Proofs for Propositional Model Counting*

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Abstract

Although propositional model counting (\#SAT) was long considered too hard to be practical, today’s highly efficient solvers facilitate applications in probabilistic reasoning, reliability estimation, quantitative design space exploration, and more. The current trend of solvers growing more capable every year is likely to continue as a diverse range of algorithms are explored in the field. However, to establish model counters as reliable tools like SAT-solvers, correctness is as critical as speed.

As in the nature of complex systems, bugs emerge as soon as the tools are widely used. To identify and avoid bugs, explain decisions, and provide trustworthy results, we need verifiable results. In this paper, we propose a novel proof system for certifying model counts. We show how proof traces can be generated efficiently for exact model counters based on dynamic programming, counting CDCL with component caching, as well as knowledge compilation to Decision-DNNF. These approaches are the most widely used techniques in exact implementations. Furthermore, we provide proof-of-concept software for emitting proofs from model counters and a parallel trace checker. Based on this, we show the feasibility of using certified model counting in an empirical experiment.

1 Introduction

Propositional model counting, also known as \#SAT, asks to output the number of satisfying assignments of a propositional formula. The problem is canonical for the complexity class \#P\footnote{Authors are stated in alphabetic order.} and by consequence of Toda’s theorem,\footnote{\texttt{johannes.fichte@tuwien.ac.at}} a polynomial-time machine can solve with a single call to a \#SAT oracle any problem in the Polynomial Hierarchy. While \#SAT was long considered impractical, the field has seen considerable advances in recent years and highly efficient solvers emerged, capable of solving larger problems each year\footnote{\texttt{hecher@dbai.tuwien.ac.at}}\footnote{\texttt{valentin.roland@secunet.de}}. The applications of these solvers are vast. In artificial intelligence and reasoning, model counting is key when using logic-based reasoning for symbolic quantitative tasks\footnote{\texttt{1144}}. Model counters are becoming a standard tool for answering quantitative queries on propositional theories, in domains like configuration analysis\footnote{\texttt{52}}\footnote{\texttt{10}}, probabilistic reasoning\footnote{\texttt{1140}}, explainable artificial intelligence\footnote{\texttt{512}}, or risk analysis\footnote{\texttt{5710}}.

In fields such as explainability, risk analysis, or verification, we need to be able to trust the output of a model counter. Similar to SAT solvers, model counters are highly efficient, but complex software systems making it hard to trust their outputs. As in any complex system\footnote{\texttt{20}}, subtle implementation errors may easily remain undiscovered. In fact, in the course of this research, we spotted such an error in the model counter \texttt{sharpsat}\footnote{\texttt{53}}, which also serves as a basis for several
other implementations, like Ganak [50] and Dsharp [41]: If preprocessing is disabled, the solver does not propagate unit clauses present in the input formula. In some cases, this leads to illegal assignments to their variables and wrong model counts. This highlights that even a widely used solver can hide implementation errors, especially in lesser-used code paths.

**Example 1 (Spotted Bug in sharpSAT).** Consider a formula consisting of two unit clauses $F = (a) \land (b)$. It is easy to see that this formula has exactly one model. However, without preprocessing, sharpSAT produces a model count of 4. Since input unit clauses are not propagated, a and b are considered unconstrained during component analysis.

Since verifying entire complex solvers, which are constantly enhanced by algorithmic innovations, is currently effectively impossible, researchers came up with a way to certify outputs from SAT solvers [5, 6, 34]. The idea is simple but extremely effective. The solver emits a trace during solving in a specific proof system focusing on simple constructs. In that way, decisions made by a solver can be externally verified. The tool to prove correctness of the trace can itself be fully verified and remains stable – even if the solver changes. While one might not be able to verify all techniques in the solver, one can ensure that every step is correct by verifying simple mathematical properties on the trace. This led to incredible stability improvements in the solvers, in particular for complex parts such as inprocessing [31]. However, propositional proof systems are tailored towards unsatisfiability [10] and there is no obvious way to encode a count in a sequence of existing clausal proofs [29], as enumerating models quickly becomes infeasible. So far no generic proof system for model counting exists.

In model counting, we are interested in properties that are between satisfiability and equivalence of the formulas. Mathematical combinatorics and number theory already provides us with basic tools to establish counting proofs [30]. One possible principle is double-counting, which checks whether two different approaches return the same answer. However, while the diversity of algorithms for model counting makes correlated errors in two different solvers less likely, there may still be a common conceptual error. In addition, double-counting limits us to instances for which more than one solver returns a count. Another classical principle is to establish a proof by bijection. Two sets are shown to have the same number of members by exhibiting a bijection, i.e., a one-to-one correspondence, between them. We employ the principle of proofs by bijection and establish a proof system that is tailored to practical model counting. To this end, we take formalize systematic search space splitting, which is a common technique in model counting, and employ existing clausal proofs for unsatisfiability.

**Contributions** Our main contributions are as follows:

1. We propose a novel proof system for certifying propositional model counts in practice.

2. We show how proof traces can be generated efficiently for exact model counters based on dynamic programming, counting version of CDCL with component caching, and knowledge compilation to Decision-DNNF.

3. We provide proof-of-concept software for emitting proofs from model counters and a parallel trace checker. Based on this, we show the feasibility of using certified model counting in an empirical experiment.

**Related Works** Over time, various proof systems were developed for certifying SAT solver outputs [27, 34, 50, 11], among them the popular format DRAT [54]. These approaches aim to show unsatisfiability as a satisfying assignment can anyways be checked efficiently. While there is no general method for verifying exact model counts, outputs of knowledge compilers can be validated by equivalence checking [7, 8]. One approach is to label unsatisfiable sub-formulas, during knowledge compilation, with a clause indicating the cause of the conflict [7]. This allows checking equivalence to a formula in conjunctive normal form (CNF) in polynomial time under restricted conditions. Since it conflicts with component caching employed in knowledge compilers
like D4 [39] and Dsharp [41], more recent works introduce a more flexible notion of syntactical equivalence [8]. While knowledge compilers, counting version of CDCL-based algorithms, and dynamic programming-based solving techniques are related [33], the two latter techniques are not accommodated in the certification approach from above. For example, the best performing solver of the 2021 Model Counting Competition did not use knowledge compilations [37]. Furthermore, the notion of equivalence used for certifying knowledge compilations is stronger than needed for preserving correctness of counting. In consequence, we expect that certain counting-specific simplifications cannot be formulated within this framework. Model counting on Decision-DNNFs can easily be done in terms of complexity [13], but technically the result is not verified. Hence, errors in the counting step will not be caught by checking equivalence to the input formula. Finally, we would like to mention that our focus is on exact model counting. Some modern counting techniques rely on probabilistic exact counting or approximate counting, which is incredibly valuable for scalable counting of very large instances in applications where the exact count is of less relevance. For approximate counting, the idea is to reduce the solution space uniformly to a small number of samples. By varying the number and length of randomly chosen XOR constraints and the number of repetitions, approximate counting can produce arbitrarily tight bounds with arbitrarily high confidence. Here, our approach is not meaningful, already existing techniques for certifying XOR constraints can be used [44] [28] instead.

2 Preliminaries

We assume that the reader is familiar with basic notions on functions, set theory, computational complexity [43], and propositional logic [39].

Propositional Satisfiability and Model Counting (#SAT) We briefly recall the used notation and define propositional satisfiability and its evaluation in the usual way [39] [8]. We consider a universe \( V \) of propositional variables. A literal \( \ell \) is a variable \( v \) or its negation \( \neg v \). We assume \( \neg \neg v = v \) and, for a set \( L \) of literals, let \( \neg L := \{ \neg v \mid v \in L \} \). A finite set \( C \) of literals is called a clause, interpreted as disjunction of literals. We say that \( C \) is unit if \( |C| = 1 \).

A formula \( F \) in conjunctive normal form (CNF) is a finite set of clauses, interpreted as conjunction of clauses. Unless otherwise stated, we assume that formulas are in CNF. We refer by \( \text{vars}(C) \) to the variables occurring in a finite set of literals, i.e., \( \text{vars}(C) := \{ \text{var}(\ell) \mid \ell \in C \} \) where \( \text{var}(v) = v \) and \( \text{var}(\neg v) = v \) for \( v \in V \), and by \( \text{vars}(F) \) to the variables occurring in a formula \( F \), i.e., \( \text{vars}(F) := \bigcup_{C \in F} \text{vars}(C) \). An assumption is a set \( A \) of literals. We let \( \hat{A} := \{ \ell \mid \ell \in A \} \), i.e., the formula enforcing assumption \( A \). A restriction of assumption \( A \) to set \( V \) of variables is \( A_V := A \cap (V \cup \neg V) \). An assignment is a mapping \( \alpha : V \to \{0, 1\} \) defined for a set of variables. A restriction of assignment \( \alpha \) to a set \( V \) of variables is defined as \( \alpha|_V := \{ v \mapsto \alpha(v) \mid v \in \text{dom}(\alpha) \cap V \} \). For \( v \in V \), we define \( \alpha(\neg v) = 1 - \alpha(v) \). By \( 2^V \) we denote the set of all total assignments \( \alpha : V \to \{0, 1\} \), by \( \alpha^{-1}(b) \) the preimage of \( \alpha \) for \( b \in \{0, 1\} \), and by \( \text{dom}(\alpha) = V \) its domain. For a formula \( F \), we let \( F[\alpha] := \{ C \setminus \alpha^{-1}(0) \mid C \in F, C \cap \alpha^{-1}(1) = \emptyset \} \) be the formula \( F \) under assignment \( \alpha \). An assignment \( \alpha \) satisfies \( F \) if \( \emptyset \in F[\alpha] \); \( \alpha \) satisfies \( F \) if \( F[\alpha] = \emptyset \), in which case we call \( \alpha \) a model of \( F \). Note for an assumption \( A \), we can satisfy \( \hat{A} \) trivially by assigning \( \tau_A := \{ v \mapsto 0 \mid \neg v \in A \} \cap \{ v \mapsto 1 \mid v \in A \cap V \} \). We let \( \text{Mod}(F) \) be the set of all models of \( F \) over variables \( \text{vars}(F) \) and \( |\text{Mod}(F)| \) be the model count of \( F \). The problem SAT asks whether \( \text{Mod}(F) \geq 1 \) and the model counting problem, or #SAT for short, asks to output \( |\text{Mod}(F)| \). In addition, for tuple \( T = (F, V) \) consisting of formula \( F \) and set \( V \) of variables, we let \( \text{Mod}_A(T) := \{ \alpha \mid \alpha \in 2^V, \alpha \text{ satisfies } F \cup \hat{A} \} \) be the models of \( T \). The model count for \( T \) under assumption \( A \) is \( |\text{Mod}_A(T)| \).

Certified SAT Solving Certified results are common in SAT solving [27] [31] [25] and support of a standardized format is mandatory for solvers taking part in the competition [25]. Popular formalisms like DRAT [31] and RUP are examples of the more general notion of propositional proof systems [10]. Let \( \Sigma \) be an alphabet, \( L \subseteq \Sigma^* \) be language, and \( \text{TAUT} \) be the class of all
propositional formulas that are tautological, encoded in a fixed alphabet. In general, a *proof system* is a polynomial-time computable function \( s : \Sigma_1^* \rightarrow \Sigma^* \) with range \( L \) that maps from words over a proof alphabet \( \Sigma_1 \) to words in \( L \). For a formula \( F \), if \( s(x) = F \) then \( x \in \Sigma_1 \) is called a *proof* in system \( s \). A *propositional proof system* is a proof system for \( \text{TAUT} \). Usual properties are *completeness* asking that every propositional tautology has a proof in system \( s \) and *soundness* asking that if a propositional formula has a proof in system \( s \) then it is a tautology. Intuitively, propositional proof systems concern *certificates* of membership in \( \text{TAUT} \) for a given formula. Since SAT solving usually works on CNF formulas as input, practical focus is on clausal proofs. *Clausal proofs* are sequences of clauses, where each clause is entailed by the original formula \([39]\). A clausal proof is called a *refutation*, or proof of unsatisfiability, if it contains the empty clause. The input formula is unsatisfiable if the empty clause is derived in this sequence. Proof variants are based on clauses that have the RUP (reverse unit propagation) \([27]\) and RAT (resolution asymmetric tautology) property. These proof formats share verifiability in polynomial time in the size of the proof and input formula and can be tightly coupled with modern solving techniques. The popular DRAT proof system uses a more general redundancy property based on extended resolution instead of entailment \([31]\).

### 3 From Clausal Proofs Towards Certifying \#SAT

By basic constructions, we can use propositional proof systems to establish correctness of an outputted model count. To this end, we systematically enumerate models and add the negation of a found model to the formula, thereby forbidding it. After solving outputs that all models were found, we can prove that no more models exist by using clausal proofs of unsatisfiability. Alternatively, we can establish equivalence using propositional proof systems when solving by means of knowledge compilation. In both cases we use decision-based proof systems for an actual function problem. Clearly, propositional proof systems lack the capability of reasoning about sets of models and their cardinality, instead, we can only reason about individual yes/no decisions. It is easy to see that enumerating models quickly becomes infeasible and an equivalence-based approach works only for very specific techniques. Moreover, a decision-based approach is not how modern model counters reason. Instead, they commonly split the search space into sub-problems where solutions can be considered independently, so-called *components*, as illustrated on a high level in the example below.

**Example 2 (Refutations are Insufficient for Counting).** Consider formula \( F = (a \lor b) \land (c \lor d) \), which has 9 models. In detail, there are 3 assignments to \( a \) and \( b \) that satisfy \( (a \lor b) \) and 3 assignments to \( c \) and \( d \) to satisfy \( (c \lor d) \). Since both clauses share no variables, we can freely combine the satisfying assignments for each clause. Hence, we obtain \( 3 \cdot 3 = 9 \) models.

Example 2 states a reasoning technique that cannot be expressed concisely in a clausal proof. In fact, we miss notions to express the combination of both sub-problems and lack the capability of reasoning about sets of models and their cardinalities. Consequently, we are interested in a natural approach to obtain certifiably correct results for practical propositional model counters. From a more theoretical perspective, we design a proof system for counting where the computable function has range \( \mathbb{N}_0 \). The challenging part is to define a system that is simple, but expressive enough that certificates can be generated by diverse model counting algorithms with low overhead. Mathematical combinatorics and number theory already provides us with basic tools to establish counting proofs \([20]\). A classical concept in proofs is to establish relations between sets using bijections, i.e., a one-to-one correspondence. Two sets have the same number of elements if there is a bijection between them.

**Example 3.** Consider again Example 2. We can easily show that there is a bijective mapping between the set obtained from the Cartesian product of models of both clauses and the set of models of the formula \( F \), i.e., \( \text{Mod}(F) = \text{Mod}(a \lor b) \times \text{Mod}(c \lor d) \). Hence, the model count of formula \( F \) can simply be expressed as the product of the cardinality of both sets, i.e., \( |\text{Mod}(F)| = |\text{Mod}(a \lor b)| \cdot |\text{Mod}(c \lor d)| \).
Figure 1: Visual representation of operations used by \#SAT solvers when exploring the solution space. Splits in the search space are composed, independent sub-instances are joined and sub-instances are extended by assigning new variables.

Below, we formalize systematic search space splitting, which is common among all exact model counting approaches. Then, we establish general reasoning rules based on this technique and we combine it with established techniques on clausal proofs for TAUT. The overall principle remains. Solvers output certificates, which can be easily verified by a program (checker). Correctness of the checker needs to be audited for fully verified results.

3.1 Search Space Splitting

State-of-the-art model counters split the input instance into sub-instances and combine the results – even implicitly in knowledge compilation-based or dynamic programming-based counters. Fig. 1 visualizes solution space exploration and operations that modern solvers use to combine solutions of sub-instances. Intuitively, we observe the following principles: (i) Solvers split the search space along an assumption, e.g., a decision literal. The corresponding reasoning operation is a composition of disjoint sets of models for a sub-instance. (ii) Solvers identify sub-instances that are independent or overlap in such a way that their models can be combined in a database-like join. (iii) Under some restrictions, models of a sub-instance can be extended to models of a larger one by adding a stricter assumption. (iv) There are sub-instances that do not have any models.

Next, we introduce components as a formal notion for sub-instances and claims as statements talking about the model count of components under an assumption. Intuitively, a component consists of a subset of the input formula and a selection of variables.

Definition 1 (Component and Sub-components). A component $C = (F, V)$ consists of a formula $F$ and a set $V$ of variables. We say that a component $C_s = (F_s, V_s)$ is a sub-component of $C$ if $F_s \subseteq F$ and $V_s \subseteq V$.

The set of models of a component can be restricted by an assumption, i.e., a set of unit literals. Then, models can be represented by stating the component and assumption, which we can directly use for counting. We call such a statement a claim, which represents a partial result. The following example illustrates sets of models for components under assumptions.

Example 4. Consider formula $F = (a \lor b) \land (c \lor d)$ from Example 3 and corresponding component $C = (F, \text{vars}(F))$. Recall that without an assumption, $F$ has 9 models. If we add assumption $A_1 = \{a \rightarrow \neg b\}$, then $\text{Mod}_{A_1}(C) = \{\{a \rightarrow 1, b \rightarrow 0, c \rightarrow 1, d \rightarrow 0\}, \{a \rightarrow 1, b \rightarrow 0, c \rightarrow 0, d \rightarrow 1\}, \{a \rightarrow 1, b \rightarrow 0, c \rightarrow 1, d \rightarrow 1\}\}$, hence $|\text{Mod}_{A_1}(C)| = 3$. With assumption $A_2 = \{\neg a, \neg b\}$ we have $\text{Mod}_{A_2}(C) = \emptyset$.

Definition 2 (Claim). Let $C = (F, V)$ be a component. A claim $I = (C, A, c)$ over $C$ consists of an assumption $A$ with $\text{vars}(A) \subseteq V$ and a count $c \in \mathbb{N}_0$. $I$ is correct if $c = |\text{Mod}_A(C)|$.

Working with claims over components avoids naive enumeration of models, which we can see in the following example.
Example 5. Consider a formula $F$ with a large number of clauses and variables. Further, we have two disjoint components $C_1 = (F_1, V_1)$ and $C_2 = (F_2, V_2)$ in formula $F$ (i.e., $F_1 \cap F_2 = \emptyset$ and $V_1 \cap V_2 = \emptyset$). Using claims, we succinctly express that any two of the models can be combined into a model of a larger component $C = (F_1 \cup F_2, V_1 \cup V_2)$: If $[C_1, \emptyset, c_1]$ and $[C_2, \emptyset, c_2]$ are correct claims, then $[C, \emptyset, c_1 \cdot c_2]$ is correct, regardless of $c_1$ and $c_2$. In contrast, enumeration would quickly become infeasible in practice, since the individual components may have exponentially many models.

3.2 (Re-)combining Claims of Search Spaces

As already demonstrated by the previous example, we can model search space splitting of modern model counters by the combination of claims. In general, however, it is not guaranteed that we do not over- or undercount when combining claims. Hence, we consider properties of sets of claims that dictate which claims can be combined correctly.

Avoid Overcounting. First, we demonstrate a case where overcounting occurs.

Example 6. Let $C = (\{\{a, b, c\}, \{a, \neg d\}\}, \{a, b, c, d\})$ be a component and $C = (\{a, b, c\}, 4)$ and $\{a, d\}, 4$ be claims of $C$. Then, both claims count the model $\{a \mapsto 1, b \mapsto 1, c \mapsto 1, d \mapsto 1\}$, i.e., they overlap and cannot be simply combined without overcounting.

Consequently, we consider non-overlapping sets of claims, which prevents overcounting.

Definition 3 (Non-overlapping and Uniform Claims). Let $C = (F, V)$ be a component and $S$ be a set of claims over $C$.

- Then, $S$ is non-overlapping if, for every two distinct claims $(\{C, A_1, c_1\})$ and $(\{C, A_2, c_2\})$ in set $S$, we have $\text{Mod}_{A_1}(C) \cap \text{Mod}_{A_2}(C) = \emptyset$.
- Let $U \subseteq V$ be a set of variables of the component. If $\text{vars}(A') = U$ for every $(C, A', c') \in S$, we call $S$ uniform for $U$.

Observe that uniformness is a special case of non-overlapping that is easier to verify.

Observation 1 (⋆). Let $C = (F, V)$ be a component, $U \subseteq V$ a set of variables, and $S$ a set of claims over $C$. If $S$ is uniform for $U$ and all claims in $S$ are correct, $S$ is non-overlapping.

Provably Prevent Undercounting. While non-overlapping sets of claims prevent overcounting, we also need to ensure the absence of further models to prevent undercounting. Before we show how to avoid this, we briefly illustrate undercounting below.

Example 7. Consider a component $C = (\{\{a, b, c\}, \{a, \neg d\}\}, \{a, b, c, d\})$ and claims $I_1 = (\{C, \{a, d\}, 4\}$, $I_2 = (\{C, \{a, \neg d\}, 4\}$ of $C$. $I_1$ and $I_2$ are non-overlapping, but cannot correctly be combined to $I = (\{C, \emptyset, 8\}$, because models with $a \mapsto 0$ are counted by neither $I_1$ nor $I_2$. However, adding a third claim $\{C, \{\neg a, d\}, 3\}$ is sufficient to cover $C$ exhaustively, since there are no models with $a \mapsto 0$ and $d \mapsto 1$.

To prove that undercounting does not occur, we need to ensure that a set of claims covers the models of a component 

Definition 4 (Exhaustive Claims). Let $C = (F, V)$ be a component and $S$ a set of claims over $C$. For an assumption $A$, we call $S$ exhaustive for $A$ if for every model $\alpha \in \text{Mod}_A(C)$ there is a claim $(C, A', c') \in S$ with $\alpha$ satisfies $A'$ and $A \subseteq A'$.

Unfortunately, already the task of checking exhaustiveness is hard.

Proposition 1 (⋆.Exhaustiveness of Claims is co-NP Hard). Let $S$ be a set of claims for a component $C = (F, V)$ that is uniform for $U \subseteq V$ and $A$ be an assumption. Then, it is co-NP complete to decide if $S$ is exhaustive for $A$. 


Despite this result, we later show that exhaustiveness can be established efficiently for the surveyed solver implementations. This is not a contradiction, since \#SAT is known to be in \#P \cite{origin}, and thus at least as hard as decision problems in NP and co-NP. Hence, when using intermediate results from the solving process for exhaustiveness checking, the “hardness” lies in computing this intermediate information.

However, if a set of claims is uniform, we can show exhaustiveness, using well-known constructs from SAT. Next, we establish a shortcut for exhaustiveness of sets of claims. We formalize this in the notion of an absence of models statement.

**Definition 5** (Absence of Models Statement). Let \( C = (F,V) \) be a component, \( S \) be a set of claims that is uniform over \( U \subseteq V \), and \( A \) be an assumption where we have \( A \subseteq A' \) for every claim \((C,A',c) \in S \). Then, given a clausal proof \( \Delta \), we call \( A = (C,A,U,\Delta) \) an absence of models statement. Such a statement \( A \) is correct if \( \Delta \) is a refutation of \( \hat{A} \sqcup \{c \mid C \in F\} \sqcup \{\neg A' \mid (C,A',c') \in S\} \).

Intuitively absence of models can be seen as acting complementarily to claims. To state that a component under some assumption is unsatisfiable, except for a set of claims, we employ the well-established concepts of unsatisfiability proofs in SAT solving. Indeed this is sufficient for exhaustiveness.

**Lemma 1** (\(*\)-Absence of Models). Let \( A = (C,A,U,\Delta) \) be an absence of models statement and \( S \) be a set of claims for component \( C \) that is uniform over \( U \) and we have \( A \subseteq A' \) for every claim \((C,A',c) \in S \). If \( A \) is correct for \( S \), then \( S \) is exhaustive for \( A \).

**Non-overlapping and Exhaustive Claims.** To avoid both over- and undercounting, we combine the properties non-overlapping and exhaustive. We say that a non-overlapping set of claims that is exhaustive for \( A \) is composable to \( A \).

**Example 8.** Consider a component \( C = (\{a,b,c\},\{a,\neg d\}) \) and a set \( S \) of claims of \( C \) with assumptions \( \{a,d\}, \{a,\neg d\}, \text{ and } \{\neg a,\neg d\} \). Since the assumptions are distinct and \( S \) is uniform for \( U = \{a,d\} \), \( S \) is non-overlapping. Since \( C \) has no model with \( a \mapsto 0 \) and \( d \mapsto 1 \), \( S \) is exhaustive for \( \emptyset \). To prove this, we can easily find a refutation \( \Delta \) for \( \{\ldots\}, \{a,\neg d\}\sqcup \{\neg a,\neg d\}, \{\neg a,d\}, \{a,d\}\}. Hence, we can construct a correct absence of models statement \((C,\emptyset,U,\Delta)\) and know that \( S \) is composable to \( \emptyset \).

Note that we can restrict a set of claims that is uniform over \( U \), that is composable to some assumption, to a subset with a stricter assumption. This subset remains composable to that stricter assumption. Hence, a single absence of models statement can be used to show compositability for multiple subsets of such a set of claims.

**Observation 2** (Composable Subset). Let \( S \) be a set of claims for a component \( C = (F,V) \), that is composable to \( A \) and uniform for \( U \). Let \( A_s \) be an assumption with \( A \subseteq A_s \) and \( \text{vars}(A_s) \subseteq U \). Then, \( S_{A_s} := \{(C,A',c') \in S \mid A_s \subseteq A'\} \) is composable to \( A_s \).

### 3.3 Inference Rules and Proof Traces for Model Counting

Next, we establish inference rules, which allow us to mechanically verify correctness of a combination of partial results, which are succinctly expressed by claims. Intuitively, these rules then establish the correctness of a claim by combining claims of sub-components or claims with stronger assumptions. In other words, we bring the combination of claims in line with splitting the search space by composition, join, and extension, as visualized in Figure 1. The resulting basic inference rules are (exactly one) model, composition, join, and extension. Composition itself is backed up by exhaustiveness, thereby using the sufficient absence of models statement via clausal proofs, cf., Lemma 2. Due to space requirements, we formalize the rules and directly show correctness in the lemmas below.

**Exactly One Model.** First, we cover the base case of exactly one model, claiming exactly one model for a component under a (total) assumption. There, no further claim is needed to check its
correctness. Since a model is a total assignment over the variables of the component, it suffices to check whether the formula of the component is satisfied by the assumption.

**Lemma 2** (Exactly One Model). Let \( C = (F, V) \) be a component and \((C, A, 1)\) be a claim for \( C \) with \( \text{vars}(A) = V \). The claim is correct if and only if \( \tau_A \) satisfies \( F \).

**Composition.** Next, we discuss the composition inference rule. If we have a set \( S \) of claims of a component that is composable to an assumption \( A \), we can directly infer a claim with the more general assumption \( A \). Intuitively, we derive a more general statement from a set of more specific claims. A matching absence of models statement represents proof that \( S \) meets the necessary conditions to avoid undercounting, i.e., that \( S \) is exhaustive for \( A \). However, by Observation 4, such a statement also implies that \( S \) is non-overlapping, thereby avoiding overcounting as well. Therefore, the following lemma assumes compositability as a requirement.

**Lemma 3** (Composition or Unsatisfiability). Let \( C = (F, V) \) be a component, \( \mathcal{I} = (C, A, c) \) be a claim with assumption \( A \), \( S \) be a set of claims over \( C \) that is composable to \( A \), and \( c := \sum_{(C, A', c) \in S} c' \). If every claim in \( S \) is correct, then \( \mathcal{I} \) is correct, i.e., \( |\text{Mod}_A(C)| = c \).

Note that if \( S = \emptyset \), composition states that the current component is unsatisfiable under assumption \( A \). Extending Example 8, the following example illustrates inference by composition of a composable set of claims.

**Example 9.** Consider a component \( C = (F, V) \) with \( F = \{ \{a, b, c\}, \{a, \neg d\} \} \) and \( V = \{a, b, c, d\} \). Recall from Example 8 that a set \( S \) of claims with assumptions \( \{a, d\} \), \( \{a, \neg d\} \), and \( \{\neg a, \neg d\} \), is composable to \( \emptyset \). By composition, we infer claim \((C, \emptyset, 11)\), as shown below.

<table>
<thead>
<tr>
<th>set ( S ):</th>
<th>composed claim:</th>
</tr>
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<tbody>
<tr>
<td>( c )</td>
<td>( A )</td>
</tr>
<tr>
<td>4</td>
<td>( {a, d} )</td>
</tr>
<tr>
<td>4</td>
<td>( {a, \neg d} )</td>
</tr>
<tr>
<td>3</td>
<td>( {\neg a, \neg d} )</td>
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</tbody>
</table>

In addition, we know that \( \emptyset \) is composable to \( A = \{\neg a, d\} \). By composition, we infer a claim \((C, A, 0)\). This is equivalent to stating that \( C \) is unsatisfiable under assumption \( A \).

**Join.** If the models of two components are independent, we can combine them arbitrarily to models of a joint component. The join inference rule generalizes this idea allowing models to overlap, whose consequences are formalized in the following lemma.

**Lemma 4** (Join). Let \( C = (F, V) \) be a component; \( \mathcal{I} = (C, A, c) \) be a claim; \( C_1 = (F_1, V_1) \) and \( C_2 = (F_2, V_2) \) be sub-components of \( C \) with \( F = F_1 \cup F_2 \), \( V = V_1 \cup V_2 \), and \( V_1 \cap V_2 \subseteq \text{vars}(A) \) where every \( C \in F \) has \( \text{vars}(C) \cap (V \setminus V_1) = \emptyset \). If \( \mathcal{I}_1 = (C_1, A|_{V_1}, c_1) \) and \( \mathcal{I}_2 = (C_2, A|_{V_2}, c_2) \) are correct claims over \( C_1 \) and \( C_2 \) and \( c = c_1 \cdot c_2 \), then \( \mathcal{I} \) is correct, i.e., \( |\text{Mod}_A(C)| = c \).

Note that if the model count of either joined claim is zero, the joint model count is zero, regardless of the second joined claim. The following example illustrates joins with overlapping assumptions.

**Example 10.** Consider components \( C_1 = (F_1, V_1) \) and \( C_2 = (F_2, V_2) \) with \( F_1 = \{\{a, b, c\}\}, V_1 = \{a, b, c\} \), \( F_2 = \{\{a, \neg d\}\} \), and \( V_2 = \{a, d\} \). The tables below represent claims for both components, along with claims for a third component \( C = (F_1 \cup F_2, V_1 \cup V_2) \). All claims of \( C \) are inferred from claims of sub-components \( C_1 \) and \( C_2 \) using Lemma 4.

---

4Proofs of statements marked with • can be found in the supplemental material.
Extension. Similar to extending models to assign additional variables, we can extend a claim to a larger component by adding additional literals to its assumption. We formalize this in the following statement.

Lemma 5 (Extension). Let $\mathcal{C} = (F, V)$ be a component, $\mathcal{I} = (\mathcal{C}, A, c)$ be a claim, $\mathcal{C}' = (F', V')$ be a sub-component of $\mathcal{C}$, and $\mathcal{I}' = (\mathcal{C}', A_{V'}, c)$ be a correct claim. If $V \setminus V' \subseteq \text{vars}(A)$, $\tau_A$ satisfies $F \setminus F'$, and $\tau_{A|V \setminus V'}$ does not satisfy $C$ for all clauses $C \in F'$, then $|\text{Mod}_A(C)| = c$.

In principle, one might ask why we cannot add arbitrary literals to the assumption when extending a claim. However, every model of a claim $\mathcal{I}'$ requires a corresponding model in the extended claim $\mathcal{I}$. We enforce this by enforcing that the clauses that are only in the larger component $\mathcal{C}$, are satisfied by the literals added to the assumption. Conversely, every model of the claim $\mathcal{I}$, when restricted to the variables of $\mathcal{C}'$, must be a model of $\mathcal{I}'$. Intuitively, this establishes a sufficient one-to-one correspondence between the models of $\mathcal{I}$ and $\mathcal{I}'$.

Example 11. We extend a component $\mathcal{C}' = (F', V')$ with $F' = \{a, b, c\}$ and $V' = \{a, b\}$ to the component $\mathcal{C} = (F, V)$ with $F = \{a, b, c\}, \{a, -d\}$ and $V = \{a, b, c, d\}$. The tables below list claims for both components, where the claims of $\mathcal{C}$ can be verified by Lemma 5 (extension). Note that the extended assumption must include $\neg c$ to satisfy extension.

Remark 1. Let $\mathcal{C} = (\emptyset, \emptyset)$ be a component. Since $\text{Mod}_\emptyset(C) = \{\emptyset\}$, claim $\mathcal{I} = (\mathcal{C}, \emptyset, 1)$ is correct. Thus, a claim satisfying exactly one model can be expressed as an extension of $\mathcal{I}$.

Proof Traces for Model Counting

In the previous section, we established principles for reasoning on correctness of claims based on claims obtained during search space splitting. From these principles, we construct a proof system for $\#\text{SAT}$ and subsequently show its soundness and completeness. To reproduce the reasoning of a solver, we use a trace, which is a sequence of steps, where each step either represents a claim (i.e., models) or an absence of models statement.

Definition 6 (Model Counting Proof Trace). A model counting proof trace $T = \langle s_1, \ldots, s_n \rangle$ is a finite sequence of steps $s_i$, which is either a claim or an absence of model statement. Trace $T$ is complete for $F$ if there is a claim $\mathcal{I} = (\mathcal{C}, \emptyset, c)$ in $T$ for a component $\mathcal{C} = (F, \text{vars}(F))$. Trace $T$ is correct if all claims in $T$ are correct.

Next, we show that a proof system using correct and complete model counting proof traces to certify the model count for a formula is sound and complete.

Theorem 1 (Soundness and Completeness). Let $F$ be a formula and $\mathcal{C} = (F, \text{vars}(F))$ a component. There is a correct model counting proof trace $T$ containing a claim $(\mathcal{C}, \emptyset, c)$ if and only if $|\text{Mod}_\emptyset(C)| = \text{Mod}(F) = c$. 

9
Algorithm 1: Simple Proof Trace Correctness Checking

\begin{algorithm}
\textbf{Input:} A model counting proof trace \(T = \{s_1, \ldots, s_n\}\)
\begin{algorithmic}[1]
\For{\text{\(i \in \{1, \ldots, n\}\)}}
\State \(P := \{s_1, \ldots, s_{i-1}\}\) \Comment{Set of predecessors.}
\EndFor
\If{\text{s}_i \text{ is an absence of models statement \((C, A, U, \Delta)\) then}}
\State \(S := \{(C, A_j, c) \in S \mid A_j \supseteq A, \text{vars}(A_j) = U\}\)
\State \(F_E := A \cup \{C[V] \mid C \in F\} \cup \{\neg A' \mid (C, A', c') \in S\}\)
\If{\(\Delta \text{ is refutation of } F_E\)} \textbf{continue}\EndIf
\EndIf
\If{\text{vars}(A) = V \text{ and } c = 1 \text{ and } \tau_A \text{ satisfies } F\)} \textbf{continue}\Comment{Lemma 2}\EndIf
\If{\text{s}_i \text{ is correct by joining } s_j, s_k \in P\)} \textbf{continue}\Comment{Lemma 1}\EndIf
\If{\text{P contains some } (C, A, U, \Delta)} \textbf{continue}\Comment{Lemma 3}\EndIf
\If{s}_i \text{ is correct by extending } s_j \in P\)
\State \(S := \{(C, A_j, c) \in S \mid A_j \supseteq A, \text{vars}(A_j) = U\}\)
\EndIf
\If{s}_i \text{ is correct by composition of } S\)
\Comment{Step verification failed.}
\EndIf
\State \textbf{return} \textit{Incorrect}
\EndAlgorithm
\end{algorithm}

3.4 Verifying Proof Traces

Having established model counting proof traces, we design an algorithm that can verify whether a trace is correct and complete. Further, we demonstrate that proofs can be verified efficiently, i.e., in polynomial time in the size of the proof and the input formula. By Theorem 1, we established that we can use traces to prove that a claimed model count of a given formula is correct. Therefore, we check that each step of a given trace can be inferred from its predecessors by using the Lemmas defined in Section 3.1.

Definition 7 (MICE). Let \(P = \{s_1, \ldots, s_n\}\) be a finite set of steps and \(s_r\) a step such that \(s_r \notin S\).

Step \(s_r\) is a model-counting induction by claim extension (MICE) step from \(P\), if

- \(s_r\) is an absence of models statement \((C, A, U, \Delta)\) that is correct for \(\{(C, A', c') \in P \mid A \subseteq A', \text{vars}(A') = U\}\) (Lemma 7); 
- \(s_r\) is a correct claim of exactly one model (Lemma 2); 
- \(s_r\) is a correct claim joining two claims \(I_1, I_2 \in P\) (Lemma 3); 
- \(s_r\) is a correct claim extending another claim \(I \in P\) (Lemma 3); or 
- \(s_r\) is a correct claim \((C, A, c)\) by composing a set \(S \subseteq P\) of claims and there is an absense of model statement \((C, A, U, \Delta)\) that is correct for \(S\), hence it is composable to \(A\) (Lemma 4).

Lemma 6 (+ Trace Correctness by MICE). Let \(T = \{s_1, \ldots, s_n\}\) be a model counting proof trace. If every \(s_i\) in \(T\) is a MICE step from \(\{s_1, \ldots, s_{i-1}\}\), \(T\) is correct.

A Simple Trace Checking Algorithm

Next, we discuss a simple polynomial-time algorithm for checking correctness of counting traces. We focus on correctness, because completeness can easily be checked by linearly searching a trace for a relevant claim.

Consider Algorithm 1 which takes as input a trace \(T\) and outputs either “Correct” or “Incorrect.” For each step \(s_i\) in \(T = \{s_1, \ldots, s_n\}\), we check if \(s_i\) is a MICE step from its predecessors \(P = \{s_1, \ldots, s_{i-1}\}\) by testing each case in Definition 7 sequentially. If no case applies to \(s_i\), we terminate with “Incorrect.” After processing all steps in \(T\) we output “Correct”. By Lemma 6 Algorithm 1 outputs “Correct”, if and only if the trace is correct.
Proposition 2 (⋆, Polynomial-Time Correctness Checking). Given a model counting proof trace $T$, Algorithm runs in polynomial time in the size of $T$ and $F$.

4 Practical Considerations and Preliminary Evaluation

In this section, we briefly explain practical improvements, solver integration, and provide preliminary data on using our certificates for model counting. We implemented Algorithm 1 into a program, which we call SharpCheck. For space reasons, we describe the trace format in Appendix C only, which adds only minor practical improvements such as identifiers for components allowing us to restrict the search for claims to a single component. If we assume that for one assumption, there exists only one claim in a component with that assumption, claims can be verified in parallel in any order. Additionally, we can ensure that a refutation for an assumption $A'$ has to be checked only once by allowing to reference an absence of models step in a more general assumption $A$, which follows directly from Observation 2.

4.1 Solver Integration

For practical use, we integrate MICE traces into existing techniques for exact model counting. Our approach works for generating proof traces in exact model counting when using a counting version of CDCL with component caching [49], dynamic programming [18], or knowledge compilation [15, 39]. For each technique, we provide a conceptual description and an implementation. We directly augmented two solvers with tracing capabilities and compile Decision-DNNFs outputted by knowledge compilers into MICE traces. Our implementations are proof-of-concepts relying on existing SAT solvers for generating clausal proofs.

sharpSAT The solver uses a counting version of CDCL with component caching for model counting [53]. It serves as basis for several model counters, e.g., Dsharp [41] or GANAK [50]. The implementation emits traces directly during the solving process. Refutations for absence of models are efficiently extracted during conflict analysis. Our implementation is available online at https://github.com/vroland/sharpSAT/tree/proof-trace.

DPDB The solver implements dynamic programming algorithms on tree decompositions using database management systems [24]. We extended the model counting implementation to translate result tables into claims and use minisat [20] to generate refutations. Our implementation is available online at https://github.com/vroland/dp_on_dbs/tree/sharpsat_proof.

D4 For prototype on knowledge compilation, we generate traces from the Decision-DNNF while using minisat for refutations. We consider D4 due to its performance in a past competition [22]. We believe that modifying D4 would result in faster trace outputs, but is out of scope for our prototypical considerations. However, we provide conceptual details for modifying a knowledge compiler in Appendix B.

Our implementation is available online at https://github.com/vroland/nnf2trace.

4.2 Empirical Evaluation

To demonstrate the capability of our approach and estimate the overhead of traces in practice, we conducted a preliminary evaluation on 400 instances of varying hardness and size.

Design of Experiment We draw a small empirical experiment to study the following questions: (Q1) Is there significant impact when solving with traces by compared solving without emitting traces? (Q2) Can we verify traces in a reasonable time? (Q3) Does the technique help to find bugs?

1https://github.com/vroland/sharptrace
Figure 2: Number of solved instances with and without tracing for sharpSAT, DPDB, and D4. Of the instances solved with tracing, we show the number of traces verified given the same timeout.

Table 1: Performance of solving and trace checking where # refers to number of instances, $t$ the total runtime, mem memory in GB, O without tracing, T with tracing, and V proof checking.

<table>
<thead>
<tr>
<th>Solver</th>
<th>#O</th>
<th>#T</th>
<th>#V</th>
</tr>
</thead>
<tbody>
<tr>
<td>sharpSAT</td>
<td>141</td>
<td>67</td>
<td>66</td>
</tr>
<tr>
<td>DPDB</td>
<td>137</td>
<td>78</td>
<td>65</td>
</tr>
<tr>
<td>D4</td>
<td>172</td>
<td>15</td>
<td>15</td>
</tr>
</tbody>
</table>

(a) Number of solved instances.

(b) Runtimes for verified instances and total proof size.

Instances We considered sets of instances from Tracks 1 and 2 of the Model Counting Competition 2020 [22]. For Track 2, we removed the weights. Instances are available on a public data repository [23].

Hardware, Measure, and Restrictions All solvers run on a server with two physical Intel Xeon Silver 4112CPUs, where each of these 16 runs at 2.60GHz and has access to 128GB RAM. Results are gathered on Ubuntu 18.04 LTS powered on kernel 4.15.0-135 with hyperthreading disabled. We allow a solving time of 600 seconds per instance. Since we do not implement traces directly into D4, we apply timeout to the combined runtime of D4 and the trace generation. We execute solvers sequentially, one at a time, limiting available memory to the maximum available on the system. Checking of the proof traces runs in parallel on 16 cores with a timeout of 600 seconds per instance.

Limitations All implementations are prototypical and are not optimized towards efficiency. Traces currently do not support pre-processing.

Performance of Solving with Trace Outputs
Towards answering the question on the impact of traces on solving time, we consider the number of solved instances with and without proof logging within the considered timeout. We survey the number of solved instances in Fig. 2 and provide more details in Table 1. sharpSAT and DPDB solved a similar number of instances when outputting traces. However, when comparing the number of solved instances with and without trace output, outputting traces reduced the number of instances to half. For D4, we could generate traces only for a small number of instances in the available time. In Table 1, we list the size of proofs. For sharpSAT, we can see that proofs grow quite large.
Discussion  To explain the results in more detail, we consider each solver individually. For sharpSAT, most overhead in generating trace outputs is introduced by I/O functions and serializing the output, which can be determined from profiling the solver. Currently, component definitions are only implicitly stored and have to be re-constructed. Refutations can directly be constructed within sharpSAT. For DPDB, fetching large intermediate results from the database and passing them to miniSAT to generate refutations causes significant overhead. For Decision-DNNFs, we currently need to construct a high number of refutations, which requires to call an external solver resulting in a notable overhead. We believe that this can be done directly inside a knowledge compiler. In comparison to DPDB, there is a much higher number of refutations, however, refutations are of much smaller size.

Summary  Overall, the results show that we are capable of generating traces with common solving techniques in practice. However, emitting proof traces results in significantly lower performance of the solvers. In fact, within the same timeouts, only half of the instances could be solved. Generating traces from Decision-DNNFs is not yet suitable as generating refutations for absence of models suffer notable overhead, which might be already outputted during solving. Still, our prototypes hopefully serve as a basis for future implementations.

Performance of Verifying Traces

Next, we consider the question of whether traces can be verified in a reasonable time. Therefore, we consider the number of verified instances in comparison to the number of solved instances with traces logging within the considered timeout. Figure 2 illustrates an overview of the number of verified traces by sharpCheck and Table 1 provides additional details. For sharpSAT, all but one trace could be verified within the timeout. The remaining trace takes \(\approx 750\) seconds to verify on our system. Similarly, a large portion of DPDB traces could be verified. The traces generated from Decision-DNNFs were all verified.

Discussion  When verifying traces from sharpSAT, most time is spent on parsing and validating the integrity of the trace, while claims and refutations are checked very quickly. In contrast, the traces emitted by DPDB are more compact and most time is spent on verifying refutations. In dynamic programming algorithms, large sets of claims are composed resulting in large refutations in absence of model steps. When traces have more than \(10^6\) refutation steps in total, checking often timed out. We expect improvement from adding deletion information to refutations, like in DRAT.

Summary  When comparing accumulated runtimes for solving and checking in Table 1b, we see that checking MICE traces is not only efficient in theory, but also in practice. Still, we expect that checking performance can be further improved when certified model counting matures.

Finding Bugs

During our experiments, we noticed a bug in sharpSAT, which we already outlined in Example 1. If preprocessing is disabled, unit clauses are violated and the solver outputs a wrong model count. The MICE trace allowed us to pinpoint the actual origin of the issue. Furthermore, we discovered that outputting traces may interfere with the two watched literals scheme used in sharpSAT. This resulted in wrong counts for some benchmark instances. We located and fixed the bug using MICE traces. Despite these bugs, which show up only under certain conditions, we did not discover major issues. This confirms common mathematical intuition that double counting already improves correctness. In addition, it supports observations made in the Model Counting Competition 2020 that state-of-the-art model counters are quite robust. However, the situation might well look different as soon as advanced preprocessing is included and instances are solved for which only one model counter gives a solution similar to SAT solving \([35, 32]\).
5 Conclusion and Future Work

Model counters are key tools for symbolic quantitative reasoning. Exact model counters need to be trustworthy, in particular, in fields such as explainability, risk analysis, or verification. While proof logging and verification approaches exist for SAT, a common proof system for exact model counting was missing. Previous approaches to correctness were either limited to a specific counting algorithm by establishing equivalence or could only be used to establish correctness of steps in approximate counting.

In this paper, we propose a novel approach to certified \#SAT based on traces that capture the solution space exploration during solving. We show that clausal proofs used for certifying unsatisfiability in SAT solvers are insufficient for \#SAT. Instead, we propose a system for certifying outputs from propositional model counters practice, where we use clausal proofs as basic building blocks. We demonstrate that our approach can be applied to solvers based on variants of CDCL with component caching, dynamic programming on tree decompositions, and knowledge compilation to Decision-DNNFs. To this end, we provide prototypes for each solving technique and a tool for automated trace checking. Finally, we illustrate preliminary results for certified model counting in actual solvers.

Our work opens up a wide variety of directions. A prime candidate for future investigations is an efficient integration into knowledge compilers and dynamic programming-based solving that uses more sophisticated data structures [17]. Further, establishing more general, but efficiently verifiable, inference rules may facilitate integration into solvers. Here, stronger proof techniques might come in handy [21, 28]. Beyond simple model counting, extending counting traces to weighted model counting or projected model counting, which is highly relevant in practical applications, seems to be a natural step for future considerations. Finally, although our implementation to verify traces is conceptually simple, an efficient, formally verified implementation might be interesting for highly sensitive applications.

References


A Omitted Proofs

(Re-)combining Claims of Search Spaces

Observation [1]. Let \( C = (F, V) \) be a component, \( U \subseteq V \) a set of variables, and \( S \) a set of claims over \( C \). If \( S \) is uniform for \( U \) and all claims in \( S \) are correct, \( S \) is non-overlapping.

Proof. Recall that since \( S \) is uniform over \( U \), we have \( \text{vars}(A) = U \) for all claims \((C, A, c)\) in \( S \). Then, for every assignment \( \alpha \) that is total over \( V \), there is exactly one assumption \( A \) with \( \text{vars}(A) = U \) and \( \alpha \) satisfies \( A \). Assume all claims in \( S \) are correct. Since there is one correct count for a component and assumption, no two claims in \( S \) have the same assumption. Hence, for every total assignment \( \alpha \) to \( V \), there is exactly one matching assumption over variables \( U \), thus there is at most one claim \((C, A, c)\) in \( S \) with \( \alpha \in \text{Mod}_A(C) \).

Proposition [1]. (Exhaustiveness of Claims is co-NP Hard). Let \( S \) be a set of claims for a component \( C = (F, V) \) that is uniform for \( U \subseteq V \) and \( A \) be an assumption. Then, it is co-NP complete to decide if \( S \) is exhaustive for \( A \).

Proof. To show hardness, we provide a reduction from UNSAT, which asks to decide whether a given formula is unsatisfiable. Let \( C = (F, \text{vars}(F)) \) be the a component. It is easy to see that a function producing \( C \) from \( F \) is computable in polynomial time. Formula \( F \) is satisfiable if and only if there is a model \( \alpha \in \text{Mod}_C(C) \). In consequence, \( F \) is unsatisfiable if and only if the set \( S = \emptyset \) is exhaustive for \( \emptyset \). Since \( S = \emptyset \), there is no matching claim for \( \alpha \) in \( S \). Consequently, \( S \) is not exhaustive.

For membership in co-NP, let \( f \) be a function that outputs the formula \( F_E := A \cup \{C|_V \mid C \in E \} \cup \{\neg A’ \mid (C, A’, c) \in S\} \) from a set \( S \) of claims for a component \( C = (F, V) \) and \( A \). By Definition [5], function \( f \) is a reduction to UNSAT.

Next, we show that \( f \) is computable in polynomial time in the size of \( S, C, \) and \( A \). It is easy to see that the inputs to \( f \) can be encoded as strings of symbols and comparison and inversion of literals can be performed in polynomial time. We first output \( A \) from \( A \). Then, we output \( \{C|_V \mid C \in F\} \), where we need \( O(|C| \cdot |V|) \) steps per clause \( C \in F \). Finally, we output \( \neg A’ \) for every \( (C, A’, c) \in S \), taking polynomial time in the size of \( S \). Hence, function \( f \) is computable in polynomial time.

Since UNSAT can be reduced to exhaustiveness of a uniform set of claims and vice versa in polynomial time, exhaustiveness is co-NP complete.

Lemma [1] (Absence of Models). Let \( A = (C, A, U, \Delta) \) be an absence of models statement and \( S \) be a set of claims for component \( C \) that is uniform over \( U \) and we have \( A \subseteq A’ \) for every claim \((C, A’, c) \in S \). If \( A \) is correct for \( S \), then \( S \) is exhaustive for \( A \).

Proof. Let \( C = (F, V) \). Recall that \( A \) is correct if \( \Delta \) is a refutation of \( E = A \cup \{C|_V \mid C \in F\} \cup \{\neg A’ \mid (C, A’, c) \in S\} \). For proof of contradiction, assume \( S \) is not exhaustive for \( A \). Then, there is a model \( \alpha \in \text{Mod}_A(C) \), but there is no claim with assumption \( A \) in \( S \), such that \( \tau_{\alpha} = \alpha|_U \). Since \( \alpha \in \text{Mod}_A(C) \), assignment \( \alpha \) satisfies \( A \cup \{C|_V \mid C \in F\} \). Hence, assignment \( \alpha \) must falsify \( \{\neg A’ \mid (C, A’, c) \in S\} \), or else \( E \) is satisfiable and \( \Delta \) is not a refutation. However, since no claim with \( A \) is in \( S \), \( \alpha \) satisfies \( \neg A’ \) for all assumptions \( A’ \) of claims in \( S \) by construction. Thus, \( \alpha \) satisfies \( E \), which contradicts that \( \Delta \) is a refutation. Hence, if \( \Delta \) is a refutation for \( E \) and hence \( A \) is correct, \( S \) is exhaustive for \( A \).

Inference Rules and Proof Traces for Model Counting

Lemma [2] (Exactly One Model). Let \( C = (F, V) \) be a component and \((C, A, 1)\) be a claim for \( C \) with \( \text{vars}(A) = V \). The claim is correct if and only if \( \tau_A \) satisfies \( F \).

Proof. Consider the assignment \( \alpha = \tau_A \). Since \( \text{vars}(A) = V \), assignment \( \alpha \) is total over \( V \) and \( \text{Mod}_A(C) \subseteq \{\alpha\} \). Consequently, \( \tau_A \) satisfies \( F \) if and only if we have \( \text{Mod}_A(C) = \{\alpha\} \) and \(|\text{Mod}_A(C)| = 1\).
Lemma 3 (Composition or Unsatifiability). Let \( C = (F, V) \) be a component, \( I = (C, A, c) \) be a claim with assumption \( A \), \( S \) be a set of claims over \( C \) that is composable to \( A \), and \( c := \sum_{(C, A', c')} e' \). If every claim in \( S \) is correct, then \( I \) is correct, i.e., \(|\text{Mod}_A(C)| = c\).

Proof. Since \( S \) is composable to \( A \), it is exhaustive for \( A \) and non-overlapping. Further, by exhaustiveness for \( A \), for all models \( \alpha \) in \( \text{Mod}_A(C) \), there is a claim \((C, A', c)\) in \( S \) with \( \alpha \in \text{Mod}_A(C) \). Hence, \( \text{Mod}_A(C) = \bigcup_{(C, A', c') \in S} \text{Mod}_A(C) \). Since \( S \) is also non-overlapping, the assignments in \( \text{Mod}_A(C) \) are mutually disjoint. Thus, if every claim in \( S \) is correct, we have \(|\text{Mod}_A(C)| = \sum_{(C, A', c') \in S} e' = c\).

Lemma 4 (Join). Let \( C = (F, V) \) be a component; \( I = (C, A, c) \) be a claim; \( C_1 = (F_1, V_1) \) and \( C_2 = (F_2, V_2) \) be sub-components of \( C \) with \( F = F_1 \cup F_2 \), \( V = V_1 \cup V_2 \), and \( V_1 \cap V_2 \subseteq \text{vars}(A) \) where every \( C \in F_1 \) has \( \text{vars}(C) \cap (V \setminus V_1) = \emptyset \). If \( I_1 = (C_1, A|_{V_1}, c_1) \) and \( I_2 = (C_2, A|_{V_2}, c_2) \) are correct claims over \( C_1 \) and \( C_2 \) and \( c = c_1 \cdot c_2 \), then \( I \) is correct, i.e., \(|\text{Mod}_A(C)| = c\).

Proof. We distinguish two cases.

Case (I), \( c_1 = 0 \) or \( c_2 = 0 \). Then, \( \text{Mod}_{A|_{V_1}}(C_1) = \emptyset \) or \( \text{Mod}_{A|_{V_1}}(C_2) = \emptyset \), since the sub-component claims are correct. Assume \( c_1 = 0 \). Then, for every possible assignment over \( V_1 \), a clause \( C \in F_1 \) exists that is not satisfied. But, because \( \text{vars}(C) \cap (V \setminus V_1) = \emptyset \), \( C \) cannot be satisfied by an assignment over \( V \setminus V_1 \). Thus, \( C \) cannot be satisfied by any assignment over \( V_2 \). Hence, \(|\text{Mod}_A(C)| = 0 = c\). If \( c_2 = 0 \), the proof works similar.

Case (II), \( c_1 \neq 0 \) and \( c_2 \neq 0 \). We will show that \( f : \alpha \mapsto (\alpha|_{V_1}, \alpha|_{V_2}) \) is a bijection from \( \text{Mod}_A(C) \) into the set \( J = \text{Mod}_{A|_{V_1}}(C_1) \times \text{Mod}_{A|_{V_2}}(C_2) \), similarly as in related work [3]. Since \(|\mathcal{J}| = |\text{Mod}_{A|_{V_1}}(C_1)| \cdot |\text{Mod}_{A|_{V_2}}(C_2)| = c_1 \cdot c_2 \), we then conclude that \(|\text{Mod}_A(C)| = c_1 \cdot c_2 \).

First, we establish that \( f \) is injection from \( \text{Mod}_A(C) \) to \( J \). For any \( \alpha \in \text{Mod}_A(C) \), \( \alpha|_{V_1} \) is in \( \text{Mod}_{A|_{V_1}}(C_1) \) because clauses \( F_1 \) cannot contain literals of variables that are in \( V \setminus V_1 \). Hence, clauses in \( F_1 \) must be satisfied by literal assignments in \( \alpha \) that are also in \( \alpha|_{V_1} \). By the same reasoning, we have \( \alpha|_{V_2} \in \text{Mod}_{A|_{V_1}}(C_2) \).

Further, \( f \) is injective, because for every \( \alpha_1, \alpha_2 \in \text{Mod}_A(C) \) with \( f(\alpha_1) = f(\alpha_2) \), we have that \( \alpha_1|_{V_1} = \alpha_2|_{V_1} \) and \( \alpha_1|_{V_2} = \alpha_2|_{V_2} \). Then, since \( V_1 \cup V_2 = V \), \( \alpha_1 = \alpha_2 \). To see that \( f \) is surjective, let \( \alpha_1, \alpha_2 \in J \). Then, the assignment \( \alpha \in \alpha_1 \alpha_2 \) is consistent, since \( V_1 \cap V_2 \subseteq \text{vars}(A) \) and \( A \) is consistent by definition. Since \( \alpha_1 \) satisfies every clause \( F_1 \alpha_2 \) satisfies every clause \( F_2 \), and \( \alpha \) is consistent, \( \alpha \) satisfies every clause in \( F_1 \cup F_2 = F \). In consequence, \( \alpha \in \text{Mod}_A(C) \).

Finally, because the sub-component claims are correct, we have \(|J| = c_1 \cdot c_2 = c \). This concludes the proof.

Lemma 5 (Extension). Let \( C = (F, V) \) be a component, \( I = (C, A, c) \) be a claim, \( C' = (F', V') \) be a sub-component of \( C \), and \( I' = (C', A|_{V'}, c) \) be a correct claim. If \( V \setminus V' \subseteq \text{vars}(A) \), \( \tau_A \) satisfies \( F \setminus F' \), and \( \tau_A|_{V \setminus V'} \) does not satisfy \( C \) for all clauses \( C \in F' \), then \(|\text{Mod}_A(C)| = c\).

Proof. We show that \(|\text{Mod}_{A|_{V'}}(C')| = |\text{Mod}_A(C)| = c\) by proving that the function \( \alpha \mapsto \alpha|_{V'} \) is a bijective mapping from model \( \text{Mod}_A(C) \) of claim \( I \) to models \( \text{Mod}_{A|_{V'}}(C') \) of \( I' \).

To see that \( f \) is a valid mapping, we show that if \( \alpha \) is a model of \( C \), \( \alpha|_{V'} \) also satisfies \( F' \). For \( \alpha|_{V'} \) to not satisfy \( F' \), there must be a clause \( C \in F' \) that is satisfied by a literal \( l \) of a variable in \( V \setminus V' \). Since we require that \( V \setminus V' \subseteq \text{vars}(A) \), \( l \) is in \( A|_{V \setminus V'} \). However, then \( \tau_A|_{V \setminus V'} \) satisfies \( C \), which is not allowed. Thus, we have \( \alpha|_{V'} \in \text{Mod}_{A|_{V'}}(C') \) for all \( \alpha \in \text{Mod}_A(C) \) and \( f \) is a valid mapping.

To show that \( f \) is injective, consider \( \alpha_i, \alpha_j \in \text{Mod}_A(C) \), where \( f(\alpha_i) = f(\alpha_j) = \alpha_i|_{V'} = \alpha_j|_{V'} \).

Then, since there is exactly one assignment \( \gamma = \tau_{A|_{V \setminus V'}} \) over variables \( V \setminus V' \) where \( \gamma \) satisfies \( A \), we have \( \alpha_i = \alpha_i|_{V'} \cup \gamma = \alpha_j|_{V'} \cup \gamma = \alpha_j \).

For surjectivity, let \( \beta \in \text{Mod}_{A|_{V'}}(C') \) be a model of claim \( I' \). Because \( V \setminus V' \subseteq \text{vars}(A) \), we can construct an assignment \( \alpha \) as \( \beta \cup \tau_A \). Then, \( \beta \) satisfies the clauses \( F' \), \( \tau_A \) satisfies the clauses \( F \setminus F' \), thus \( \alpha \) is an assignment over the variables \( V \) that satisfies the clauses \( F \). Hence, we have \( \alpha \in \text{Mod}_A(C) \).
Since $f$ is bijective and $T' = (C', A_{\mathcal{V}'}, c)$ is correct, we have that $|\text{Mod}_{A_{\mathcal{V}'}, (C')}| = |\text{Mod}_A(C)| = c$. This concludes the proof.

\textbf{Theorem 1} (Soundness and Completeness). Let $F$ be a formula and $C = (F, \text{vars}(F))$ a component. There is a correct model counting proof trace $T$ containing a claim $(C, \emptyset, c)$ if and only if $|\text{Mod}_C(C)| = \text{Mod}(F) = c$.

\textbf{Proof.} “$\Rightarrow$”: If $T$ is correct and contains claim $T = (C, \emptyset, c)$, then $T$ is correct. Hence, $|\text{Mod}_C(C)| = \text{Mod}(F) = c$.

“$\Leftarrow$”: Given the set $N := \text{Mod}_C(C)$ of models for formula $F$. By enumeration, it is easy to construct a, though impractically large, correct and complete counting trace as follows: First, we construct a set of claims $S := \{[C, A_{\alpha}, 1] \mid \alpha \in N\}$ where $\tau_{A_{\alpha}} = \alpha$. Each claim in $S$ is verifiable using Lemma 2. Since $N$ contains all models of $C$, there is no assignment that satisfies $E = \{C|_{\mathcal{V}} \mid C \in F\} \cup \{\neg A' \mid (C, A', c) \in S\}$. Hence, there is a refutation $\Delta$ for $E$ and a correct absence of models step $A = (C, A, V, \Delta)$. Such a refutation $\Delta$ must exist for every unsatisfiable formula $E$.

Finally, we construct a trace $T = (T_1, \ldots, T_n, A, \emptyset)$ with $T_1, \ldots, T_n \in S$. It is easy to see that $T$ is complete, because it contains $I$. Since $T_1, \ldots, T_n$ are correct by Lemma 2 and $T$ is correct by Lemma 3, $T$ is correct. This concludes the statement.

\textbf{Verifying Proof Traces}

\textbf{Lemma 6} (Trace Correctness by MICE). Let $T = (s_1, \ldots, s_n)$ be a model counting proof trace. If every $s_i$ in $T$ is a MICE step from $\{s_1, \ldots, s_{i-1}\}$, $T$ is correct.

\textbf{Proof.} We prove correctness of a trace $T$ by induction over its steps. The empty trace $\langle \rangle$ is correct by Definition 6.

Next, assume the sub-trace $T_{i-1} = (s_1, \ldots, s_{i-1})$ of $T$ is correct and $s_i$ is a MICE step from $P = \{s_1, \ldots, s_{i-1}\}$. If $s_i$ is a claim that can be inferred by Lemmas 2, 4, or 5 it is correct and thus $T_i = (s_1, \ldots, s_{i-1}, s_i)$ is correct.

If $s_i$ is an absence of models statement $(C, A, U, \Delta)$ that is a MICE step from $P$, we know that there is a subset $S = \{[C, A', c'] \in P \mid A \subseteq A', \text{vars}(A') = U\}$ of $P$ that is composable to $A$ by Lemma 1. Then, trace $T_i$ is correct because trace $T_{i-1}$ is correct and $s_i$ is not a claim.

Finally, consider the case where $s_i$ is a claim $(C, A, c)$ and there is an absence of models step $A = (C, A, U, \Delta)$ in $P$. Since $A$ is a MICE step from its predecessors, $A$ is correct for some set $S \subseteq P$. Then, $s_i$ can be inferred as composition of $S$ by Lemma 3. Thus, trace $T_i = (s_1, \ldots, s_{i-1}, s_i)$ is correct. This concludes the proof.

\textbf{Proposition 2} (Polynomial-Time Correctness Checking). Given a model counting proof trace $T$, Algorithm 1 runs in polynomial time in the size of $T$ and $F$.

\textbf{Proof.} In the following, we assume a naive sequential encoding of $F$ and the steps of $T$, separated by separator symbols. We process each step $s_i$ in $T$ sequentially. Consequently, if processing each $s_i$ takes polynomial time, Algorithm 1 runs in polynomial time.

For a step $s_i$, let the trace prefix $P$ be the set of steps preceding $s_i$ in $T$. If $s_i$ is an absence of models statement $(C, A, U, \Delta)$, we first construct a set of claims $S$ from $P$. To construct $S$, we search through $P$ and collect all claims for the current component with assumptions over $U$ that are a subset of $A$. Then, we construct the formula $F_E := A \cup \{C|_{\mathcal{V}} \mid C \in F\} \cup \{\neg A' \mid (C, A', c) \in S\}$. Both steps take polynomial time in the size of $P$ and the size of $F_E$ is bounded by the size of $P$. Hence, checking if $\Delta$ is a refutation of $F_E$ takes at polynomial time in the size of $P \subseteq T$.

In the case that $s_i$ is a claim, we check the prerequisites for Lemmas 2, 4, and 5 sequentially. It is easy to see that comparing sets of variables and clauses, calculating $\text{vars}(\cdot)$ for clauses and assumptions, checking subset relations and whether clauses are satisfied by an assignment are polynomial-time operations. Hence, checking Lemma 3 is easy. To check for correctness by Lemma 5 (extension), we need to search $P$ for a claim to infer $s_i$ from. We can check Lemma 5 in polynomial
time, and this search takes at most \(|P| - 1\) checks. Analogously, searching for two claims to join by Lemma 3 takes \((|P| - 1)^2\) polynomial-time checks of Lemma 4.

Finally, we check for correctness by Lemma 3. We can find an absence of models step \((C, A, U, \Delta)\) in \(|P| - 1\) steps. Finding one such step is sufficient, because using another absence of models statement must lead to the same inference, since all absence of models steps in \(P\) are correct. Constructing a set of claims \(S \subseteq P\) as before and checking the claim count by Lemma 3 takes polynomial time.

Overall, checking if \(s_i\) is a MICE step from \(P\) case-by-case takes polynomial time in the size of \(P\). Hence, Algorithm 1 checks correctness of a trace \(T\) in polynomial time.

\section{B Solver Integration}

For the sequel of this section, we assume that the reader is familiar with basic notions in graph theory and propositional logic such as entailment, otherwise we refer to standard texts \cite{10, 59}. Knowledge compilation converts a formula into a normal form on which queries are tractable that are hard on CNFs \cite{15}. Specifically, we are interested in propositional formulas in Decision-DNNF (or FBDD). Such a formula has following properties:

- **NNF**: A propositional formula is in negation normal form, NNF for short, if negations (\(\neg\)) occur only directly in front of variables and the only other operators are conjunction (\(\land\)) and disjunction (\(\lor\)) \cite{44}.

- **Decomposability**: For any two distinct subformulas \(C_i, C_j\) in a conjunction \(C = C_1 \land \cdots \land C_n\) with \(i \neq j\), we have \(\text{vars}(C_i) \cap \text{vars}(C_j) = \emptyset\), where \(\text{vars}(C_i)\) denotes the set of variables that occur in subformula \(C_i\) \cite{15}.

- **Decision**: Disjunctions are of the form \(C = (x \land C_1) \lor (\neg x \land C_2)\), where \(x\) is called decision variable. Decision is a stronger version of determinism, which requires that subformulas in the disjunction must be logically contradictory \cite{15}. Note that \(x\) does not occur in \(C_1\) and \(C_2\) because of decomposability. \(C_1\) and \(C_2\) may be conjunctions.

- **Smoothness**: A disjunction \(C = C_1 \lor C_2\) is smooth, if \(\text{vars}(C_1) = \text{vars}(C_2)\). A formula in \(C\) in Decision-DNNF is smooth if all conjunctions in \(C\) are smooth. Smoothness is required for model counting \cite{12}.

**Graph Representation** An NNF formula can be represented as a directed acyclic graph, DAG for short, where nodes represent \(\land\), \(\lor\), \(\top\) (verum) or \(\bot\) (falsum) \cite{12, 59}. \(\land\)-nodes are labeled with a set of literals \(\text{lits}(N) = \{l_1, \ldots, l_n\}\). For a \(\lor\)-node \(N = N_1 \lor N_2\), there is a decision variable \(x\), such that \(x \in \text{lits}(N_1)\) and \(\neg x \in \text{lits}(N_2)\). We let the decision literals of \(N_1\) be \(\text{declit}(N_1) := x\) and of \(N_2\) be \(\text{declit}(N_2) := \neg x\) \cite{11}. For node \(N\), we let \(\text{vars}(N)\) be the set of variables that occur in the subgraph rooted at \(N\).

Naturally, a conjunction \(C = C_1 \land \cdots \land C_n\) in an NNF formula corresponds to an \(\land\)-node \(N\) in the DAG representation. Node \(N\) is labeled with literals occurring in \(C\) and has a child node corresponding to each non-literal conjunct. If \(C\) is a conjunction of only literals then \(N\) has a \(\top\)-node as its child. A disjunction \(C = C_1 \lor C_2\) corresponds to a \(\lor\)-node \(N = N_1 \lor N_2\) with child nodes \(N_1\) and \(N_2\) for subformulas \(C_1\) and \(C_2\). An unsatisfiable disjunct corresponds to an \(\land\)-node labeled with the decision literal and that has child node \(\bot\). Fig. 3 gives an example of a formula in DAG representation. In the following, we use DAG and Decision-DNNF interchangeably.

**From Decision-DNNFs to MICE Traces** Assume that the formula is in smooth Decision-DNNF, which can be obtained in polynomial time from a Decision-DNNF \cite{15, Lem A.2}. Then, counting works by evaluating each node in post-order: A \(\top\)-node has a model count of 1 and a \(\bot\)-node a count of 0. The model count of an \(\land\)-node is the product of the counts of its children. The model count of an \(\lor\)-node is the sum of the counts of its children. The count of the root node...
is the model count of the input formula. To generate a counting trace, we construct a mapping of Decision-DNNF nodes to components and generate claims such that their correctness can be inferred from claims of child nodes.

**Annotating and Smoothing the Decision-DNNF**

Intuitively, nodes in a Decision-DNNF represent components. While the Decision-DNNF is logically equivalent to the input formula $F$ in CNF, nodes of the Decision-DNNF and clauses in the CNF might not directly be in relation.

**Clause Annotation** In DNNFs we can check in polynomial time whether a clause $C$ is entailed. To annotate the NNF, we start at the root node and proceed in preorder. We maintain a set $\mathcal{R}$ of clauses, initially set to $F$, indicating which clauses are remaining. If the current node $N$ is an $\lor$-node, we compute the set of entailed clauses $E = \{C | C \in \mathcal{R}, N \text{ entails } C\}$ and annotate node $N$ by $E$. Then, we apply the annotation to the child nodes of $N$ with $\mathcal{R} := E$. For $\land$-nodes, we construct $\mathcal{R}' = \mathcal{R} \setminus \{C \mid C \in \mathcal{R}, \hat{L} \text{ satisfies } C\}$ by removing the clauses that are solved by $L := \text{lits}(N)$ from the set $\mathcal{R}$. Then, we annotate $N$ by $\mathcal{R}'$ and proceed to the children of $N$ with set $\mathcal{R}'$. If we encounter a child node that was already annotated by a different set of clauses, we copy the sub-tree under the child node and proceed with annotating the copy. For $\top$ and $\bot$ nodes, we set $\emptyset$. We denote the annotated clauses for a node $N$ as $\text{clauses}(N)$.

Note that this annotation is not equal to the set of entailed clauses of a node. For instance, if an $\lor$-node $N = N_1 \lor N_2$ entails some clauses $E_N$, the child nodes $N_1$ and $N_2$ also entail $E_N$. But $\land$-nodes $N_1$ and $N_2$ may be annotated differently because clauses solved by their literals are removed from their annotations.

In the following, we require non-interfering annotations for $\land$-nodes. We say that annotations of two child nodes interfere if one child is annotated by a clause that shares variables with its sibling. The knowledge compilers D4 [39], DSharp [41], and C2D [13] produce such formulas by construction, as they decompose the input clauses into disjoint components under some partial assignment (dynamic decomposition). Hence, we assume non-interfering annotations for $\land$-nodes in the following. Note that we can easily construct non-interfering annotations as follows: If either subformula is unsatisfiable, which is easy to check in Decision-DNNF, it can be replaced by a $\bot$-node, which is not annotated. Otherwise, we apply a transformation from interfering $\land$-nodes.

**Smoothing** After annotation, we transform the Decision-DNNF into a smooth Decision-DNNF. Therefore, we add additional nodes, such that the children of $\lor$-nodes cover the same variables. While an $\land$-node $N$ with variable $v \not\in \text{vars}(N)$ can be transformed such that $v$ is contained by replacing $N$ with node $((v \land N) \lor (\neg v \land N))$ [13], we need a slightly more sophisticated procedure to avoid “invalidating” the annotation. To this end, we preserve the annotation and use one decision node per input variable as done in Algorithm 2. The algorithm takes a node $N$ and a set $M$ of missing variables as input. If $N$ is an $\lor$-node, we run smoothing for both decision branch nodes where set $M$ consists of variables that occur exclusively in the sibling node. If $N$ is an $\land$-node, we map children $N'$ of $N$ to sets of variables, where $P(N')$ is the subset of missing variables that
After annotation and smoothing, we have a smooth Decision-DNNF where each node is annotated with a set of clauses. Intuitively, we translate an $\text{vars}(\text{claim } I \lor \text{and an } P)$ generating counting traces that claim generated from child nodes of an $\land$-node as a child of $I$. We label $\text{vars}(\text{children}(N))$ are disjoint since they have non-interfering annotations. If no child is a $\bot$-node as child, we output claim $I$.

 Generating Counting Traces

After annotation and smoothing, we have a smooth Decision-DNNF where each node is annotated with a set of clauses. Intuitively, we translate an $\land$-node to a composition. To generate a counting trace, we apply the following procedure recursively starting with the root node, which we assume to be a decision.

**Tracing $\land$-nodes** If $N$ is an $\land$-node with parent node $N_P$, we let $C_P := (\text{clauses}(N_P), \text{vars}(N_P))$ be the parent component. We consider three cases. In each case, we construct a claim $I_P$ for the parent component $C_P$ with assumption $\text{vars}(N)$.

1. If $N$ has a $\top$-node as child, we output claim $I_P = (C_P, \text{vars}(N), 1)$.
2. If $N$ has a $\bot$-node as a child, we output a claim $I = (C_\bot, 0, 0)$ for component $C_\bot = (F, \emptyset)$. The absence of models statement for $I$ is trivial since a component with $F \neq \emptyset$ and $V = \emptyset$ is always unsatisfiable. Additionally, we output claim $I_P = (C_P, \text{vars}(N), 0)$.
3. Otherwise, $N$ has non-trivial child nodes. We define a join component $C_J = (\text{clauses}(N), \text{vars}(N) \setminus \text{vars}(\text{vars}(N)))$. If the parent node is not a decision node, we may have $\text{vars}(N) = \emptyset$ and $C_J = C_P$. First, we output traces for the children of $N$. From that, we obtain claim $I' = (\text{vars}(N_P), c')$ for each child $N'$ of $N$ with child component $C_{N'} = (\text{clauses}(N'), \text{vars}(N'))$. Because the Decision-DNNF is deterministic and we have non-interfering annotations, variables and clauses of child components for distinct children are mutually disjoint. As a result of our smoothing algorithm, no variable in $\text{vars}$(clauses$(N')$) occurs in another child of $N$. Hence, we can infer a joint claim $I_J = (C_J, 0, c)$ by Lemma[3] (Join), where $c$ is the product of the counts in claims $I'$. If $\text{vars}(N) \neq \emptyset$, we emit a claim $I_P = (C_P, \text{vars}(N), c)$ that extends $I_J$ to the parent component $C_P$ by Lemma[5].
Finally, if $|\text{lits}(N)| > 1$, we additionally output claim $(C_P, \{\text{declit}(N)\}, c)$, which can be inferred from $I_P$ by Lemma 3 (Composition). In our prototype, we use miniSAT to generate a refutation for an absence of models statement supporting this composition. A derivation of $\text{lits}(N)$ from $\text{declit}(N)$ might be obtained directly in a knowledge compiler.

Transforming a Decision-DNNF with Interfering Annotations

Current knowledge compilers do not generate Decision-DNNFs where clause annotations of children of $\land$-nodes share variables. Nevertheless, we can transform a Decision-DNNF with interfering annotations into one without interfering annotations.

Lemma 7. Let $N$ be an $\land$-node with two satisfiable children $N_1$ and $N_2$, where $N_1$ entails a clause $C$ that shares variables $I = \text{vars}(C) \cap \text{vars}(N_2)$ with $N_2$. Then, we can transform $N$ into a node Decision-DNNF $N'$ with non-interfering annotations. The sub-tree rooted at $N'$ has at most $|I| \cdot (2n + 1)$ nodes, where $n$ is the number of nodes in the subtree rooted at $N$.

Proof. Consider a literal $\ell \in C \cap I$. We construct decision node $N' = N_1 \lor N_{\neg \ell}$ from $N$, where $N_1 = N_1[\ell] \land N_2$ and $N_{\neg \ell} = N_1[\neg \ell] \land N_2$ with $\text{lits}(N_1) = \{\ell\}$ and $\text{lits}(N_{\neg \ell}) = \{\neg \ell\}$. For a node $N_1$, $N_1[\ell]$ denotes conditioning of $N_1$ with literal $\ell$. If either $N_1[\ell]$ or $N_1[\neg \ell]$ become $\top$ by conditioning, we omit $\top$ and have $N_1 = N_2$ or $N_{\neg \ell} = N_2$, respectively. We omit $N_2$ if either $N_1[\ell]$ or $N_1[\neg \ell]$ becomes $\bot$ by conditioning.

As a result, $N_1[\ell]$ is not annotated with the interfering clause $C$, because $C$ is solved by $\ell$ in $\text{lits}(N_1)$. Further, $N_1[\neg \ell]$ does not contain the interfering variable by construction. If the annotations of $N_1[\neg \ell]$ and $N_2$ still share variables, we replace $N_1[\neg \ell]$ by $N_1[\neg \ell]'$ obtained in the same way as $N'$. We proceed until no interfering variable is left, where the node conditioned with $\neg \ell$ becomes $\bot$.

It is easy to see that the result is still a Decision-DNNF. For the number of added nodes, consider that we replace a node $N_1$ by $N_1[\ell]$ and $N_1[\neg \ell]$ each with at most as many nodes as the subtree rooted at $N$. Hence, the replacement for each $\ell \in C \cap I$ has at most $2n$ additional nodes, plus one decision node. For each replacement, only one decision branch must be processed further. Thus, $N'$ is replaced by at most $|I| \cdot (2n + 1)$ nodes.

Example 12. Consider formula $F = \{\{e, a, b\}, \{a, b, c, d\}, \{c, d, f\}, \{\neg f\}\}$, which has an equivalent Decision-DNNF with interfering annotations (left). The first child of the $\land$-node labeled with $\{e\}$ is annotated with the first and second clause. The second child is annotated with the second and third clause. Hence, the second clause is an interfering clause. By applying the transformation in Lemma 7 with literal $a$ of the second clause, we obtain a Decision-DNNF (right) that does not lead to interfering annotations.
C A Plain-Text Trace Format

In this section, we describe a trace format that can be efficiently generated by solvers and is simple to implement and process. The format is line-based, where each line starts with a string indicating its type and ends with 0, followed by the newline character. Within lines, only spaces are allowed as whitespace. Lines starting with c are ignored.

PREFIX space separated items 0
c ignored line

As in the DIMACS format for formulas in CNF, variables are written as positive integers n and literals as signed integers -n or n. A trace starts with a header line, indicating the number of variables Nv and clauses Nc of the original DIMACS CNF instance.

p st Nv Nc 0

Clause Definitions To refer to clauses of the input formula in the trace, each clause is given a unique index. This positive integer index must be unique to the clause. A clause definition is written as a line starting with f, followed by the index idx and the literals l1,...,ln of the clause.

f idx l1 l2 ... ln 0

Component Definitions Let C = (F,V) be a component. As with clauses, we assign each component a unique positive integer identifier id. This allows referring to the component from other statements in the trace. A component definition line starts with tag d, followed by the component identifier id, the component’s variables V = v1,...,vn, separator symbol 0, and the clause indices c1,...,cn of clauses F.

d id v1 v2 ... vn 0 c1 c2 ... cn 0

Model Claims An (exactly one) model claim (C, A, 1) for component C = (F,V) is a claim that can be inferred by Lemma 2. It is written as a line starting with tag m, followed by the component identifier id for C and the literals l1,...,ln of its assumption A.

m id 1 l1 l2 ... ln 0
Join Claims  A join claim $\mathcal{I} = (C, A, c)$ for component $C = (F, V)$ is a claim that can be inferred by Lemma 4 or a generalization to $n$-ary joins. First, we define the subcomponents of $C$. Note that these definitions apply to all join claims of $C$, not just a $\mathcal{I}$. A join subcomponent definition starts with tag $jc$ followed by the identifier $child$ of a subcomponent of $C$ and the identifier $id$ of $C$.

\begin{verbatim}
jc child1 id 0
jc child2 id 0
jc child3 id 0
\end{verbatim}

Then, we write a claim line starting with tag $j$, followed by component identifier $id$, the claimed count $c$, and the literals $l_1, \ldots, l_n$ of the assumption $A$.

\begin{verbatim}
j id c l_1 l_2 \ldots l_n 0
\end{verbatim}

If one claim $\mathcal{I}$ of a joined subcomponent has model count 0, the claims for sibling components of $\mathcal{I}$ may be omitted.

Extension Claims  An extension claim $\mathcal{I} = (C, A, c)$ for component $C = (F, V)$ is a claim that can be inferred by Lemma 5 from a claim of subcomponent $C'$ of $C$. The claim is written as a line starting with tag $e$, followed by the identifier $id$ of $C$, the identifier $subid$ of $C'$, the claimed count $c$, and the literals $l_1, \ldots, l_n$ of the assumption $A$.

\begin{verbatim}
e id subid c l_1 l_2 \ldots l_n 0
\end{verbatim}

Absence of Models Statements  An absence of models statement $\mathcal{A} = (C, A, U, \Delta)$ for a component $C$ is written as multiple lines. First, the refutation $\Delta$ is declared with a unique positive integer identifier $aid$. The declaration line starts with tag $xp$ which is followed by the refutation identifier $aid$.

\begin{verbatim}
xp aid 0
\end{verbatim}

Then, the steps $\delta$ of $\Delta$ are given with one step $\delta = \{l_1, l_2, \ldots, l_n\}$ per line. Each line starts with tag $xs$, followed by the refutation identifier $aid$ and the literals $l_1, \ldots, l_n$ of the refutation step $\delta$.

\begin{verbatim}
xs idx l_1 l_2 \ldots l_n 0
\end{verbatim}

Finally, we write a statement line finishes. All claims in the set $\mathcal{A}$ states composability for must occur before the statement line. There may be multiple statement lines referencing the same refutation. The statement line starts with tag $xf$, followed by identifier $id$ of $C$, refutation identifier $aid$ of $\Delta$, the variables $v_1, \ldots, v_n U$, the separator symbol $0$, and the literals $l_1, \ldots, l_n$ of the assumption $A$.

\begin{verbatim}
xf id aid v_1 v_2 \ldots v_n 0 l_1 l_2 \ldots l_n 0
\end{verbatim}

Composition Claims  A composition claim $(C, A, c)$ for component $C = (F, V)$ is a claim that can be inferred by Lemma 3. It is written as a line starting with tag $a$, followed by identifier $id$ of $C$, the refutation identifier of an absence of models statement $aid$, the claimed count $c$ and the literals $l_1, \ldots, l_n$ of the assumption $A$.

\begin{verbatim}
a id aid c l_1 l_2 \ldots l_n 0
\end{verbatim}