold by Example 2, therefore $\Sigma_{stage} \subset \Sigma_{sem}$. $\Sigma_{cf} \subset \Sigma_{adm}$ follows in the same manner by Lemma 2 and the fact that $\mathcal{S} = \{\emptyset, \{a, b\}\}$ is adm-closed but not downward-closed and therefore $\mathcal{S} \in \Sigma_{adm}$, but $\mathcal{S} \notin \Sigma_{cf}$. The remaining relations in the second line follow from the definition of $dcl(\cdot)$ and respectively Lemma 7. □

5 Strict Realizability

Inspecting the proofs of Propositions 4 and 5 shows that for each extension set $\mathcal{S}$ that is realizable w.r.t. conflict-free sets (or naive semantics), there is an AF of the form $F = (Args_{\mathcal{S}}, R)$ (that is, without additional arguments) with the same outcome. Given a semantics σ, let us thus call an extension set $\mathcal{S} \subseteq 2^{\mathcal{A}}$ *strictly* σ-realizable, if there exists an AF $F = (Args_{\mathcal{S}}, R)$ such that $\sigma(F) = \mathcal{S}$. Next, we show that such a property does not hold for the remaining semantics.

Example 5. Consider $\mathcal{S} = \{\emptyset, \{a\}, \{a, b\}\}$. $\mathcal{S}$ is adm-closed, cf. Definition 4. Indeed for $F = (\{a, b, c\}, \{(a, c), (c, b)\})$, we have $adm(F) = \mathcal{S}$, thus $\mathcal{S} \in \Sigma_{adm}$. However, there does not exist an $F' = (A, R)$ with $\sigma(F') = \mathcal{S}$ and $A = \{a, b\}$, since by $\{a, b\} \in \mathcal{S}$ there cannot be any attack in F'. But then $adm(F') = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \neq \mathcal{S}$ is obvious.

Example 6. Consider $\mathcal{S} = \{\{a, b\}, \{a, c, e\}, \{b, d, e\}\}$. Figure 2 shows an AF (with additional arguments) realizing $\mathcal{S}$ as its semi-stable, and respectively, preferred extensions. Suppose there exists an AF $F = (Args_{\mathcal{S}}, R)$ such that $\sigma(F) = \mathcal{S}$. Since $\{a, c, e\}, \{b, d, e\} \in \mathcal{S}$, it is clear that R must not contain an edge involving e. But then, e is contained in each $E \in \sigma(F)$ (for the case of semi-stable extensions, since e is not attacked in such F). It follows that $\sigma(F) \neq \mathcal{S}$.

The previous example does not apply to stable and stage semantics ($\mathcal{S}$ is not tight cf. Definition 3). In fact, we need a different, slightly more involved, argument.

Example 7. Consider the extension-set $\mathcal{S} = \{\{a, b, c\}, \{a, b, c'\}, \{a, b', c\}, \{a, b', c'\}, \{a', b, c\}, \{a', b, c'\}, \{a', b', c\}\}$. It is easy to verify that $\mathcal{S}$ is non-empty, incomparable and tight. Hence, by Proposition 6, $\mathcal{S}$ is *stb*-realizable. However the AF provided by Proposition 6 makes use of an argument not in $Args_{\mathcal{S}} = \{a, b, c, a', b', c'\}$. We now show that there is no AF $F = (Args_{\mathcal{S}}, R)$ such that $stb(F) = \mathcal{S}$ or $stage(F) = \mathcal{S}$. First, given that the sets in $\mathcal{S}$ must be conflict-free the only possible attacks in R are (a, a'),

Table 1. Complexity of the recasting problem.

	stb	*stage*	*pref*	*sem*
stb	-	NP-c	NP-c	NP-c
stage	trivial	-	trivial	trivial
pref	Π_2^P-c	Π_2^P-c	-	trivial
sem	Π_2^P-c	Π_2^P-c	trivial	-

(a', a), (b, b'), (b', b), (c, c'), (c', c). We next argue that all of them must be in R. First consider the case of *stb*. As $\{a, b, c\} \in \mathit{stb}(F)$ it attacks a' and the only chance to do so is $(a, a') \in R$ and similar as $\{a', b, c\} \in \mathit{stb}(F)$ it attacks a and the only chance to do so is $(a', a) \in R$. By symmetry we obtain $\{(b, b'), (b', b), (c, c'), (c', c)\} \subseteq R$. Now let us consider the case of *stage*. As $\{a, b, c\} \in \mathit{stage}(F) \subseteq \mathit{naive}(F)$ either $(a, a') \in R$ or $(a', a) \in R$. Consider $(a, a') \notin R$ then $\{a', b, c\}^+ \supset \{a, b, c\}^+$, contradicting that $\{a, b, c\}$ is a stage extension. The same holds for pairs (b, b') and (c, c'); thus for both cases we obtain $R = \{(a, a'), (a', a), (b, b'), (b', b), (c, c'), (c', c)\}$. However, for the resulting framework $F = (A, R)$, we have that $\{a', b', c'\} \in \mathit{stb}(F) = \mathit{stage}(F)$, but $\{a', b', c'\} \notin \mathcal{S}$.

6 Complexity

In this section we exploit our results to give a preliminary complexity analysis in terms of the problem of *recasting*: given an AF $F_1 \in AF_{\mathcal{A}}$ and semantics σ_1 and σ_2, decide whether there exists an $F_2 \in AF_{\mathcal{A}}$, such that $\sigma_1(F_1) = \sigma_2(F_2)$. By the very nature of signatures, this is equivalent to test $\sigma_1(F_1) \in \Sigma_{\sigma_2}$. Table 1 shows our results: an entry for row σ_1 and column σ_2 gives the complexity of deciding whether $\sigma_1(F) \in \Sigma_{\sigma_2}$. $\mathcal{C}$-c abbreviates completeness for class $\mathcal{C}$; "trivial" means that each instance is a "Yes"-instance.

Theorem 3. *The complexity results depicted in Table 1 hold.*

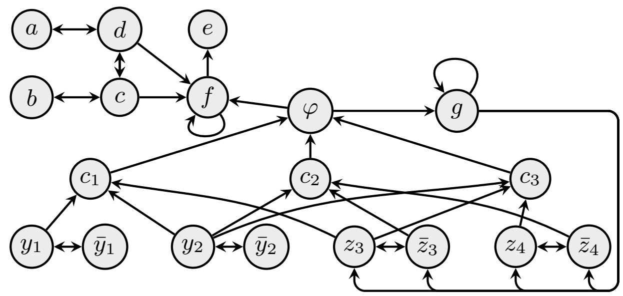
Proof (Sketch). The "trivial" results are immediate by Theorem 2. Further using that $\Sigma_{\mathit{stb}} = \Sigma_{\mathit{stage}} \cup \{\emptyset\}$ and $\Sigma_{\mathit{stage}} \subset \Sigma_{\mathit{sem}} = \Sigma_{\mathit{pref}}$ we have that $\mathit{stb}(F) \in \Sigma_\sigma$ ($\sigma \in \{\mathit{stage}, \mathit{pref}, \mathit{sem}\}$) iff $\mathit{stb}(F) \neq \emptyset$. Deciding whether an AF has a stable extension is well-known to be NP-complete.

Finally, we consider the Π_2^P-entries, i.e. $\sigma_1 \in \{\mathit{pref}, \mathit{sem}\}$, $\sigma_2 \in \{\mathit{stb}, \mathit{stage}\}$. Since $\sigma_1(F) \neq \emptyset$ for any AF F, and $\Sigma_{\mathit{stb}} = \Sigma_{\mathit{stage}} \cup \{\emptyset\}$, we can stick to $\sigma_2 = \mathit{stb}$. Membership is by an algorithm disproving, given an $F = (A, R)$, $\sigma_1(F) \in \Sigma_{\mathit{stb}}$: guess sets $B \subseteq A$, $\{A_s \subseteq A \mid s \in B\}$ and $a \in A \setminus B$; use an NP-oracle to check $B \in \sigma_1(F)$ [14, 16]; for all $s \in B$ check $A_s \in \mathit{adm}(F)$, $\{a, s\} \subseteq A_s$. Intuitively, the algorithm accepts (i.e. all checks holds), if $B \in \sigma_1(F)$ violates tightness for $\sigma_1(F)$.

We show Π_2^P-hardness for $\sigma_1 = \mathit{pref}$ (as *pref* semantics can be efficiently reduced to *sem* semantics [17], the result for $\sigma_1 = \mathit{sem}$ follows): Given QBF $\Phi = \forall Y \exists Z \varphi(Y, Z)$, where φ is a CNF $\bigwedge_{c \in C} c$ with each $c \in C$ a disjunction of literals from $X = Y \cup Z$, let $F_\Phi = (A_\Phi, R_\Phi)$ with $A_\Phi = \{\varphi, g\} \cup C \cup X \cup \bar{X} \cup \{a, b, c, d, e, f\}$ and

$$
\begin{aligned}
R_\Phi = &\{\langle c,\varphi\rangle \mid c \in C\} \cup \{\langle x,\bar{x}\rangle, \langle \bar{x},x\rangle \mid x \in X\} \cup \\
&\{\langle x,c\rangle \mid x \text{ occurs in } c\} \cup \{\langle \bar{x},c\rangle \mid \neg x \text{ occurs in } c\} \cup \\
&\{\langle \varphi,g\rangle, \langle g,g\rangle\} \cup \{(g,z),(g,\bar{z}) \mid z \in Z\} \cup \\
&\{(a,d),(d,a),(b,c),(c,b),(c,d),(d,c),(c,f), \\
&(d,f),(f,e),(f,f),(\varphi,f)\}
\end{aligned}
$$

We illustrate F_Φ for the QBF $\Phi = \forall y_1, y_2 \exists z_3, z_4 \big((y_1 \vee y_2 \vee z_3) \wedge (y_2 \vee \neg z_3 \vee \neg z_4) \wedge (y_2 \vee z_3 \vee z_4)\big)$.



F_Φ links the reduction from [14] with the AF from Figure 2 via $\varphi \mapsto f$. One can show that $\mathit{pref}(F_\Phi) \in \Sigma_{\mathit{stb}}$ iff φ is contained in each $E \in \mathit{pref}(G_\Phi)$. □

7 Discussion

In this work, we initiated a study on the characteristics the set of extensions w.r.t. a given semantics satisfy. For the semantics naive, stable, stage, preferred, and semi-stable we have an exact picture fully describing their signatures Σ_σ. These results also tell about the limits of global disagreement (a notion introduced in [8]) that can be modelled within AFs, e.g. our results show that preferred and semi-stable semantics are able to express more disagreement than stage semantics: $\Sigma_{\mathit{stage}} \subset \Sigma_{\mathit{pref}} = \Sigma_{\mathit{sem}}$.

We have also touched the concept of strict realizability, i.e. the question whether a set $\mathcal{S}$ of extensions can be realized by an AF F having no additional arguments (all arguments of F appear in $\mathcal{S}$). Exact characterizations for strict signatures are important foundations for simplifications of AFs and thus a natural next step for our studies. In general, we believe that results on signatures yield useful methods for pruning the search space in algorithms for abstract argumentation.

Further directions of future work are an investigation of other important semantics, in particular complete [13], resolution-based grounded [2], and cf2-semantics [4], and an according extension of our complexity analysis. Finally, since we have viewed semantics here only in an extension-based way, it would also be of high interest to extend our studies to labelling-based semantics [11].

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