Full Proofs for Submission #1565

Additional definitions and lemmas

Definition 15. Let $R$ be a relation on a set $X$. Then we write $R^{-1}$ for the inverse relation defined by $xR^{-1}y$ if $yRx$.

Observe that $R^{-1}$ is a total order if and only if $R$ is a total order and $R^{-1}$ is a linear order if and only if $R$ is a linear order.

Lemma 16. Let $X$ be a set of objects and $X \subseteq P(X) \setminus \{\emptyset\}$ a family of sets. Assume that $X$ is $DI^S$-orderable with respect to a linear order $\leq$. Then $X$ is $DI^S$-orderable with respect to $\leq^{-1}$.

Similarly, if we assume that $X$ is $DI$-orderable with respect to a linear order $\leq$, then $X$ is $DI$-orderable with respect to $\leq^{-1}$.

Proof. Let $\leq$ be an order on $X$ that satisfies dominance and strict independence with respect to $\leq$. Then we claim that $\leq^{-1}$ satisfies dominance and strict independence with respect to $\leq^{-1}$. Assume $A, \cup \{x\} \in X$, then $\forall y \in A(y \prec x)$ implies $\forall y \in A(y > x)$, which implies $A \cup \{x\} \prec A$ by assumption, hence $A \prec^{-1} A \cup \{x\}$. Similarly, $\forall y \in A(x \prec^{-1} y)$ implies $A \cup \{x\} \prec^{-1} A$.

Now, assume $A, B, \cup \{x\}, \cup \{x\} \in X$ and $A \prec^{-1} B$.

Then $B \prec A$ and hence by assumption $B \cup \{x\} \prec \cup \{x\}$ which implies $A \cup \{x\} \prec^{-1} B \cup \{x\}$.

The argumentation for independence is the same. $\square$

The following definition is taken from (Murray and Williams 2017).

Definition 17. An algorithm $R: \{0, 1\}^* \times \{0, 1\}^* \rightarrow \{0, 1, *, \}$ is a polylog-time reduction from $L$ to $L'$ if there are constants $c \geq 1$ and $k \geq 1$ such that for all $x \in \{0, 1\}^*$,

- $R(x, i)$ has random access to $x$ and runs in $O((\log|x|)^k)$ time for all $i \in \{0, 1\}^{\lceil 2c \log(|x|) \rceil}$
- there is an $l_x \leq |x|^{c} + c$ such that $R(x, i) \in \{0, 1\}$ and for all $i \leq l_x$ and $R(x, i) = * \Rightarrow \forall i > l_x$.
- $x \in L$ iff $R(x, 0) \cdot R(x, 2) \cdots R(x, l_x) \in L'$.

Here $\cdot$ is the string concatenation and $*$ is the out of bounds character that marks the end of a string.

Proof of Proposition 4

Let $\phi$ be a instance of SAT with $n$ variables and $m$ clauses. Then we produce an instance $(X, X')$ of STRONG $DI^S$-ORDERABILITY. We produce this instance in a way that there is a linear order $\leq$ such that $X$ is $DI^S$-orderable with respect to $\leq$ only if $\phi$ is satisfiable. Then, we sketch how to use a satisfying assignment of $\phi$ to construct for any linear order $\leq$ on $X$ a linear order $\leq'$ on $X$ that satisfies dominance and strict independence.

The set of elements $X$ contains for every variable $V_i$ elements $x_{i,1}, x_{i,2}, x_{i,3}$ and $x_{i,4}$. Furthermore, it contains for every clause $C_i$ variables $x_{i,3}, y_{i,3}, \min_{a}^i$ and $\max_{a}^i$ for $a \in \{1, 2, 3\}$. We call the elements $\min_{a}^i$ and $\max_{a}^i$ the extremum-elements. Finally, it contains two elements $v_1$ and $v_2$. Then we define the following linear order $\leq'$ on $X$:

\begin{align*}
\min_{1}^1 < \min_{1}^2 < \cdots < \min_{3}^3 < x_{1,1} < x_{1,2} < \cdots < x_{n,2} < v_1 < v_2 < z_1 < z_2 \cdots < z_{n}^2 < y_{1}^2 < y_{2}^2 < \cdots < y_{m}^2 < x_{1,1}^2 < x_{1,2}^2 < \cdots < x_{n,2}^2 < \max_{1}^1 < \max_{2}^2 < \cdots < \max_{3}^3
\end{align*}

Next, we will construct a family $X'$ that is only $DI^S$-orderable with respect to $\leq$ if $\phi$ is satisfiable. In the following, we write

\[
Y := \{x \in X \mid v_1 \leq x \leq v_2^2\}
\]

First, we add for every variable $V_i$ sets $X_i^1 = Y \cup \{x_{i,1}, x_{i,1}^2\}$ and $X_i^2 = Y \cup \{z_{i,1}, x_{i,1}^2\}$. We call these the class 1 sets and write $CL_1$ for the collection of all class 1 sets.

Intuitively, the truth value of $\phi$ will be coded by the preference between $X_i^1$ and $X_i^2$, where $X_i^1 < X_i^2$ equals $V_i$ is false and $X_i^2 < X_i^1$ equals $V_i$ is true.

We will now add for every clause new sets that lead to a contradiction if the clause is not satisfied. Let $C_i$ be a clause with variables $V_j, V_k, V_l$. We add

\[
X_i^1 \setminus \{y_{j}^1\}, X_i^2 \setminus \{y_{j}^1\}, X_i^1 \setminus \{y_{j}^2\}, X_i^2 \setminus \{y_{j}^2\}, X_i^1 \setminus \{y_{k}^1\}, X_i^2 \setminus \{y_{k}^1\}, X_i^1 \setminus \{y_{l}^2\}, X_i^2 \setminus \{y_{l}^2\}.
\]

We call these the class 2 sets and write $CL_2$ for the collection of all class 2 sets.

By “reverse strict independence” we know that the preference between $X_i^1 \setminus \{y_{j}^1\}$ and $X_i^2 \setminus \{y_{j}^1\}$ must be the same as the preference between $X_i^1$ and $X_i^2$. The same holds for the other two variables. Now, if all variables occur positively in $C_i$, we add sets such that $X_i^2 \setminus \{y_{j}^1\} < X_i^1 \setminus \{y_{j}^1\}, X_i^2 \setminus \{y_{k}^1\} < X_i^1 \setminus \{y_{k}^1\}$, and $X_i^2 \setminus \{y_{l}^2\} < X_i^1 \setminus \{y_{l}^2\}$ must hold in any order $\leq'$ on $X$ that satisfies dominance and strict independence with respect to $\leq$. We call this enforcing these preferences. Then we get a contradiction if $V_j, V_k$ and $V_l$ are false because

\[
X_i^1 \setminus \{y_{j}^1\} < X_i^2 \setminus \{y_{j}^1\} < X_i^1 \setminus \{y_{j}^2\} < X_i^2 \setminus \{y_{j}^2\} < X_i^1 \setminus \{y_{k}^1\} < X_i^2 \setminus \{y_{k}^1\} < X_i^1 \setminus \{y_{l}^2\} < X_i^2 \setminus \{y_{l}^2\}
\]

holds. If a variable, say $V_j$, occurs negatively in $C_i$, we switch $X_i^1$ and $X_i^2$ and enforce $X_i^1 \setminus \{y_{j}^1\} < X_i^1 \setminus \{y_{j}^2\}$ and $X_i^2 \setminus \{y_{j}^2\} < X_i^2 \setminus \{y_{j}^1\}$.

Next, we show how we can enforce these preferences. Assume we want to enforce $X_i^2 \setminus \{y_{j}^1\} < X_i^2 \setminus \{y_{j}^1\}$ for $a, b \in \{1, 2\}$. We add \{$z_{1}^1, \{\infty, \max_{1}^a\}$\} and \{$x_{j}^1 \setminus \{y_{j}^1\} \cup \{\max_{1}^a\}\}$. Our goal is to enforce \{$x_{j}^1 \setminus \{y_{j}^1\} \cup \{\max_{1}^a\}\} < \{z_{1}^1\}$ which forces by reverse strict independence $X_i^2 \setminus \{y_{j}^1\} < \{z_{1}^1\}$. Then we enforce \{$z_{1}^1\} < X_i^2 \setminus \{y_{j}^2\}$ to get by transitivity $X_i^2 \setminus \{y_{j}^1\} < X_i^2 \setminus \{y_{j}^2\}$ as desired. To enforce $X_i^2 \setminus \{y_{j}^1\}$

\footnote{Every linear order satisfying strict independence has to satisfy reverse strict independence, i.e. $A \cup \{x\} < B \cup \{x\}$ implies $A < B$: Assume otherwise $A < B$, holds, then by strict independence $B \cup \{x\} < A \cup \{x\}$ must hold, contradicting $A \cup \{x\} < B \cup \{x\}$. Hence by the totality of $\leq'$ we have $A < B$.}
\{y^1_i\} \cup \{\text{max}^2_i\} \times \{z_i, \text{max}^1_i\} we add a sequence of sets \(A_1, A_2, \ldots, A_k\) such that \(A_1 = (X^0_i \setminus \{y^1_i, z_i\}) \cup \{\text{max}^2_i\}\). 
\[A_{i+1} = A_i \setminus \min_2(A_i) \text{ and } A_k = \{\text{max}^1_i\}\]. This enforces by dominance \(A_1 < A_2 < \cdots < A_k\) which enforces by transitivity \(A_1 = (X^0_i \setminus \{y^1_i, z_i\}) \cup \{\text{max}^2_i\} = A_k\). Finally, this enforces by strict independence the desired \(\{X^0_i \setminus \{y^1_i\}\} \cup \{\text{max}^1_i\} < \{z_i, \text{max}^2_i\}\). Using the same idea and \(\min_1^{\prime}\) we enforce \(\{z_i\} < X^k \setminus \{y^3_i\}\) finishing the construction for \(X^0_i \setminus \{y^1_i\} \setminus X^k \setminus \{y^3_i\}\). We enforce \(X^k \setminus \{y^3_i\} < X^1_i \setminus \{y^2_i\}\) and \(X^2_i \setminus \{y^3_i\} < X^1_i \setminus \{y^2_i\}\) similarly using \(z^2_i, \text{max}^3_i\) and \(\min_2\) resp. \(z^3_i, \text{max}^3_i\) and \(\min_3\).

We repeat this procedure for every clause. We call the sets added in this step the class 3 sets and write \(C_3\) for the collection of all class 3 sets. Now, by construction, \(X\) can only be \(DI^2\)-orderable with respect to \(\leq\) if \(\phi\) is a positive instance of SAT.

Next, we pick an arbitrary linear order \(\preceq\) on \(X\). We distinguish two cases: \(v_1 < v_2\) and \(v_2 < v_1\). By Lemma 16 it suffices to show \(DI^2\) orderability in the first case, because \(v_2 < v_1\) implies \(v_2 < v_1 < v_2\) and we only need to show \(DI^2\) orderability for one of these two orders. Hence, we can assume in the following w.l.o.g. \(v_1 < v_2\). Now, we want to construct an order \(\succeq\) on \(X\) that satisfies dominance and strict independence with respect to \(\leq\) if \(\phi\) is satisfiable. First, we order the sets \(X^1_i, X^2_i\) according to a satisfying assignment of \(\phi\), i.e. \(X^1_i < X^2_i\) if \(V_i\) is false in the assignment and \(X^2_i < X^1_i\) if it is true. Then, we project this order down to the class 2 sets by reverse strict independence. Finally, we take the transitive closure of this order. It is clear by construction, that this is an acyclic strict partial order if and only if \(\phi\) is satisfiable. Now, for any clause \(C_i\), we fix any linear order on the sets \(X^1_j \setminus \{y^1_i\}, X^2_j \setminus \{y^1_i\}, X^1_k \setminus \{y^2_i\}, X^2_k \setminus \{y^3_i\}, X^1_i \setminus \{y^2_i\}\) and \(X^2_i \setminus \{y^3_i\}\) that extends this order.

For the class 1 sets we have ordered all pairs \((X^1_i, X^2_i)\) but we still have to fix an order among these pairs. For the class 2 sets, we have fixed an order on between all sets introduced for a single clause, but we have to fix an order between sets from different clauses. Now, we observe that \(A, A \cup \{x\} \in Cl_1 \cup Cl_2\) implies that \(A \in Cl_1\), \(B \in Cl_2\) and \(x = y^m_i\) for \(j \leq m\) and \(k \leq 3\) as all Class 1 sets differ from all other class 1 sets in at least two elements and all class 2 sets differ from all other class 2 sets in at least two elements. Hence the only possible application of strict independence on class 1 and 2 is the one already covered by construction. Dominance is applicable only if \(y^m_i\) for some \(i\) and \(a\) is the minimal or maximal element of the set it gets removed from. We fix an order on the pairs and clauses that is compatible with these applications of dominance. First, assume the minimal element \(y^m_i\) of the form \(y^m_i\) and the maximal element \(y^m_i\) of the form \(y^m_i\) are used for the same clause. Let \(X^b_i\) and \(X^c_i\) be the sets such that \(X^b_i \setminus \{y^m_i\} \in X\) and \(X^c_i \setminus \{y^m_i\} \in X\) holds. Then, by construction \(j \neq k\). In that case we fix any linear order \(\preceq\) on the pairs \((X^1_i, X^2_i)\) such that \((X^1_i, X^2_i) \preceq (X^1_j, X^2_j)\) holds and an arbitrary order on the clauses. We set \(X^b_i\) is smaller than any class 2 set if \((X^1_i, X^2_i) \preceq (X^1_j, X^2_j)\). Furthermore, any set \(X^b\) is bigger than any class 2 set if \((X^1_i, X^2_i) \preceq (X^1_j, X^2_j)\). This is obviously a linear order and we have \(X^b_j < X^b_i \setminus \{y^m_i\}\) and \(X^c_i \setminus \{y^m_i\} < X^c_i\) for \(b, c \leq 2\). Hence the constructed order on \(C_1 \cup C_2\) satisfies dominance.

Now, assume the minimal element \(y^m_i\) and the maximal element \(y^m_i\) of the form \(y^m_i\) and \(y^m_i\) are used for different clauses \(C_i\) and \(C_j\). We fix any order on the clauses such that \(C_i\) and \(C_j\) are an arbitrary order on the pairs. Additionally we set all sets from clauses smaller or equal \(C_i\) smaller than any set from class 1 and any set from a clause larger than \(C_i\) larger than any set from class 1. This is obviously a linear order and we have \(X^b_i < X^b_i \setminus \{y^m_i\}\) and \(X^c_i \setminus \{y^m_i\} < X^c_i\) for \(b, c \leq 2\). Hence the constructed order on \(C_1 \cup C_2\) satisfies dominance.

Next, we add the class 3 sets. First, we observe that if we have a set \(A \in Cl_1 \cup Cl_2\) and \(A \cup \{x\}\) then we know \(A \in Cl_2\) and \(x\) is an extremum-element. On the other hand, there is no set \(A \in Cl_3\) such that \(A \cup \{x\} \in Cl_1 \cup Cl_2\) holds, as every set in \(C_3\) either contains an extremum-element or it is a singleton and no set in class 1 and 2 contains an extremum-element and every set in class 1 and 2 has more than three elements.

Hence, for the interaction of class 3 with the other classes, we only have to consider dominance if we add an extremum element to a class 2 set that is smaller/larger than any element already in the class 2 set. This is achieved by the following construction. Let \(A\) be a set in \(Cl_3\) containing an extremum-element \(mm\). Then, \(A\) is in \(Cl_3^\ast\) if \(v_1 \prec \{mm\}\) holds and in \(Cl_3^\sim\) otherwise. Then we set \(A < B\) if

- \(A \in Cl_1 \cup Cl_2\) and \(B \in Cl_3^\ast\),
- \(A \in Cl_3^\sim\) and \(B \in Cl_1 \cup Cl_2\).

Next, we order the sets in \(Cl_1^\ast\) and \(Cl_3^-\). First, we define an order between sets containing the same extremum element. An extremum-element \(mm\) (which is, by construction unique) \(A_1 := \{x \in A \mid x \preceq \{mm\}\}\) for the set of elements in \(A\) that are smaller than \(mm\) and \(A_L := \{x \in A \mid \{mm\} < x\}\) for the set of elements in \(A\) that are larger than \(mm\).

We set \(A < B\) for sets \(A, B\) that both contain the same extremum element of the form \(y^m_i\) if:

- \(\max_{<}(A_L \triangle B_L) \in B\),
- \(A_2 = B_2\) and \(\min_{<}(A_S \triangle B_S) \in A\).

Here, \(\triangle\) is the symmetric difference operator, i.e. \(A \triangle B := (A \cup B) \setminus (A \cap B)\). We claim that this order satisfies dominance and strict independence. It satisfies strict independence because for all sets \(S, T\) by definition \(S \cup \{x\} \setminus T \cup \{x\} = S \setminus T\) for any \(x \not\in S \cup T\). For dominance, assume \(x \prec \{mm\}\) and \(\max_{<} \in A \cup A \cup \{x\}\). Then \(A_{<} = (A \cup \{x\})_{<}\) and \(\min_{<}(A_S \triangle (A \cup \{x\})_{<}) = x\). Hence, \(A \cup \{x\} < A\). The case \(\max_{<}(A) < x\) is similar.
We observe that we may have either $X_i^q \setminus \{y_i^q\} \cup \{\max_i^q\} \prec \{z_i^q, \max_i^q\}$ or $\{z_i^q, \max_i^q\} \prec X_i^q \setminus \{y_i^q\} \cup \{\max_i^q\}$. In the first case, we add $z_i^q$ in the order exactly after $X_i^q \setminus \{y_i^q\}$ and in the second case exactly before $X_i^q \setminus \{y_i^q\}$.

Now let $X_i^q \setminus \{y_i^q\}$ be the set for which we enforce the preference $X_i^q \setminus \{y_i^q\} \prec X_i^q \setminus \{y_i^q\}$. Then, this construction implies $(z_i^q, \min_i^q) \prec X_i^q \setminus \{y_i^q\}$. Therefore, we have to make sure that $(z_i^q, \min_i^q) \prec X_i^q \setminus \{y_i^q\} \cup \{\min_i^q\}$ holds as intended by the construction to avoid a contradiction. For this we use the fact that $v_1 < v_2$ holds. We set $A < B$ for elements $A, B$ if they both contain an element of the form $\min_i^q$ if:

- $v_2 \in B$ and $v_2 \not\in A$ (\star),
- $v_2 \in A, B$ or $v_2 \not\in A, B$ and $\max_i^q(A_i \triangle B_i) \in B$,
- $v_2 \in A, B$ or $v_2 \not\in A, B$, $A_L = B_L$ and $\min_i^q(A_i \triangle B_i) \in A$.

It is clear that (\star) implies $(z_i^q, \min_i^q) \prec X_i^q \setminus \{y_i^q\} \cup \{\min_i^q\}$. It is also clear that it satisfies strict independence because the (\star) implies a preference between sets $A \cup \{x\}$ and $B \cup \{x\}$ for $x \not\in A \cup B$ if it implies the same preference for $A$ and $B$. If (\star) is not applicable, strict independence is satisfied by the same argument as above. Now, for dominance $v_2 \in (A \triangle (A \cup \{x\}))$ implies $x = v_2$. Then, $x < \min_i(A)$ is not possible because by construction $v_1 \not\in A$ holds and we assume $v_1 < v_2$. If we have $\max_i(A) < x$ then dominance is satisfied because $A < A \cup \{x\}$ holds by (\star). If $x \not\in v_2$, then (\star) is not applicable and dominance is satisfied by the same argument as above.

We observe that this is the only application of strict independence with sets from class 3 and sets not from class 3 because if we have a set $A \in Cl_1 \cup Cl_2$ and $A \cup \{x\} \in Cl_3$ there is no other set $B \in Cl_1 \cup Cl_2$ such that $B \cup \{x\} \in Cl_3$ holds. Finally, we have to extend the order $\preceq$ to the whole class 3. However, any two sets not yet comparable differ in at least two elements. Hence, any completion of $\preceq$ satisfies dominance and strict independence. □

**Proof of Theorem 5**

$\Pi_2^P$-membership is clear as we can universally guess a linear order $\preceq$ on $X$ and then check via the NP-oracle if $X$ is $D1^S$-orderable with respect to $\preceq$.

It remains to show that $\text{STRONG } D1^S$-ORDERABILITY is $\Pi_2^P$-hard. We do this by reducing the above ordering to a reduction from a $\Pi_2$-SAT instance $\phi = \forall W \exists V \psi(W, V)$. Let $w_1 \ldots w_l$ be the universally quantified variables. Then $X$ contains the same elements as in the construction above and additionally for every universally quantified variable $w_i$ elements $w_i^1$ and $w_i^2$ as well as elements $y_i^1, \min_i^q, \max_i^q, y_i^2, \min_i^q, \max_i^q$. Then we add the same sets as in the reduction above, except that the elements $w_i^1, w_i^2, y_i^1$ and $y_i^2$ are included in the set $Y$ used in the construction. Furthermore, we add for every universally quantified variable $w_i$ sets $X_i^1 \setminus \{y_i^1\}, X_i^2 \setminus \{y_i^1\}$ and $\{w_i^1, w_i^2\}$. Furthermore, we enforce as described above $X_i^1 \setminus \{y_i^1\} \prec \{w_i^1\}$ and $\{w_i^2\} \prec X_i^2 \setminus \{y_i^1\}$ using $\min_i^q$ and $\max_i^q$. Now, let $\preceq$ be a linear order on $X$ such that $w_i^1 < w_i^2$ holds. Then $X_i^1 \prec X_i^2$ must hold for every order $\preceq$ on $X$ that satisfies dominance and strict independence with respect to $\preceq$. Now, we add additionally sets $X_i^1 \setminus \{y_i^2\}$ and $X_i^2 \setminus \{y_i^2\}$. Then, we enforce $X_i^1 \setminus \{y_i^2\} \prec \{w_i^1\}$ and $\{w_i^2\} \prec X_i^2 \setminus \{y_i^2\}$ using $\min_i^q$ and $\max_i^q$. Analogously to above $X_i^2 \prec X_i^1$ must hold for every order $\preceq$ on $X$ that satisfies dominance and strict independence with respect to a linear order $\preceq$ on $X$ such that $w_i^2 < w_i^1$ holds.

We claim that we $(X, X)$ can only be a positive instance of $\text{STRONG } D1^S$-ORDERABILITY, if $\phi$ is a positive instance of $\Pi_2$-SAT. First, we fix the same order $\preceq$ as above on the elements that occur already in the first reduction. Then, for every truth assignment $T$ to the variables in $\tilde{W}$ there is a linear order $\preceq^*$ on $X$ that coincides with $\preceq$ on the old elements such that $w_i^1 < w_i^2$ if $w_i$ is assigned false in $T$ and $w_i^2 < w_i$ if $w_i$ is assigned true in $T$. Now, if there is no satisfying assignment on $\phi$ that extends $T$, then there can be no order on $X$ satisfying dominance and strict independence with respect to $\preceq^*$. Hence $(X, X)$ can only be $D1^S$-orderable with respect to every linear order $\preceq^*$ if $\phi$ is a positive instance of $\Pi_2$-SAT.

It remains to show that if $\phi$ is satisfiable then $(X, X)$ is a positive instance of $\text{STRONG } D1^S$-ORDERABILITY. This can be done by using nearly the same construction as above treating $X_i^1 \setminus \{y_i^1\}$ and $X_i^2 \setminus \{y_i^2\}$ as Class 2 sets, all other new sets as Class 3 sets and inserting $\{w_i^1\} < \{w_i^1\} < \{w_i^2\}$ resp. $\{w_i^1\} < \{w_i^1\} < \{w_i^2\}$ where we would insert $z_i^q$. The only exception has to be made if there is an $i$ such that $y_i^q = \min(X_i^1)$ and $y_i^q = \max(X_i^1)$ or $y_i^q = \min(X_i^2)$ and $y_i^q = \max(X_i^2)$. In the first case, we set $A < B$ for the sets containing $\min_i^q$ if:

- $y_i^q \in A$ and $y_i^q \not\in B$,
- $y_i^q \in A, B$ or $y_i^q \not\in A, B$ and $\max_i^q(A_i \triangle B_i) \in B$,
- $y_i^q \in A, B$ or $y_i^q \not\in A, B$, $A_L = B_L$ and $\min_i^q(A_i \triangle B_i) \in A$.

where $A_L := \{x \in A \mid \min_i^q < x\}$ and $A_S := \{x \in A \mid x < \min_i^q\}$. And for $A < B$ for the sets containing $\max_i^q$ if:

- $\forall_i^q \in A$ and $\forall_i^q \not\in A$,
- $\forall_i^q \in A, B$ or $\forall_i^q \not\in A, B$ and $\max_i^q(A_i \triangle B_i) \in B$,
- $\forall_i^q \in A, B$ or $\forall_i^q \not\in A, B$, $A_L = B_L$ and $\max_i^q(A_i \triangle B_i) \in A$.

where $A_L := \{x \in A \mid \max_i^q < x\}$ and $A_S := \{x \in A \mid x < \max_i^q\}$. It is clear that these orders satisfy dominance and strict independence, similarly to the orders on the class 3 sets defined above. Furthermore, we have $\{w_i^1, \max_i^q\} \prec (X_i^1 \setminus \{y_i^1\}) \cup \{\max_i^q\}$ and $(X_i^1 \setminus \{y_i^1\}) \cup \{\max_i^q\} \prec \{w_i, \min_i^q\}$ which allows us to set $X_i^1 \setminus \{y_i^1\} < \{w_i\} < X_i^1 \setminus \{y_i^1\}$ which is consistent with the enforced $X_i^1 \setminus \{y_i^1\} < X_i^1 \setminus \{y_i^1\}$. The second case can be treated analogously. □

**Proof of Theorem 6**

We need to change the reduction from a $\Pi_2$-SAT instance $\phi$ in two places compared to Theorem 5. First, we need to
modify the way we enforce a strict preference \( X_i^a \setminus \{ y_i^j \} \prec X_i^b \setminus \{ y_i^j \} \) using independence instead of strict independence. We replace every element \( z_i^j \) by two elements \( z_i^{j±} \) and \( z_i^{j±} \), set \( z_i^{j±} < z_i^{j±} \) and add the sets \( \{ z_i^{j±}, z_i^{j±}, z_i^{j±} \} \) to \( X_i \). Then, to enforce \( X_i^a \setminus \{ y_i^j, x_i^j \} \npreceq \{ z_i \} \) we add the same sequence \( A_1, \ldots, A_l \) as in the proof of Proposition 4 and, additionally, the set \( \{ X_i^a \setminus \{ y_i^j, x_i^j \} \} \cup \{ \max_i \} \). We observe \( l > 3 \) and that the following preference enforced by dominance
\[
A_2 = (X_i^a \setminus \{ y_i^j, z_i^{j±}, x_i^j \}) \cup \{ \max_i \} \prec \{ \max_i \} = A_l
\]
which enforces by independence
\[
(X_i^a \setminus \{ y_i^j \}) \cup \{ \max_i \} \prec \{ \max_i \}
\]
and hence by dominance
\[
(X_i^a \setminus \{ y_i^j \}) \cup \{ \max_i \} \prec \{ z_i^{j±}, \max_i \}
\]

This gives us by "reverse independence"5 the desired \( X_i^a \setminus \{ y_i^j, x_i^j \} \npreceq \{ z_i \} \). Now, we can enforce \( \{ z_i \} \preceq X_e^i \setminus \{ y_i^j \} \) similarly. Then, this enforces by dominance
\[
X_i^a \setminus \{ y_i^j \} \npreceq \{ z_i^{j±}, z_i^{j±}, z_i^{j±} \} \npreceq \{ z_i \} \preceq X_e^i \setminus \{ y_i^j \}
\]
and hence by transitivity \( X_i^a \setminus \{ y_i^j \} \prec X_e^i \setminus \{ y_i^j \} \).

Second, we have to make sure that all preferences between sets \( X_i^a \) and \( X_i^b \) are strict. We borrow an idea from (Maly and Woltran 2017) to achieve this. We add for every variable \( X_i \) new elements ordered in \( \leq \) as follows
\[
a_i^− < b_i^− < c_i^− < d_i^− < r_i < s_i < d_i^+ < c_i^+ < b_i^+ < a_i^+. \]
such that \( v_1 \) and \( v_2 \) lie between \( d_i \) and \( r_i \) in the order \( \leq \). Then, we add new sets \( A_i := \{ a_i^−, v_1, v_2, r_i, s_i, a_i^+ \} \),
\[
B_i := \{ b_i^−, v_1, v_2, r_i, s_i, b_i^+ \}, C_i := \{ c_i^−, v_1, v_2, r_i, s_i, c_i^+ \}
\]
and
\[
D_i := \{ d_i^−, v_1, v_2, r_i, s_i, d_i^+ \}.
\]
Now, let \( z_i^{j±}, z_i^{j±}, \max_i \) and \( \min_i \) be new elements where we set \( z_i^{j±} \in Y \). Then, we enforce with the method described above \( A_i \prec X_i^b \) using these new elements. Furthermore, we enforce \( X_i^a \prec B_i \), \( X_i^b \prec C_i \) and \( D_i \prec X_i^a \). Finally, we add the sets \( A_i \setminus \{ r_i \} \),
\[
B_i \setminus \{ r_i \}, C_i \setminus \{ s_i \} \) and \( D_i \setminus \{ s_i \} \) and enforce \( B_i \setminus \{ r_i \} \prec D_i \setminus \{ s_i \} \) and \( C_i \setminus \{ s_i \} \prec A_i \setminus \{ r_i \} \). We call the sets added in this step the class 4 sets. These enforced preference are shown as solid arrows in Figure 2.

Now, we claim that it is not possible for a weak order \( \preceq \) to satisfy dominance and independence with respect to \( \leq \) if \( X_i^a \prec X_i^b \) holds. Assume otherwise that \( \preceq \) is a weak order that satisfies dominance and independence with respect to \( \leq \) such that \( X_i^a \prec X_i^b \) holds. Then \( D_i < X_i^a \prec X_i^b < C_i \) implies \( D_i < C_i \) by transitivity and hence \( D_i \setminus \{ s_i \} \prec C_i \setminus \{ s_i \} \) by reverse independence. Similarly, \( A_i < X_i^a \prec X_i^b \prec B_i \) implies \( A_i < B_i \) by transitivity and hence \( A_i \setminus \{ r_i \} \prec A_i \setminus \{ r_i \} \)

\[5\text{Every linear order satisfying independence has to satisfy reverse independence, i.e. } A \cup \{ x \} \prec B \cup \{ x \} \text{ implies } A \preceq B. \text{ Assume otherwise } B \prec A \text{ holds, then by independence } B \cup \{ x \} \preceq A \cup \{ x \} \text{ must hold, contradicting } A \cup \{ x \} \prec B \cup \{ x \}. \text{ Hence by the totality of } \preceq \text{ we have } A \preceq B. \]
Proof of Corollary 7

We claim that Stronger DI-Orderability would be in coNP if there exists a polynomial time algorithm that produces on input $(X, X', \leq)$ a weak order $\preceq$ on $X$ that satisfies dominance and independence. Observe that there exists a linear order $\preceq$ on $X$ that can not be lifted if and only if $(X, X', \leq)$ is negative instance of Stronger DI-Orderability. Hence $\preceq$ is a certificate (of polynomial size) for the fact that $(X, X', \leq)$ is a negative instance. Furthermore, one can check the certificate by running $\mathcal{A}$ on $(X, X', \leq)$. Then, one only needs to check that the produced order does not satisfy dominance and strict independence. By definition, this can only be the case if $(X, X', \leq)$ is a negative instance of Stronger DI-Orderability. The argument for strict independence is analog. □

Proof of Proposition 10

Let $\phi$ be an instance of Taut. We assume w.l.o.g. that no variable occurs twice in the same clause. We construct an instance $(S, X')$ of the Strong Partial $\text{DIS}_{\leq}$-Orderability. For every variable $X_i$ in $\phi$ we add new elements $x_i$ and $x'_{i}$ to $S$. We call the set of these elements $X$. We will treat every order on $S$ as encoding a truth assignment by equating $x_i < x'_{i}$ to "$X_i$ is true" and $x_i > x'_{i}$ to "$X_i$ is false". Furthermore, we add for every clause new variables $y_j, y'_j$. We call the set of these elements $Y$. We also add for every clause $C_j$ elements $c_i$ as well as $d^{k}_{j}$ and $e^{k}_{j}$ for $k \leq 3$. Finally, we add new variables $u, v, z_1$ and $z_2$. In the following we call any linear order on $S$ that is derived by replacing $X$ with an arbitrary linear order on the elements in $X$ in the following linear order

$$u < c_1 < \cdots < c_m < y_1 < \cdots < y_m < d_1^2 < \cdots < d_m^2 < X < c_1^2 < \cdots < c_m^2 < y_2^2 < \cdots < y_m^2 < z_1 < z_2 < \omega$$

a critical linear order. In the following, we write $\preceq_{\text{min}}$ for the minimal partial order satisfying dominance and strict independence with respect to some linear order on $S$.

Next, we build the family $X$. We do this in a way such that $X$ is not strongly $\text{DIS}_{\leq}$-orderable if there is a non-satisfying assignment of $\phi$. First, we add singletons for all elements of $X$ and $y_i$ and $(x_i^1, x_i^2)$ for all elements of $X$. Then, for every linear order $\preceq$ we have $(x_i^1) \preceq_{\text{min}} (x_i^1, x_i^2) \preceq_{\text{min}} (x_i^2)$ and hence $(x_i^2) \preceq_{\text{min}} (x_i^1, x_i^2)$ if $x_i^1 < x_i^2$ and, on the other hand, $(x_i^2) \preceq_{\text{min}} (x_i^1) \preceq_{\text{min}} (x_i^1, x_i^2)$ and hence $(x_i^1, x_i^2) \preceq_{\text{min}} (x_i^1)$. Next, we add sets such that there is a critical linear order $\preceq$ on $S$ such that we have $(y_i^j) \preceq_{\text{min}} (y_i^j)$ for all $i \leq m$ if and only if $\phi$ is not a tautology. For every clause $C_j = X_{i_1} \land X_{i_2} \land X_{i_3}$ we add sets

$$\{y_j^i, d_j^i, \{y_j^i, d_j^i, x_{i_1}\}, \{d_j^i, x_{i_2}\}\}$$

for all $k \in \{1, 2, 3\}$ as well as

$$\{x_{i_1}^2, e_{j_1}^i, x_{i_2}^2, e_{j_2}^i, z_{1}, x_{i_2}^2, e_{j_2}^i, z_{1}, z_{2}, 2\}$$

$$\{e_{j_1}^i, z_{1}, z_{2}, e_{j_1}^i, z_{1}, z_{2}, y_j^i, 2\}, \{z_{1}, z_{2}, y_j^i, 2\}, \{z_{2}, y_j^i, 2\}.$$
and finally
\{u, c_1, \ldots, c_j, y_1, \ldots, y_{j-1}\} and
\{u, c_1, \ldots, c_j, y_2, \ldots, y_1\},
as well as
\{u, c_1, \ldots, c_j, y_1, \ldots, y_{j-1}, v\} and
\{u, c_1, \ldots, c_j, y_2, \ldots, y_1, v\}.

By construction
\{u, c_1, \ldots, c_j, y_1, \ldots, y_{j-1}, v\} \prec_{\min}
\{u, c_1, \ldots, c_j, y_2, \ldots, y_1, v\}
holds for the minimal partial order satisfying dominance and
strict independence for any linear order on \(X\) if and only if
\(\{y_j\} \prec_{\min} \{y_i\}\) holds for that partial order.

Next we add \(\{u, c_1, \ldots, c_j\}, \{u, c_1, \ldots, c_{j+1}\}\) and
\(\{u, c_1, \ldots, c_{j+1}, y_{j+1}\}\). Then we add new sets derived as
above by adding to both sets first all elements \(y_j\) to \(y_1\) and
then \(v\), one by one, in that order until we reach
\{u, c_1, \ldots, c_j, y_2, \ldots, y_{j-1}, v\} and
\{u, c_1, \ldots, c_{j+1}, y_{j+1}, y_1, \ldots, y_{j-1}, v\}.

Then
\{u, c_1, \ldots, c_j, y_2, \ldots, y_{j-1}, v\} \prec_{\min}
\{u, c_1, \ldots, c_{j+1}, y_{j+1}, y_1, \ldots, y_{j-1}, v\}
holds for the critical linear order by strict independence be-
cause
\{u, c_1, \ldots, c_j\} \prec \{u, c_1, \ldots, c_{j+1}\} \prec
\{u, c_1, \ldots, c_{j+1}, y_{j+1}\}
holds by dominance. Finally, we add \(\{v\}\) and then \(\{y_2, v\},\)
\(\{y_2, y_1, v\}\) and so on till we reach
\{c_1, \ldots, c_m, y_2, \ldots, y_1, v\}.

This forces for any critical linear order
\{u, c_1, \ldots, c_m, y_2, \ldots, y_1, v\} \prec_{\min} \{u, v\}.

Now, by construction and transitive we have for any critical linear order
\{u, v\} \prec_{\min} \{u, c_1, y_1, v\} \prec_{\min}
\{u, c_1, y_2, v\} \prec_{\min} \{u, c_1, c_2, y_2, y_1, v\} \prec_{\min} \ldots
\prec_{\min} \{u, c_1, c_2, \ldots, c_m, y_2, \ldots, y_1, v\} \prec_{\min} \{u, v\}
if (and only if) \(\{y_1\} \prec_{\min} \{y_2\}\) holds for all clauses, i.e.
if the critical linear order codes an unsatisfying assignment.
It follows that if \(\phi\) is not a tautology, then \((X, \mathcal{X})\) is not
strongly partial \(D^{+}\)-orderable.

It remains to show that \((X, \mathcal{X})\) is strongly partial \(D^{+}\)-
orderable if \(\phi\) is a tautology. Let \(\leq\) be a linear order on \(S\).
We construct a partial order \(\preceq\) that satisfies dominance and
strict independence with respect to \(\leq\). To avoid unnecessary
case distinctions, we will describe the construction only for
clauses with all positive variables. The only change in con-
struction required for negative variables is switching \(x_i^1\) and
\(x_i^2\). By Lemma 16, we can assume w.l.o.g. that \(z_1 < z_2\).
First we add the forced preferences between \(\{x_i^1\}, \{x_i^2, x_i^2\}\)
and \(\{x_i^2, x_i^2\}\). Next, we consider the sets containing an element
\(d_i^1\). We add all preferences that are implied by dominance
between sets from
\{\(u_i^1\), \(y_1, d_i^1\), \(y_i^1, d_i^1, x_1^1\), \(d_i^1, x_1^1\), \(x_1^1\)\}
and close under transitivity. The only possible application
of strict independence on these sets is that any preference
between \(\{y_i^1\}\) and \(\{x_i^1\}\) has to be lifted to \(\{y_i^1, d_i^1\}\)
and \(\{d_i^1, x_i^1\}\). By construction however, there can only
be a preference between \(\{y_i^1\}\) and \(\{x_i^1\}\) forced by domi-
nance and transitivity if the same preference holds between
\(\{y_i^1, d_i^1\}\) and \(\{d_i^1, x_i^1\}\). Because we assume that
no variable occurs twice in a clause, a preference between
\(\{y_i^1\}\) and \(\{x_i^1\}\) can not later be introduced through sets containing
another \(d_i^1\). Finally, to satisfy dominance and transitivity we have to add for all \(x_i^1\) the preference \(\{x_i^1, d_i^1\} \prec \{x_i^1, d_i^2\}\)
for all \(d_i^1, d_i^2\) such that \(d_i^1 < d_i^2\) holds.

Using a similar construction, we can order all sets contain-
ing an element \(c_i^1\) if we replace \(x_i^1\) by \(x_i^2\) and \(y_i^1\) by
\(z_i, z_2, y_i^2\). Finally, we add the enforced preference be-
 tween \(\{z_2, y_i^2\}\) and \(\{y_i^2\}\) as well as \(\{z_1, z_2, y_i^2, y_i^2\}\).
The later is enforced by dominance as we assume \(z_1 < z_2\).
Then we close everything under transitivity. By construc-
tion, this does not produce any new instances of strict inde-
pendence.

Now, we observe that \(\{z_1, z_2, y_i^2\} \prec \{z_2, y_i^2\}\) implies that
\(\{y_1^1\} \prec \{y_2^1\}\) can only hold if \(\{x_1^1\} \prec \{x_2^1\}\) holds for a vari-
able occurring in clause \(C_j\), i.e. if clause \(C_j\) is not satisfied.
As \(\phi\) is a tautology, there is clause \(C_i\) that is satisfied by
the assignment coded by \(\leq\). Hence, \(\{y_1^1\} \prec \{y_2^1\}\) does not hold.
We now consider the sets containing an element \(c_i\) for some
\(i\). We partition these sets in partitions \(P_1, \ldots, P_m\), based on
largest \(i\) for which they contain \(c_i\). We set \(S_1 \prec S_2\) if
\(S_1 \in P_1, S_2 \in P_2\) and one of the following holds:
\[
\begin{align*}
&c_i \prec c_{i+2} \text{ and } i_1, i_2 < l \\
&c_{i+2} \prec c_i \text{ and } l < i_1, i_2
\end{align*}
\]

Then any set that contains \(c_i\) also contains \(y_i\) ex-
cept \(\{u, c_1, \ldots, c_i\}\). Hence the only possible application
of dominance between sets of different partitions is
\(\{u, c_1, \ldots, c_{i+1}\} \prec \{u, c_1, \ldots, c_{i+1}\}\) which is satisfied by
construction for \(i, i+1 \neq l\). Now for any set in any par-
ition \(P_i\), such that \(i \neq l\) we set \(S \prec S'\) if \(y_i^1 \in S\) and
\(y_i^1 \not\in S'\). This covers all applications of strict independence
in a partition. Finally, we add all preferences that are forced
by dominance in a partition and close under transitivity. We
observe that \(S, S' \cup \{x\} \in P_i\) implies either \(y_i^1 \in S, S' \cup \{x\}\)
or \(y_i^1 \not\in S, S' \cup \{x\}\), hence this can not lead to a contradiction.
Now, for a set \(S \in P_i\) such that \(y_i^1 \in S\) we set
\[
\begin{align*}
&S' \prec S \text{ if } S' \in P_i \text{ for } i < l \text{ and } c_i < c_l
\end{align*}
\]
\[ S \prec S' \text{ if } S' \in P_i \text{ for } i < l \text{ and } c_l < c_i \]

Furthermore, for a set \( S \) in \( P_i \) such that \( y_1 \notin S \) we set
\[ S' \prec S \text{ if } S' \in P_l \text{ for } l < i \text{ and } c_l < c_i \]

\[ S \prec S' \text{ if } S' \in P_i \text{ for } i < l \text{ and } c_l < c_i \]

And finally, we add again all preferences forced by dominance and close by transitivity. As \( \{y_1\} \) and \( \{y_2\} \) are incomparable in \( \leq \) this order is consistent. Furthermore, \( \{u,c_1,y_1,z_1,z_2,v\} \) and \( \{u,c_1,\ldots,c_m,y_{m_1},\ldots,y_{m_2},v\} \) are incomparable in \( \leq \). This allows us to add all preferences forced by dominance and strict independence regarding \( \{u\}, \{v\} \) and \( \{u,v\} \) without creating a contradiction. By construction, \( \leq \) is now a partial order that satisfies dominance and strict independence. \( \square \)

**Proof of Theorem 13**

**SUCCINCT DI\(^3\)-ORDERABILITY** can be solved in NEXP-time by explicitly computing the family \( X \) and then solving the (exponentially larger) explicit problem in NP-time.

For the hardness, we only have to check that the presented reduction is computable in polytime. Then, by the Conversion Lemma, there is a ptme reduction from **SUCCINCT SAT** to **SUCCINCT DI\(^3\)-ORDERABILITY** resp. **SUCCINCT strong DI\(^3\)-ORDERABILITY**. The NEXP-hardness of both problems then follows as **SUCCINCT SAT** is known to be NEXPT-complete (Papadimitriou 1994). We have to show that we can compute a single bit of the output in polylog-time if we have random access to the input. For this, we have to take the binary representation of the SAT into account. Unfortunately, (Papadimitriou 1994) does not specify a binary representation for the NEXP-hardness proof. However, the proof given in the book is not sensitive to the representation as long as it is reasonable. The same is true for our proof. Reasonable means in our context that it is possible to determine the number of variables \( n \) and clauses \( m \) in polylog-time. For any sensible encoding of 3-CNF this is either explicitly encoded or can be determined via binary search. Furthermore, we assume that one only needs polylog-time to read the \( i \)-th variable in the \( j \)-th clause. This is trivially true if we assume that every clause is encode by the same amount of bits. It is easy to see that the proof in (Papadimitriou 1994) of the NEXP-hardness of **SUCCINCT SAT** works for such an encoding.

Now, we fix a binary representation for instances of **DI\(^3\)-ORDERABILITY** resp. **STRONG DI\(^3\)-ORDERABILITY**. First, we encode the number of elements \( k \) of \( X \) in binary. Then, the family \( X \) is encoded as a series of strings of length \( k \), where a 1 in position \( l \) means the \( l \)-th element of \( X \) is in the set and a 0 in position \( l \) means the \( l \)-th element is not in the set. For an instance of **DI\(^3\)-ORDERABILITY**, the linear order \( \leq \) is given by the natural order on these positions.

First observe that the size of \( X \) is \( 4n + 12m + 3 \) and the size of \( X \) is \( p(n,m) \) for some polynomial \( p(x,y) \). Therefore, we can determine it in polylog-time. Now, assume we want to decide whether the \( i \)-th bit of the output is 0 or 1. It is clear that this can be done in polylog-time if the \( i \)-th bit is part of the representation of the size of \( X \). Assume that the \( i \)-th bit determines if the \( l \)-th element \( x \) is part of a \( k \)-th set \( A \). We can assume that we fixed an order in which we generate the sets in \( X \) such that we can compute from \( m, n \) and \( i \) which set \( A \) is supposed to be. Observe that if \( x \) is not of the form \( x_j^+ \) or \( x_j^- \) then, this already suffices to decide whether \( x \) is in \( A \). On the other hand, if \( x = x_j^+ \) or \( x = x_j^- \) and \( A \) is a class 1 set, then this already suffices. Finally, if \( x = x_j^+ \) or \( x = x_j^- \) and \( A \) is not a class 1 set then the question whether \( x \) is in \( A \) only depends on the question if \( X_j \) occurs (positively or negatively) in that clause in the right position.

The properties of the reduction from **SAT** to **DI\(^3\)-ORDERABILITY** resp. **DI\(^3\)-ORDERABILITY** used in the proof above hold also for the reduction from **SAT** to **DI\(^3\)-ORDERABILITY** resp. **DI-ORDERABILITY**. Therefore, Theorem 14 can be proven using the same argumentation as above. \( \square \)

**Proof of Theorem 14**

We observe that the reduction from **TAUT** to **STRONG PARTIAL DI\(^3\)-ORDERABILITY** satisfies the same properties as the reduction from **SAT** to **STRONG DI\(^3\)-ORDERABILITY**, i.e. the number of elements in \( S \) as well as the number and size of the sets in \( X \) only depends on the size and not on the structure of the formula \( \phi \). Furthermore, if a element is in a set or not only depends on one specific clause. Therefore, the reduction can be done in polylog-time. By the Conversion Lemma and the coNEXP-completeness of **TAUT**, this suffices to show that **STRONG PARTIAL DI\(^3\)-ORDERABILITY** is coNEXP-complete. \( \square \)