Complexity of Semi-Stable and Stage Semantics in Argumentation Frameworks[☆]

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Abstract

In this work, we answer two questions about the complexity of semi-stable semantics for abstract argumentation frameworks: we show Π_2^P-completeness for the problem of deciding whether an argument is skeptically accepted, and respectively, Σ_2^P-completeness for the problem of deciding whether an argument is credulously accepted under the semi-stable semantics. Furthermore, we extend these complexity bounds to the according decision problems for stage semantics and discuss two approaches towards tractability.

Keywords: computational complexity, abstract argumentation

1. Introduction

In Artificial Intelligence (AI), the area of argumentation [2] has become one of the central issues during the last years. Argumentation provides a formal treatment for reasoning problems arising in a number of interesting applications fields, including Multi-Agent Systems and Law Research. In a nutshell, so-called abstract argumentation frameworks formalize statements[1] together with a relation denoting attacks between them, such that the semantics gives an abstract handle to solve the inherent conflicts between statements by selecting acceptable subsets of them. Several such semantics have already been proposed by Dung in his seminal paper [5], but there are several others which received significant interest lately.

One such approach is known as stage semantics and was introduced by Verheij [17] more than ten years ago. With the work on semi-stable semantics by Caminada [3], who revived Verheij's basic concepts, stage semantics are nowadays mentioned as an

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[1]In general, arguments are not considered as simple statements but contain a number of reasons that lead to a conclusion. However, for the purpose of this work, we will treat arguments as abstract entities (following [5]), thus abstracting from their internal structure.

important alternative (see, e.g. [1]) to Dung's original semantics. The underlying idea of stage semantics is to maximize not only the arguments included in an extension but also those attacked by such an extension. While the complexity of Dung's basic extension based semantics is well studied (see, e.g. [10]) there are open issues for stage semantics. In this work, we give exact complexity bounds for typical decision problems assigned to argumentation frameworks (AFs). In particular, we prove Σ_2^P-hardness, and respectively Π_2^P-hardness, for the problems of deciding whether a given argument is contained in one (credulous acceptance), respectively in all (skeptical acceptance), semi-stable extensions of a given argumentation framework. The respective membership results have been shown by Dunne and Caminada [9], but matching lower bounds have been left as an open problem. We also show that stage semantics (defined in terms of conflict-free sets) are of the same complexity as semi-stable semantics. In order to identify tractable subclasses, we finally investigate the effect of bounding parameters as tree-width and cycle-rank.

2. Background

An *argumentation framework* (AF, for short) is a pair (A, R) where A is a finite[2] set of arguments and $R \subseteq A \times A$ represents the attack-relation. For an AF $AF = (A, R)$, $S \subseteq A$, and $a \in A$, we call (i) S *conflict-free* in AF, if there are no $b, c \in S$ such that $\langle b, c \rangle \in R$; (ii) a *attacked* by S in AF, if there is a $b \in S$ such that $\langle b, a \rangle \in R$; (iii) a *defended* by S in AF, if for each $b \in A$ such that $\langle b, a \rangle \in R$, b is attacked by S in AF; (iv) S *admissible* in AF, if S is conflict-free in AF and each $a \in S$ is defended by S in AF. To define the concepts of stage and resp. semi-stable extensions, we basically follow the conventions used in [1]. Let for an AF $AF = (A, R)$ and $S \subseteq A$, $S_R^+ = S \cup \{b \mid \exists a \in S$, such that $\langle a, b \rangle \in R\}$. Moreover, let us say that $S \leq_R^+ T$ holds if $S_R^+ \subseteq T_R^+$ and $S <_R^+ T$ holds if $S_R^+ \subset T_R^+$.

Definition 1. *Let $AF = (A, R)$ be an AF. A set S is a* stage *(resp. a* semi-stable*)* extension *of AF, if S is maximal conflict-free (resp. admissible) in AF wrt. $\leq_R^+$.*

Example 1. The AF $(\{a, b, c\}, \{\langle a, b \rangle, \langle b, c \rangle, \langle c, a \rangle\})$ has as its only semi-stable extension of AF the empty set, while it possesses stage extensions $\{a\}$, $\{b\}$, $\{c\}$. Thus, the set of stage extensions and the set of semi-stable extensions are in general incomparable. Concerning the relations to the semantics proposed by Dung, we refer to [3, 17].

We consider the following decision problems, for given AF $AF = (A, R)$ and $a \in A$:

- StageC: is a contained in at least one stage extension of AF?
- StageS: is a contained in every stage extension of AF?

[2]Usually, an argumentation framework may be infinite, but for the complexity analysis carried out in this paper, it is sufficient to restrict to finite frameworks.

- SemiC: is a contained in at least one semi-stable extension of AF?
- SemiS: is a contained in every semi-stable extension of AF?

For our forthcoming reductions, we require a particular class of quantified Boolean formulas (QBFs) which we introduce next. A $QBF_\forall^2$ formula is of the form $\forall Y \exists Z C$ where Y and Z are sets of propositional atoms from a countable domain U, and C is a collection of clauses (which we shall represent as sets) over literals built from atoms $Y \cup Z$. A clause itself is interpreted as a disjunction over its literals. For a variable y, we use $\bar{y}$ to represent its negation. Moreover, $\bar{\bar{y}}$ stands for y, etc. We say that a QBF $\forall Y \exists Z C$ is true iff, for each $I_Y \subseteq Y$ there exists an $I_Z \subseteq Z$, such that for each $c \in C$,

$$\big(I_Y \cup I_Z \cup \{\bar{x} \mid x \in (Y \cup Z) \setminus (I_Y \cup I_Z)\}\big) \cap c \neq \emptyset. \tag{1}$$

Example 2. Consider QBF $\Phi = \forall y_1 y_2 \exists z_3 z_4 \{\{y_1, y_2, z_3\}, \{\bar{y}_2, \bar{z}_3, \bar{z}_4)\}, \{\bar{y}_1, \bar{y}_2, z_4\}\}$ which we will use as running example. It can be checked that this QBF is true.

We recall that the problem $QSAT_\forall^2$ (i.e. given a $QBF_\forall^2$ formula Φ, decide whether Φ is true) is Π_2^P-complete.

3. Complexity of Semi-Stable and Stage Semantics

As already mentioned, we consider a countable set U of propositional atoms (in what follows, we use atoms and arguments interchangeably). Moreover, we have the following pairwise disjoint sets of arguments $\bar{U} = \{\bar{u} \mid u \in U\}$, $U' = \{u' \mid u \in U\}$, $\bar{U}' = \{\bar{u}' \mid u \in U\}$. For any set $V \subseteq U$, we use $\bar{V}$, V', $\bar{V}'$, also as renaming schemes in the usual way (for instance, V' denotes the set $\{v' \mid v \in V\}$). Finally, we use further new arguments $t, \bar{t}, b$ and $\{c_1, c_2, \ldots\}$.

We make use of the following reduction from $QBF_\forall^2$ formulas to AFs.

Reduction 1. Given a $QBF_\forall^2$ formula $\Phi = \forall Y \exists Z C$, we define $AF_\Phi = (A, R)$, where

$$\begin{aligned}
A &= \{t, \bar{t}, b\} \cup C \cup Y \cup \bar{Y} \cup Y' \cup \bar{Y}' \cup Z \cup \bar{Z} \\
R &= \{\langle c, t\rangle \mid c \in C\} \cup \{\langle t, \bar{t}\rangle, \langle \bar{t}, t\rangle, \langle t, b\rangle, \langle b, b\rangle\} \cup \\
&\quad \{\langle x, \bar{x}\rangle, \langle \bar{x}, x\rangle \mid x \in Y \cup Z\} \cup \\
&\quad \{\langle y, y'\rangle, \langle \bar{y}, \bar{y}'\rangle, \langle y', y'\rangle, \langle \bar{y}', \bar{y}'\rangle \mid y \in Y\} \cup \\
&\quad \{\langle l, c\rangle \mid \text{literal } l \text{ occurs in } c \in C\}.
\end{aligned}$$

Figure 1 illustrates the corresponding AF AF_Φ for Φ from Example 2.

Lemma 1. *For every stage (resp. semi-stable) extension S of an AF $AF_\Phi = (A, R)$, the following propositions hold: (i) $b \notin S$, as well as $y' \notin S$ and $\bar{y}' \notin S$ for each $y \in Y$ and (ii) $x \notin S \Leftrightarrow \bar{x} \in S$ for each $x \in \{t\} \cup Y \cup Z$.*

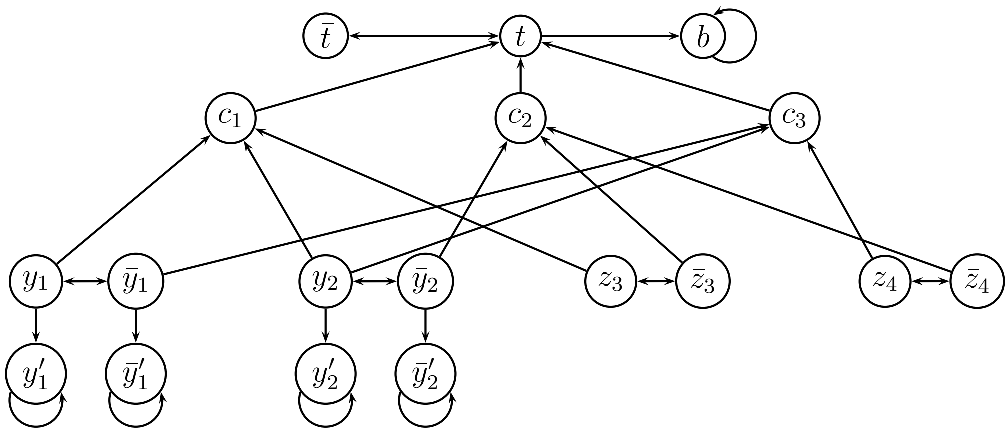

Figure 1: Argumentation framework AF_Φ for QBF Φ given in Example 2.

Proof. Let $\Phi = \forall Y \exists Z C$ and $AF_\Phi = (A, R)$ be the corresponding AF.

ad (i) Clear, since all these arguments are self-attacking.

ad (ii) Obviously, for each $x \in \{t\} \cup Y \cup Z$, $\{x, \bar{x}\} \subseteq S$ cannot hold, since S has to be conflict-free in AF_Φ. It remains to show $\{x, \bar{x}\} \cap S \neq \emptyset$. Towards a contradiction, let us assume there exists such an x, such that $\{x, \bar{x}\} \cap S = \emptyset$ holds for a stage (resp. semi-stable) extension S of AF_Φ.

Let us first assume $x = t$. Then the set $T = S \cup \{\bar{t}\}$ is conflict-free and we have $S <^+_R T$. The argument $\bar{t}$ defends itself and therefore T is admissible if S is. This already shows that S then cannot be a stage or semi-stable extension.

Let us thus assume that $x \in Y \cup Z$ and let $T = (S \setminus \{c \in C \mid \langle \bar{x}, c\rangle \in R\}) \cup \{\bar{x}\}$. One can check that T is conflict-free and that if S is admissible then T is admissible. Moreover, we again have $S <^+_R T$. In fact, for the removed arguments $c \in C$, we have $c \in T^+_R$. Moreover, the only argument attacked by such c is t, but $t \in T^+_R$, since we can already assume $\{t, \bar{t}\} \cap S \neq \emptyset$. This shows that S cannot be a stage (resp. semi-stable) extension. $\square$

Lemma 2. *Let $Y^* = Y \cup \bar{Y} \cup Y' \cup \bar{Y}'$ and S, T be conflict-free sets in $AF_\Phi = (A, R)$. Then $S \cap Y^* \subseteq T \cap Y^*$ iff $(S \cap Y^*)^+_R \subseteq (T \cap Y^*)^+_R$ and further $S \cap Y^* = T \cap Y^*$ iff $(S \cap Y^*)^+_R = (T \cap Y^*)^+_R$.*

Proof. First, assume $S \cap Y^* \subseteq T \cap Y^*$. By the monotonicity of $(.)^+_R$ we get $(S \cap Y^*)^+_R \subseteq (T \cap Y^*)^+_R$. So, assume now $(S \cap Y^*)^+_R \subseteq (T \cap Y^*)^+_R$ and let $l \in S \cap Y^*$. By Lemma 1(i), l is either of form y or $\bar{y}$. If $l \in S \cap Y^*$, then $l, \bar{l}, l' \in (S \cap Y^*)^+_R$. Using our assumption we get $l, \bar{l}, l' \in (T \cap Y^*)^+_R$. But then, $l \in T \cap Y^*$ follows from $l' \in (T \cap Y^*)^+_R$. This shows $S \cap Y^* \subseteq T \cap Y^*$ iff $(S \cap Y^*)^+_R \subseteq (T \cap Y^*)^+_R$. By symmetry, $S \cap Y^* = T \cap Y^*$ iff $(S \cap Y^*)^+_R = (T \cap Y^*)^+_R$ follows. $\square$

Lemma 3. *Let Φ be a $QBF^2_\forall$ formula. If Φ is true, then t is contained in every stage and in every semi-stable extension of AF_Φ.*

Proof. Suppose $\Phi = \forall Y \exists Z C$ is true and let, towards a contradiction, S be a stage or a semi-stable extension of $AF_\Phi = (A, R)$ with $t \notin S$. By Lemma 1(ii), we know that for each $y \in Y$, either y or $\bar{y}$ is in S. Let $I_Y = Y \cap S$. Since Φ is true we know there exists an $I_Z \subseteq Z$, such that (1) holds, for each $c \in C$. Consider now the set $T = I_Y \cup I_Z \cup \{\bar{x} \mid x \in (Y \cup Z) \setminus (I_Y \cup I_Z)\} \cup \{t\}$. We show that T is admissible in AF_Φ and that $S <^+_R T$ holds. This will contradict our assumption in both cases, i.e. that S is a stage or a semi-stable extension of AF_Φ. It is easily verified that T is conflict-free in AF_Φ. Next we show that each $a \in T$ is defended by T in AF_Φ. This is quite obvious for each $a \in T$ except t, since all those arguments defend themselves. To have t defended by T in AF_Φ, each argument $c \in C$ has to be attacked by an element from T. But this is the case since (1) holds and by the construction of AF_Φ, i.e. by the definition of attacks $\{\langle l, c\rangle \mid \text{literal } l \text{ occurs in } c\}$, each such attacker c is attacked by an argument $x \in Y \cup \bar{Y} \cup Z \cup \bar{Z}$. It remains to show $S <^+_R T$. By Lemma 2, $(S \cap Y^*)^+_R = (T \cap Y^*)^+_R$, for $Y^* = Y \cup \bar{Y} \cup Y' \cup \bar{Y}'$. Moreover, by Lemma 1(ii) either z or $\bar{z}$ in S, for each $z \in Z$; the same holds for T, by definition. We observe that $S^+_R \cap (Z \cup \bar{Z}) = T^+_R \cap (Z \cup \bar{Z}) = Z \cup \bar{Z}$. Moreover, we already have argued that each $c \in C$ is attacked by some argument in T. Let $A^- = A \setminus \{t, \bar{t}, b\}$. So far, we thus have shown that $S^+_R \cap A^- \subseteq T^+_R \cap A^- = I_Y \cup I'_Y \cup (\bar{Y} \setminus \bar{I}_Y) \cup (\bar{Y}' \setminus \bar{I}'_Y) \cup Z \cup \bar{Z} \cup C$. We finally observe that $S^+_R \cap \{t, \bar{t}, b\} = \{t, \bar{t}\} \subset \{t, \bar{t}, b\} = T^+_R \cap \{t, \bar{t}, b\}$, since $t \notin S$ by assumption and $t \in T$ by definition. This shows $S <^+_R T$ as desired. $\square$

We are now prepared to give our first main result.

Theorem 1. SemiS *is* Π^P_2*-hard.*

Proof. We use our reduction from $QBF^2_\forall$ formulas to AFs and show that, for each such QBF Φ, it holds that t is contained in all semi-stable extensions of AF_Φ iff Φ is true. Since AF_Φ can be constructed from Φ in polynomial time, the claim then follows.

Let $\Phi = \forall Y \exists Z C$ and $AF_\Phi = (A, R)$ be the corresponding AF. The if direction is captured by Lemma 3. We prove the only-if direction by showing that if Φ is false, then there exists a semi-stable extension S of AF_Φ such that $t \notin S$.

In case Φ is false, there exists an $I_Y \subseteq Y$, such that for each $I_Z \subseteq Z$, there exists a $c \in C$, such that

$$\big(I_Y \cup I_Z \cup \{\bar{x} \mid x \in (Y \cup Z) \setminus (I_Y \cup I_Z)\}\big) \cap c = \emptyset. \tag{2}$$

Consider now a maximal (wrt. $\leq^+_R$) admissible (in AF_Φ) set S, such that $I_Y \subseteq S$ (note that such a set exists, since I_Y itself is admissible in AF_Φ). Using Lemma 2, one can show that S then has to be a semi-stable extension of AF_Φ. To wit, let T be an admissible (in AF_Φ) set such that $I_Y \not\subseteq T$. By Lemma 2 it holds that $(S \cap Y^*)^+_R \not\subseteq (T \cap Y^*)^+_R$ and therefore $S^+_R \not\subseteq T^+_R$. Putting this together with the maximality of S in the set $\{T \mid T \text{ is admissible in } AF_\Phi \text{ and } I_Y \subseteq T\}$ we get that there is no admissible (in AF_Φ) set T, such that $S^+_R \subset T^+_R$. Hence, S is a semi-stable extension of AF_Φ.

It remains to show $t \notin S$. We prove this by contradiction and assume $t \in S$. As S is admissible in AF_Φ, S defends t and therefore it attacks all $c \in C$. As all attacks against arguments in C come from $Y \cup \bar{Y} \cup Z \cup \bar{Z}$, the set $U = \big(I_Y \cup (S \cap (Z \cup \bar{Z})) \cup \{\bar{y} \mid y \in Y \setminus I_Y\}\big)$ attacks all $c \in C$. By Lemma 1(ii), for each $z \in Z$, either z or $\bar{z}$ is contained in S. We get an equivalent characterization for U by $U = \big(I_Y \cup I_Z \cup \{\bar{x} \mid x \in (Y \cup Z) \setminus (I_Y \cap I_Z)\}\big)$ with $I_Z = S \cap Z$. Thus, for all $c \in C$,

$$\big(I_Y \cup I_Z \cup \{\bar{x} \mid x \in (Y \cup Z) \setminus (I_Y \cup I_Z)\}\big) \cap c \neq \emptyset,$$

which contradicts assumption (2). □

Theorem 2. SemiC *is* Σ_2^P*-hard.*

Proof. In the proof of Theorem 1, we have shown that a $QBF_\forall^2$ formula Φ is true iff t is contained in each semi-stable extension of AF_Φ. According to Lemma 1(ii), this holds iff $\bar{t}$ is not contained in any semi-stable extension of AF_Φ. Thus, the complementary problem of SemiC is also Π_2^P-hard. Σ_2^P-hardness of SemiC follows immediately. □

We now turn our attention to the stage semantics.

Theorem 3. StageS *is* Π_2^P*-hard.*

Proof. We again use our reduction from $QBF_\forall^2$ formulas to AFs and show that, for each such QBF Φ, it holds that t is contained in all stage extensions of AF_Φ iff Φ is true. Thus, let $\Phi = \forall Y \exists Z C$ and $AF_\Phi = (A, R)$ be the corresponding AF. The if direction is already captured by Lemma 3. We prove the only-if direction by showing that, if Φ is false, then there exists a stage extension S of AF_Φ such that $t \notin S$.

If Φ is false, there is an $I_Y \subseteq Y$, such that for each $I_Z \subseteq Z$, we have a $c \in C$ with

$$\big(I_Y \cup I_Z \cup \{\bar{x} \mid x \in (Y \cup Z) \setminus (I_Y \cup I_Z)\}\big) \cap c = \emptyset. \tag{3}$$

Consider the collection $W = \{S \mid I_Y \subseteq S, S \text{ is conflict-free in } AF_\Phi\}$ of conflict-free sets in AF_Φ. Using Lemma 2, we can show that for every conflict-free (in AF_Φ) set T, $S \leq_R^+ T$ implies $I_Y \subseteq T$. For verifying $\leq_R^+$-maximality of a set $S \in W$ we thus can restrict ourselves to sets $T \in W$.

It remains to show that there is a stage extension S in W with $t \notin S$. We prove that (i) for every set $S \in W$ with $t \in S$, there exists a $c \in C$, such $c \notin S_R^+$; and (ii) existence of a set $S \in W$ such that $C \subseteq S_R^+$. Note that (i)+(ii) imply existence of a stage extension S of AF_Φ with $t \notin S$.

We prove (i) by contradiction and assume that $C \subseteq S_R^+$. As S is conflict-free in AF_Φ and $t \in S$, we get $C \cap S = \emptyset$. Since $C \subseteq S_R^+$, S attacks all $c \in C$. As all attacks against C come from $Y \cup \bar{Y} \cup Z \cup \bar{Z}$, the set $U = \big(I_Y \cup (S \cap (Z \cup Z')) \cup \{\bar{y} \mid y \in Y \setminus I_Y\}\big)$ attacks all $c \in C$. By Lemma 1(ii), for each $z \in Z$, either z or $\bar{z}$ is contained in S and so we get $U = \big(I_Y \cup I_Z \cup \{\bar{x} \mid x \in (Y \cup Z) \setminus (I_Y \cup I_Z)\}\big)$ with $I_Z = S \cap Z$. Thus, for each $c \in C$, $U \cap c \neq \emptyset$, which contradicts assumption (3).

To show (ii) we just construct such a set $S = U \cup V$ using $U = I_Y \cup \{\bar{y} \in Y \setminus I_Y\} \cup Z$ and $V = \{c \in C \mid \nexists u \in U \text{ with } \langle u, c \rangle \in R\}$. It is easy to verify that S is conflict-free in AF_Φ. It remains to show $c \in S_R^+$, for all $c \in C$. Note that for each $c \in C$ we have that either c is attacked by U or contained in V. In both cases, $c \in S_R^+$ is clear. □

The following result is proven analogously to Theorem 2.

Theorem 4. StageC *is* Σ_2^P*-hard.*

Our hardness results can be extended to AFs without self-attacking arguments. To this end, we adapt our reduction by replacing all self-attacking arguments in the framework AF_Φ by cycles of odd length (for instance, of length 3). Figure 2 illustrates such a framework AF_Φ^m for our example QBF. In case of semi-stable extensions, we use the fact that the only admissible set of an (unattacked) odd-length cycle is the empty set. Indeed, S is a semi-stable extension of AF_Φ iff S is a semi-stable extension of AF_Φ^m.

The same construction can be used for stage semantics, although the argumentation is slightly different: As stage extensions only require conflict-freeness and not admissibility, the arguments of the introduced cycles may now be part of stage extensions. However, to repair the correctness proofs for the modified reduction, we use the observation that for each cycle of length 3 at most one argument can be in a stage extension S (see also Example 1) and at least one argument in the cycle is not attacked by S. Thus each such cycle contributes in three different but incomparable ways to stage extensions. More formally, let A^m be the set of arguments in AF_Φ^m, $X = \{b\} \cup Y' \cup \bar{Y}'$ and denote by x^- be the (unique) attacker of an argument $x \in X$ in the original framework $AF_\Phi = (A, R)$. Then, we get that (i) if S is a stage extension of AF_Φ, then each $S' \subseteq A^m$, such that $S' \cap A = S$ and for each $x \in X$,

$$card(S' \cap \{x_1, x_2, x_3\}) = \begin{cases} 1 & \text{if } x^- \notin S \\ 0 & \text{otherwise} \end{cases}$$

is a stage extension of AF_Φ^m; and (ii) if S is a stage extension of AF_Φ^m, then $S \cap A$ is a stage extension of AF_Φ. This correspondence between extensions suffices to show that our hardness results carry over to self-attack free AFs.

We summarize our results in terms of completeness results. The matching upper bounds for semi-stable semantics have been reported in [9]; for the stage semantics we give them in the proof of the following theorem.

Theorem 5. StageC *and* SemiC *are* Σ_2^P*- complete;* StageS *and* SemiS *are* Π_2^P*- complete. For all problems, hardness holds even for AFs without self-attacking arguments.*

Proof. Hardness is by Theorems 1– 4 and by the observations above.

For the matching upper bounds, we first consider the following problem which we show to be in coNP: Given an AF $AF = (A, R)$ and a set $S \subseteq A$, is S a stage (resp. a semi-stable) extension of AF? Let $cf(AF)$ denote the collection of conflict-free sets $S \subseteq A$ of AF and $adm(AF)$ denote the collection of sets $S \subseteq A$, admissible in AF. By definition, S is a stage (resp. a semi-stable) extension of AF iff (i) $S \in \sigma(AF)$ and (ii) $\forall\ T \subseteq A$, $T \in \sigma(AF)$ only if $S_R^+ \not\subset T_R^+$, for $\sigma = cf$ (resp. $\sigma = adm$). Given S, we can decide $S \in \sigma(AF)$ in polynomial time, for $\sigma \in \{cf, adm\}$. For the complement of (ii), we guess a set T and then we verify (again, in polynomial time),

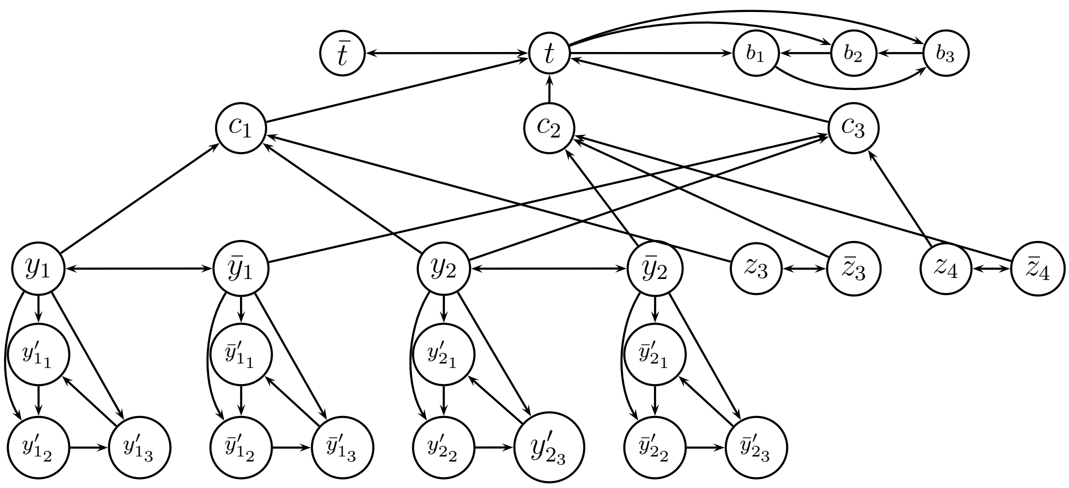

Figure 2: The modified framework AF_Φ^m for Φ from Example 2.

whether $S_R^+ \subset T_R^+$ and $T \in \sigma(AF)$, for $\sigma \in \{cf, adm\}$. This yields membership in NP for the complement of (ii), thus, given set S, (ii) is in coNP, and thus the entire problem is in coNP.[3]

We now can give algorithms for StageC and SemiC as follows. We have given an AF $AF = (A, R)$ and an argument $a \in A$. We guess a set $S \subseteq A$ with $a \in S$ and then use an NP-oracle (we recall that oracle calls are closed under complement), to check whether S is a stage (resp. semi-stable) extension of AF. Obviously this algorithm correctly decides the considered problems. Hence, these problems are in Σ_2^P.

For StageS and SemiS we argue as follows: Given an AF $AF = (A, R)$, to decide if an argument $a \in A$ is contained in each stage (resp. semi-stable) extension of AF, we have to prove that every set S with $a \notin S$ is *not* a stage (resp. semi-stable) extension of AF. Thus, for the complementary problem, we can guess a set S with $a \notin S$ and check whether S is a stage (resp. semi-stable) extension of AF. Again, this check can be done with a single call to an NP-oracle, and thus the complementary problems of StageS and SemiS are in Σ_2^P. Π_2^P-membership of StageS and SemiS follows immediately. $\square$

4. Fixed Parameter Tractability

As we have shown in the previous section, all considered problems are highly intractable. A natural task is now to identify tractable subclasses. We focus on particular graph parameters and check whether bounding such parameters leads to the desired tractable fragments. One such parameter for graph problems is tree-width [16]. Intuitively, the tree-width of a graph measures the tree-likeness of the graph.

Definition 2. *Let $G = (V, E)$ be a digraph. A* tree decomposition *of G is a pair $\langle \mathcal{T}, \mathcal{X} \rangle$ where $\mathcal{T} = \langle V_\mathcal{T}, E_\mathcal{T} \rangle$ is a tree and $\mathcal{X} = (X_t)_{t \in V_\mathcal{T}}$ such that:*

[3] For semi-stable semantics, this problem is also coNP-complete, cf. [9].

1. $\bigcup_{t \in V_{\mathcal{T}}} X_t = V$, *i.e. $\mathcal{X}$ is a cover of V,*
2. *for each $v \in V$ the subgraph of $\mathcal{T}$ induced by $\{t \mid v \in X_t\}$ is connected,*
3. *for each edge $(v_i, v_j) \in E$ there exists an X_t with $\{v_i, v_j\} \subseteq X_t$.*

The width *of a decomposition $\langle \mathcal{T}, \mathcal{X} \rangle$ is given by* $\max\{|X_t| : t \in V_{\mathcal{T}}\} - 1$. *The* tree-width *of a graph G is the minimum width over all tree decompositions of G.*

Many graph properties can be defined by formulas of monadic second-order logic (MSOL) and by Courcelle's Theorem [4] such properties can be efficiently decided on arbitrary relational structures with bounded tree-width (see also [12]). Proposition 1 gives a more specific version of this meta-theorem, namely for digraphs with a set of distinguished vertices (we express that a vertex x is distinguished by a predicate $q(x)$).

Proposition 1. *Let K be a class of digraphs with a set of distinguished vertices, for which the tree-width is bounded by some constant k and Π be a MSOL-definable property. For each such $G \in K$, $G \in \Pi$ is decidable in linear time wrt. the size of G.*

This result is a powerful tool to classify graph problems as fixed-parameter tractable.

Theorem 6. *Let K be a class of AFs, which, when interpreted as undirected graphs, have tree-width $\leq k$ (for fixed k). For each $AF \in K$ the problems* SemiS, SemiC, StageS *and* StageC *are decidable in linear time.*

Proof. One can show that the MSOL formula $\mathrm{semi}_{AF}(U)$ (resp. $\mathrm{stage}_{AF}(U)$) as given below characterizes the semi-stable (resp. stage) extensions of an AF AF.

$$\begin{aligned}
U \subseteq_R^+ V &= \forall x\Big(\big(x \in U \vee \exists y(y \in U \wedge \langle y, x\rangle \in R)\big) \rightarrow \\
&\qquad\quad \big(x \in V \vee \exists y(y \in V \wedge \langle y, x\rangle \in R)\big)\Big) \\
U \subset_R^+ V &= U \subseteq_R^+ V \wedge \neg(V \subseteq_R^+ U) \\
\mathrm{cf}_R(U) &= \forall x, y\big(\langle x, y\rangle \in R \rightarrow (\neg x \in U \vee \neg y \in U)\big) \\
\mathrm{adm}_R(U) &= \mathrm{cf}_R(U) \wedge \forall x, y\Big((\langle x, y\rangle \in R \wedge y \in U) \rightarrow \exists z(z \in U \wedge \langle z, x\rangle \in R)\Big) \\
\mathrm{semi}_{(A,R)}(U) &= \mathrm{adm}_R(U) \wedge \neg\exists V(V \subseteq A \wedge \mathrm{adm}_R(V) \wedge U \subset_R^+ V) \\
\mathrm{stage}_{(A,R)}(U) &= \mathrm{cf}_R(U) \wedge \neg\exists V(V \subseteq A \wedge \mathrm{cf}_R(V) \wedge U \subset_R^+ V)
\end{aligned}$$

The required checks for the considered decision problems are easily added to these formulas. For instance, if we label the argument which we check for acceptance by q, SemiC (with input (A, R) and $a \in A$) can be decided by $\exists U \exists u(\mathrm{semi}_{(A,R)}(U) \wedge u \in U \wedge q(u))$. Proposition 1 then yields the desired result. $\square$

As argumentation frameworks are directed graphs it seems natural to consider directed graph measures to get larger tractable fragments than those we capture with bounded tree-width, which doesn't take the direction of the edges into account. Unfortunately, it turns out that the considered problems remain hard when bounding typical directed graph measures. We illustrate this fact by using cycle rank [11] as a parameter.

Definition 3. *Let $G = \langle V, E\rangle$ be a directed graph. The cycle rank of G, $cr(G)$, is defined as follows: An acyclic graph has $cr(G) = 0$. If G is strongly connected then $cr(G) = 1 + min_{v \in V_G} cr(G \setminus v)$. Otherwise, $cr(G)$ is the maximum cycle rank among all strongly connected components of G.*

Theorem 7. *The problems* $\mathsf{SemiS}, \mathsf{StageS}$ *(resp.* $\mathsf{SemiC}, \mathsf{StageC}$*) remain Π_2^P- hard (resp. Σ_2^P- hard), even if restricted to AFs which have a cycle-rank of* 1.

Proof. Its easy to see that every framework of the form AF_Φ has cycle-rank 1 and therefore we have an reduction from $QBF_\forall^2$ formulas to an AF with cycle-rank 1. In fact, the strongly connected components of AF_Φ are induced by the following sets: $\{y_i, \bar{y}_i\}, \{z_i, \bar{z}_i\}, \{t, \bar{t}\}, \{y_i'\}, \{\bar{y}_i'\}, \{z_i'\}, \{\bar{z}_i'\}, \{b\}$. As each of these components can be made acyclic by removing one vertex, the cycle-rank of AF_Φ is thus 1. □

By results in [13, 14, 15] it follows that a problem which is hard for bounded cycle-rank remains hard for bounding other directed graph measures, i.e. directed path-width, Kelly-width, DAG-width and directed tree-width.

5. Conclusion

In this note, we provided novel complexity results for abstract argumentation frameworks in terms of skeptical and resp. credulous acceptance under semi-stable and stage semantics (as defined in [1]). In case of semi-stable semantics, we improved the existing $P_{||}^{NP}$-lower bound [9] to hardness for classes Π_2^P (resp. Σ_2^P). Together with existing upper bounds, we thus obtained completeness for classes on the second level of the polynomial hierarchy, answering an open question raised by Caminada and Dunne [9].

Furthermore, we showed that stage semantics leads to the same complexity. To the best of our knowledge, no complexity results for this semantics have been obtained so far. We emphasize that the level of complexity the considered semantics show is unique for credulous reasoning not only among the semantics proposed by Dung[4] but also with respect to more recent proposals such as the ideal semantics where acceptance can be decided within $P_{||}^{NP}$ [7].

Finally, we gave some results in terms of bounding some problem parameter. As a positive result, bounded tree-width leads to tractable subclasses of the problems under consideration (similar results for other semantics have been shown by Dunne [6]).

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