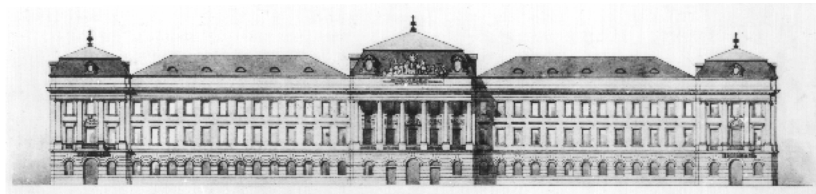

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A General Notion of Equivalence for Abstract Argumentation

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A General Notion of Equivalence for Abstract Argumentation

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Abstract. We introduce a parametrized equivalence notion for abstract argumentation that subsumes standard and strong equivalence as corner cases. Under this notion, two argumentation frameworks are equivalent if they deliver the same extensions under any addition of arguments and attacks that do not affect a given set of core arguments. As we will see, this notion of equivalence nicely captures the concept of local simplifications. We provide exact characterizations and complexity results for deciding our new notion of equivalence.

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1 Introduction

Argumentation has become one of the major fields within AI over the last two decades [24, 13]. In particular, Dung's argumentation frameworks [15], AFs for short, are widely used and act as integral concepts in several advanced argumentation formalisms. They focus entirely on conflict resolution among arguments, treating the latter as abstract items without logical structure. Hence, the only information available in AFs is the so-called attack-relation that determines whether an argument is in a certain conflict with another one. As already outlined by Dung, AFs provide a formally simple basis to capture the essence of different nonmonotonic formalisms. Therefore, several so-called semantics are typically considered for AFs, see also [2]. A semantics delivers several sets of arguments (called extensions) that can be jointly accepted in order to satisfy certain properties. One such property is given by admissible sets which consist of arguments that do not attack each other and attack each argument attacking the set itself.

Bearing the nonmonotonic nature of AFs in mind, it is evident that the *standard* notion of *equivalence* (i.e., do two AFs possess the same sets of extensions?) is a rather weak concept. In particular, it is *not* the case that replacing an AF by an equivalent one is a faithful manipulation. As an example consider the AFs $F_{abc} = (\{a,b,c\}, \{(a,b),(b,c),(c,a)\})$ and $F_{ab} = (\{a,b\}, \{(a,a),(a,b)\})$, which are equivalent for most semantics, including admissible sets. However, replacing F_{abc} by F_{ab} in a larger AF G might not be an equivalence-preserving action. Suppose G expands F_{abc} via an attack from some argument d to b. Then, the mentioned replacement would change each admissible set $S \cup \{d,c\}$ into $S \cup \{d\}$. On the other hand, if F_{abc} is embedded in G only via an attack (d,a) – see Figure 1 – the replacement of F_{abc} by F_{ab} is faithful. More formally, we then have that the admissible sets of G and $G[F_{abc}/F_{ab}]$ are the same.[1]

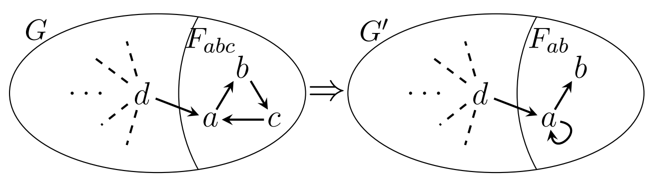


Figure 1: Replacing F_{abc} in G to obtain $G' = G[F_{abc}/F_{ab}]$.

Observations of this kind gave rise to more restricted notions of equivalence [23, 5, 12]. *Strong equivalence* (also called expansion equivalence) between two AFs F and F' holds (w.r.t. a semantics σ) if and only if for all AFs H the expanded AFs $F \cup H$ and $F' \cup H$ have the same σ-extensions. By definition, this notion of equivalence guarantees that F can be replaced by a strongly equivalent (w.r.t. σ) AF F' in any framework G without changing the σ-extensions of G. Interestingly, the characterization results for strong equivalence are surprisingly simple and can be given via so-called kernels, syntactic modifications of the involved AFs. From a theoretical perspective, it is thus open how this conceptual difference between standard and strong equivalence can be captured via a uniform formal characterization which has these two notions as corner cases.

[1] A formal definition of replacements $G[\cdot/\cdot]$ is given in Section 6.

From a computational point of view, strong equivalence (and related versions) seem to be an appealing notion, since checks for replacements, and thus also for simplifications in AFs, would become easy. However, strong equivalence is too restricted for practical purposes. Even obvious simplifications are not captured: an example are isolated self-loops, which can be safely removed from AFs for many standard semantics. However, AF $F = (\{a\}, \{(a, a)\})$ is not strongly equivalent to the empty AF $F' = (\emptyset, \emptyset)$ for admissible semantics; just take $H = (\{a\}, \emptyset)$. Then, $\{a\}$ is admissible for $F' \cup H$ but not for $F \cup H$. This indicates that a suitable equivalence notion for replacement needs a particular treatment for those arguments which are directly involved in the change.

Hence, what we require is an equivalence notion that compares two AFs such that

1. the relations between *core* arguments are fixed, while
2. the remaining arguments are allowed to interact arbitrarily with possible expansions of the compared AFs.

Our proposal is to define, given a set of core arguments C and a semantics σ, *C-relativized equivalence* between two AFs F and F' w.r.t. σ (in symbols, $F \equiv^{\sigma}_{C} F'$) to hold, if $F \cup H$ and $F' \cup H$ have the same σ-extensions, for each AF H not containing arguments from C. Observe that this notion indeed captures strong equivalence (set $C = \emptyset$) and standard equivalence (set C to be the universe of all arguments).

Coming back to our example with F_{abc} and F_{ab}, the idea is to set $C = \{a, b, c\}$ and compare the two AFs plus their interaction with the AF G where F_{abc} occurs in. In our case, we compare $F^{G}_{abc} = (\{a, b, c, d\}, \{(a, b), (b, c), (c, a), (d, a)\})$ and $F^{G}_{ab} = (\{a, b, d\}, \{(a, a), (a, b), (d, a)\})$. Then, $F^{G}_{abc} \equiv^{\sigma}_{C} F^{G}_{ab}$ implies that G and $G[F_{abc}/F_{ab}]$ are equivalent under σ, i.e., replacing F_{abc} by F_{ab} in G is safe for semantics σ. As we will see later, this is the case for all standard semantics.

Our main contributions are as follows:

- We first define restrictions for the main semantics of stable, admissible, preferred, complete and grounded extensions. These identify extensions of an AF F that are acceptable in some expansion $F \cup H$ and are integral for equivalence characterizations.
- We give exact characterizations of C-relativized equivalence for the five semantics mentioned above; in addition we also show results for conflict-free and naive sets.
- We provide a complexity analysis for deciding C-relativized equivalence; as corollaries we also obtain insight to the complexity of standard equivalence.
- Finally, we give a formal notion of replacement in AFs and illustrate how our equivalence notion can be employed for local simplifications within AFs.

2 Preliminaries

In this section, we introduce argumentation frameworks [15] and recall the semantics we study (for an overview, see [2]). We fix U as countable infinite domain of arguments.

Definition 1. *An* argumentation framework (AF) *is a pair $F = (A, R)$ where $A \subseteq U$ is a finite set of arguments and $R \subseteq A \times A$ is the attack relation. The pair $(a, b) \in R$ means that a attacks b. We use $A(F)$ to refer to A and $R(F)$ to refer to R. We say that an AF is given over a set B if $A(F) \subseteq B$.*

Given an AF F and $S \subseteq U$, we define $S_F^+ = \{x \mid \exists y \in S : (y, x) \in R(F)\}$, $S_F^- = \{x \mid \exists y \in S : (x, y) \in R(F)\}$), and the range of *$S$ in F as $S_F^\oplus = (S \cap A(F)) \cup S_F^+$.*

Given an AF $F = (A, R)$, an argument $a \in A$ is defended *(in F) by a set $S \subseteq A$ if $\{x\}_F^- \subseteq S_F^+$. The* characteristic function *$\mathcal{F}_F : 2^A \to 2^A$ of F is defined as $\mathcal{F}_F(S) = \{x \in A \mid x$ is defended by S in $F\}$.*

Given AFs $F = (A, R)$, $F' = (A', R')$, and $S \subseteq U$, we denote the union of AFs as $F \cup F' = (A \cup A', R \cup R')$, and define $F \setminus S = (A \setminus S, R \cap ((A \setminus S) \times (A \setminus S)))$ and $F \cap S = (A \cap S, R \cap ((A \cap S) \times (A \cap S)))$.

Semantics for argumentation frameworks are defined as functions σ which assign to each AF F a set $\sigma(F) \subseteq 2^{A(F)}$ of extensions. We consider for σ the functions $naive$, grd, stb, adm, com, and prf, which stand for naive, grounded, stable, admissible, complete, and preferred extensions, respectively.

Definition 2. *Let $F = (A, R)$ be an AF. A set $S \subseteq A$ is* conflict-free (in F)*, if there are no $a, b \in S$, such that $(a, b) \in R$. $cf(F)$ denotes the collection of conflict-free sets of F. For a conflict-free set $S \in cf(F)$, it holds that*

- *$S \in naive(F)$, if there is no $T \in cf(F)$ with $T \supset S$;*
- *$S \in stb(F)$, if $S_F^\oplus = A$;*
- *$S \in adm(F)$, if $S \subseteq \mathcal{F}_F(S)$;*
- *$S \in com(F)$, if $S = \mathcal{F}_F(S)$;*
- *$S \in grd(F)$, if $S \in com(F)$ and there is no $T \subset S$ such that $T \in com(F)$;*
- *$S \in prf(F)$, if $S \in adm(F)$ and there is no $T \supset S$ such that $T \in adm(F)$.*

We recall that for each AF F, the grounded semantics yields a unique extension, which is the least fixed-point of the characteristic function $\mathcal{F}_F$.

3 Notions of Equivalence

We first review two equivalence notions for AFs from the literature, namely standard and strong equivalence.

Definition 3. *Given a semantics σ. Two AFs F and G are* (standard) equivalent *w.r.t. σ ($F \equiv^\sigma G$) iff $\sigma(F) = \sigma(G)$.*

Definition 4. *Given a semantics σ. Two AFs F and G over U are* strongly equivalent *w.r.t. σ ($F \equiv^\sigma_S G$) iff $F \cup H \equiv^\sigma G \cup H$ holds for each AF H over U.*

In this work we introduce the new notion of C-relativized equivalence, which is parametrized by the set C of *core* arguments which will not be directly touched by the possible expansions (i.e., AFs H added to the compared AFs are not arbitrary anymore).

Definition 5. *Given a semantics σ and $C \subseteq U$. Two AFs F and G over U are C-*relativized equivalent *w.r.t. σ ($F \equiv^\sigma_C G$) iff $F \cup H \equiv^\sigma G \cup H$ holds for each AF H over $U \setminus C$.*

Notice that (i) for $C = \emptyset$ the C-relativized equivalence coincides with strong equivalence and (ii) when $C = U$ then C-relativized equivalence is just standard equivalence (the only AF over $U \setminus C = \emptyset$ is $(\emptyset, \emptyset)$ and $F \cup (\emptyset, \emptyset) = F$ for all AFs F).

The following observation expresses the fact that C-relativized equivalence survives if we extend the core C with further untouchable arguments. Since in general standard equivalence ($C = U$) does not imply strong equivalence ($C = \emptyset$) the assertion does not hold for shrinking the core.

Observation 1. *For any two AFs F, G, any two sets $C, D \subseteq U$ and any semantics σ, if $C \subseteq D$ and $F \equiv^\sigma_C G$, then $F \equiv^\sigma_D G$.*

An immediate consequence of the observation above is that strong (standard) equivalence is more (less) demanding than relativized equivalence, no matter which core C is considered. This is simply due to the fact that for any core C, $\emptyset \subseteq C \subseteq U$. The next proposition gives more refined conditions for the coincidence between C-relativized equivalence and strong or standard equivalence, respectively. Since we consider finite AFs only we restrict our considerations to finite cores too.

Proposition 1. *Let F, G be AFs, $C \subseteq U$ a finite core, $\sigma \in \{stb, adm, com, grd, prf\}$, and $B = C \cap (A(F) \cup A(G))$.*

1. *If $B = \emptyset$, then $F \equiv^\sigma_C G$ iff $F \equiv^\sigma_S G$.*
2. *If $B = A(F) \cup A(G)$, then $F \equiv^\sigma_C G$ iff $F \equiv^\sigma G$.*

Proof. 1) $\Leftarrow$: By Observation 1 and the fact that $F \equiv^\sigma_S G$ is equivalent to $F \equiv^\sigma_\emptyset G$. $\Rightarrow$: Obvious since C is finite (via renaming one may model any distinguishing AF H).
2) $\Rightarrow$: By Observation 1 and the fact that $F \equiv^\sigma G$ is equivalent to $F \equiv^\sigma_U G$. $\Leftarrow$: Observe that any suitable AF H constitutes new weakly connected components, i.e., components that are not connected to the original AF. Consequently, computing the σ-extensions of $F \cup H$ as well as $G \cup H$ can be reduced to computing the σ-extensions of H as well as F or G, respectively (cf. [10, Lemma 46]). $\square$

4 Characterization Results

In what follows, we aim for giving characterizations for deciding $F \equiv_C^\sigma G$ with finite $C \subset U$, such that an explicit consideration of all possible expansions is avoided. In other words, we need semantical concepts that are solely defined on the AFs F and G, but take the core C into account. To this end, we start with the concept of C-restricted semantics. Our main result for exactly characterizing $F \equiv_C^\sigma G$ then requires that the C-restricted extensions coincide for the compared AFs. As we will see in Section 4.3, some further semantics-dependent conditions must be met for this purpose.

4.1 C-restricted Semantics

In this section we introduce so called C-restricted variants of the semantics under consideration, which will nicely characterize the sets of arguments in an AF F that are a projection of an expansion $F \cup H$. C-restricted semantics will be a fundamental concept in the characterizations of our equivalence notion.

The overall idea for all C-restricted semantics is that we restrict the relevant properties of the original semantics to the core arguments. Conflict-freeness is the only exception from the above, i.e., we always require an extension to be conflict-free on the whole AF. This is because a conflict present in the current AF F will also be present in every expansion $F \cup H$.

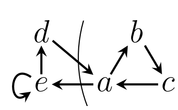
4.1.1 C-restricted Stable Semantics

For stable semantics we have two conditions: (a) the set must be conflict-free and (b) all arguments are either in the extension or attacked by some argument in the extension. While we cannot relax the former (a conflict present in the current AF will also be present in every expansion), we relax the latter to only hold for arguments in the set C. The intuition behind this is that arguments not in C might be attacked in expansions of the framework by newly introduced attacks while arguments in C can only be attacked by the already present attacks.

Definition 6. *Let F be an AF, $C \subseteq U$, and $E \subseteq A(F)$. We define*

- *$E \in stb_C(F)$ if $E \in cf(F)$ and $A(F) \cap C \subseteq E_F^\oplus$.*

Example 1. *For AF F_{abc}^G from the introduction and $C = \{a, b, c\}$, we have $stb_C(F_{abc}^G) = \{\{d, b\}\}$. In this particular case, standard extensions and restricted ones coincide. Let us thus extend F_{abc}^G by $(\{a, d, e\}, \{(a, e), (e, e), (e, d)\})$ to the AF F as depicted below.*



We observe that $stb_C(F) = stb_C(F_{abc}^G)$ although $stb(F) = \emptyset$. $\Diamond$

Another crucial feature of C-restricted semantics is that $\sigma_C(F)$ returns all the argument sets that are projections of σ-extensions in some $F \cup H$ with H defined over $U \setminus C$. We next show that $stb_C(F)$ exactly characterizes the sets of arguments that can be extended to a stable extension in some expansion $F \cup H$.

Lemma 1. *Let F be an AF, $C \subseteq U$, and $E \subseteq A(F)$. Then, $E \in stb_C(F)$ iff there exists an AF H over $U \setminus C$ and $T \in stb(F \cup H)$ such that $T \cap A(F) = E$.*

Proof. $\Rightarrow$: Let $B = A(F) \setminus (E \cup C)$ and consider

$$H = (\{t\} \cup B, \{(t, b) \mid b \in B\})$$

with $t \in U \setminus C$ a fresh argument (not occurring in F). Clearly H is given over $U \setminus C$. We show that $S = E \cup \{t\}$ is a stable extension of $F \cup H$. We observe that S is conflict-free in $F \cup H$ (since $E \in stb_C(F)$ is conflict-free in F and by construction of H) and moreover that $S^{\oplus}_{F \cup H} = (S^{\oplus}_{F \cup H} \setminus C) \cup (S^{\oplus}_{F \cup H} \cap C) = (A(F \cup H) \setminus C) \cup (E^{\oplus}_F \cap C)$. From $E \in stb_C(F)$, we obtain $A(F) \cap C \subseteq E^{\oplus}_F \cap C$. It follows that $S^{\oplus}_{F \cup H} = A(F \cup H)$, hence $S \in stb(F \cup H)$.

$\Leftarrow$: Consider $T \in stb(F \cup H)$ for some H an define $E = T \cap A(F)$. We have to show that $E \in stb_C(F)$. Clearly, E is conflict-free in F; moreover each $c \in A(F) \cap C$ that is not contained in E is attacked by E in F, since $c \in E^{+}_{F \cup H}$, but we are not allowed to have $(b, c) \in H$. Thus, $E \in stb_C(F)$. □

Example 2. *Recall F from Example 1. For $C = \{a, b, c\}$ we had $\{b, d\} \in stb_C(F)$. The construction in the proof of Proposition 1 just adds an argument t attacking e (note that t and e are not from C). For the resulting AF it is easily checked that $\{t, b, d\}$ is its only stable extension.* ◊

4.1.2 C-restricted Admissible Semantics

For a set S being admissible we have two conditions: (a) the set must be conflict-free and (b) all arguments in S are defended by S. While we cannot relax the former, we relax the latter to (b') all arguments in S are defended against attackers from C by S. The intuition behind this is that arguments not in C might be attacked in expansions of the framework by newly introduced attacks while arguments in C have to be attacked by the already present attacks.

Definition 7. *Let F be an AF, $C \subseteq U$, and $E \subseteq A(F)$. We define*

- *$E \in adm_C(F)$ if $E \in cf(F)$ and $E^{-}_F \cap C \subseteq E^{+}_F$.*

Example 3. *For AF F^G_{abc} from the introduction and $C = \{a, b, c\}$, we have $adm_C(F^G_{abc}) = \{\emptyset, \{d\}, \{d, b\}\}$. In this particular case, standard extensions and restricted ones coincide. Let us thus again extend F^G_{abc} to the AF F from Example 1. We observe that $adm_C(F) = adm_C(F^G_{abc})$ but $adm(F) = \{\emptyset\}$.* ◊

We next show that $adm_C(F)$ exactly characterizes the sets of arguments that can be extended to admissible sets in some expansion $F \cup H$ of F with H over $U \setminus C$.

Lemma 2. *Let F be an AF, $C \subseteq U$, and $E \subseteq A(F)$. It holds that $E \in adm_C(F)$ iff there exists an AF H over $U \setminus C$ and $T \in adm(F \cup H)$ such that $T \cap A(F) = E$.*

Proof. $\Rightarrow$: We use the same construction as in the proof of Lemma 1. Let $B = A(F) \setminus (E \cup C)$ and consider

$$H = (\{t\} \cup B, \{(t,b) \mid b \in B\})$$

with $t \in U \setminus C$ a fresh argument (not occurring in F). Clearly H is given over $U \setminus C$. We show that $S = E \cup \{t\}$ is admissible in $F \cup H$. We observe that S is conflict-free in $F \cup H$. Let $b \in A(F \cup H) \setminus S$ be an attacker of some element in S. If $b \in A(H)$, b is attacked by t in $F \cup H$. Otherwise $b \in C$. Since $E \in adm_C(F)$, E attacks b in F, and so does S in $F \cup H$. This shows that each $a \in S$ is defended by S in $F \cup H$.

$\Leftarrow$: Consider $T \in adm(F \cup H)$ for some H an define $E = F \cap A(F)$. We have to show that $E \in adm_C(F)$. Clearly, E is conflict-free in F; moreover each $c \in E^-_F \cap C$ is attacked by E in F, since $c \in T^+_{F \cup H}$ but we are not allowed to have $(b,c) \in H$. Thus, $E \in adm_C(F)$. $\square$

Example 4. *Recall F from Example 3. For $C = \{a,b,c\}$, we had $adm_C(F) = \{\emptyset, \{d\}, \{b,d\}\}$. For $\{d\}$ and $\{b,d\}$, the construction in the proof of Lemma 2 just adds an argument t attacking e (note that t and e are not from C). For the resulting AF it is easily checked that $\{t,b,d\}$ and $\{t,d\}$ are among its admissible sets. For $\emptyset \in adm_C(F)$, H contains an additional attack (t,d). The admissible sets of $F \cup H$ do not contain any argument from F, as desired.* $\Diamond$

4.1.3 C-restricted Preferred Semantics

Preferred extensions are defined as maximal admissible sets. We can consider C-restricted admissible sets instead of admissible sets, but also have to consider a different version of maximality. That is, (a) we only compare different extensions on the set C and (b) only compare extensions if they coincide outside of C w.r.t. the arguments in the set, attacked by the set, and undefeated attackers. The former is by the reasons discussed above, the latter is because any difference outside C can cause the acceptance of an argument in some expansion of the framework and thus make the two sets incomparable.

Definition 8. *Let F be an AF, $C \subseteq U$ and $E \subseteq A(F)$. We define*

- *$E \in prf_C(F)$ if $E \in adm_C(F)$ and for all $D \in adm_C(F)$ with $E \setminus C = D \setminus C$, $E^+_F \setminus C \subseteq D^+_F \setminus C$, and $E^-_F \setminus E^+_F \supseteq D^-_F \setminus D^+_F$ we have $E \cap C \not\subset D \cap C$.*

Example 5. *Recall the AF F^G_{abc} from the introduction and the AF F from Example 1. We have $prf_C(F^G_{abc}) = prf_C(F) = \{\emptyset, \{d,b\}\}$. The C-restricted admissible set $\{d\}$ is not C-restricted preferred in F as $\{d\} \setminus C = \{d,b\} \setminus C = \{d\}$, $\{d\}^+_F \setminus C = \{d,b\}^+_F \setminus C = \emptyset$, $\{d\}^-_F \setminus \{d\}^+_F = \{d,b\}^-_F \setminus \{d,b\}^+_F = \{e\}$, but $\{d\} \cap C = \emptyset \subset \{d,b\} \cap C = \{b\}$.* $\Diamond$

We next show that $prf_C(F)$ exactly characterizes the sets of arguments that can be extended to preferred extensions in some expansion $F \cup H$ of F with H over $U \setminus C$.

Lemma 3. *Let F be an AF, $C \subseteq U$, and $E \subseteq A(F)$. Then, $E \in \mathit{prf}_C(F)$ iff there exists an AF H over $U \setminus C$ and $T \in \mathit{prf}(F \cup H)$ such that $T \cap A(F) = E$.*

Proof. $\Rightarrow$: Let $B = A(F) \setminus (E \cup C)$ and consider

$$H = (\{t\} \cup B, \{(b, b) \mid b \in B\} \cup \{(t, b) \mid b \in E_F^- \setminus E_F^+\} \cup \{(a, t) \mid a \in E_F^+ \setminus C\})$$

with $t \in U \setminus C$ being a fresh argument (not occurring in F). Observe that H is given over $U \setminus C$, in particular since $E_F^- \cap C \subseteq E_F^+$, i.e., $(E_F^- \setminus E_F^+) \cap C = \emptyset$, holds for $E \in \mathit{adm}_C(F)$. We show that $S = E \cup \{t\}$ is preferred. First, S is conflict-free in $F \cup H$. Second, the argument t is only attacked by $E_F^+ \setminus C$ in $F \cup H$ and thus defended by S in $F \cup H$. Since $E \in \mathit{adm}_C(F)$, S defends itself against all attackers from C and t defends S against all attackers $E_F^- \setminus E_F^+$. This shows that each $a \in S$ is defended by S in $F \cup H$, i.e., $S \in \mathit{adm}(F \cup H)$.

Finally, consider the maximality of S. Towards a contradiction assume there is a $T \in \mathit{adm}(F \cup H)$ such that $S \subset T$. Notice that all arguments in B are self-attacking and therefore $S \setminus C = T \setminus C$. Moreover, by Lemma 2 it holds that $D = T \cap A(F)$ is a C-restricted admissible set of F and, as $S \subset T$ and $S \setminus C = T \setminus C$, we have $E \subset D$, (a) $E \setminus C = D \setminus C$, and (b) $E \cap C \subset D \cap C$. By the monotonicity of $(.)_F^+$ we have (c) $E_F^+ \setminus C \subseteq D_F^+ \setminus C$ and as T is admissible we have that t attacks all arguments in $D_F^- \setminus D_F^+$, i.e., (d) $E_F^- \setminus E_F^+ \supseteq D_F^- \setminus D_F^+$. Combining (a)-(d) we obtain a contradiction to $E \in \mathit{prf}_C(F)$.

$\Leftarrow$: Consider $T \in \mathit{prf}(F \cup H)$ for some H an define $E = F \cap A(F)$. We have to show that $E \in \mathit{prf}_C(F)$. By Lemma 2, $E \in \mathit{adm}_C(F)$. Towards a contradiction assume there is a $D \in \mathit{adm}_C(F)$ with $E \setminus C = D \setminus C$ such that $E_F^+ \setminus C \subseteq D_F^+ \setminus C$, $E_F^- \setminus E_F^+ \supseteq D_F^- \setminus E_F^+$ and $E \cap C \subset D \cap C$. Then $D \cup (T \setminus A(F))$ is admissible in $F \cup H$ and $T \subset D \cup (T \setminus A(F))$. A contradiction to $T \in \mathit{prf}(F \cup H)$. $\square$

4.1.4 C-restricted Complete Semantics

In order to define the C-restricted complete and grounded semantics we need the concept of the *C-restricted characteristic function* $\mathcal{F}_{F,C,E}(S)$ for an AF F and $E, S \subseteq A(F)$.

$$\begin{aligned}\mathcal{F}_{F,C,E}(S) = &\{a \in E \mid \forall c \in C : (c, a) \in R(F) \to c \in S_F^+\} \cup \\ &\{c \in C \cap A(F) \mid \forall (b, c) \in R(F) : b \in S_F^+ \cup (S_F^- \setminus C)\}\end{aligned}$$

The C-restricted characteristic function (1) tests arguments in E to be acceptable w.r.t. C-restricted admissible conditions, i.e., whether it is defended against all attackers from C, and (2) tests arguments in C whether they can have undefeated attackers when assuming that S is admissible. The intuition for the former is that (a) attackers outside C can be counter-attacked via attacks in the expansion and (b) arguments outside C can be disabled by self-attacks in the expansion and thus the characteristic function can be restricted to arbitrary sets E using the right expansion. However, the attacks to arguments in C are fixed and thus any extension containing S must also contain all arguments satisfying the latter condition.

Now C-restricted complete semantics can be characterized as conflict-free fixed-points of a C-restricted characteristic function.

Definition 9. *Let F be an AF, $C \subseteq U$ and $E \subseteq A(F)$. We define*

- $E \in com_C(F)$ *if* $E \in cf(F)$ *and* $E = \mathcal{F}_{F,C,E}(E)$.

Example 6. *Recall the AF F^G_{abc} from the introduction and the AF F from Example 1. We have $com_C(F^G_{abc}) = com_C(F) = \{\emptyset, \{d, b\}\}$. The C-restricted admissible set $E = \{d\}$ is not C-restricted complete as $b \in C$ is defended by E in the sense of $\mathcal{F}_{F,C,E}(E)$, i.e., we have $a \in E^+_F \cup (E^-_F \setminus C)$. Thus, $\mathcal{F}_{F,C,E}(E) = \{b, d\} \neq E$.*

We next show that $com_C(F)$ exactly characterizes the sets of arguments that can be extended to complete extensions in some expansion $F \cup H$ of F with H over $U \setminus C$.

Lemma 4. *Let F be an AF, $C \subseteq U$, and $E \subseteq A(F)$. $E \in com_C(F)$ iff there exists an AF H over $U \setminus C$ and $T \in com(F \cup H)$ such that $T \cap A(F) = E$.*

Proof. $\Rightarrow$: Let $B = A(F) \setminus (E \cup C)$ and consider the AF

$$H = (\{t\} \cup B,\ \{(t, b) \mid b \in E^-_F \setminus C\} \cup \{(b, b) \mid b \in B\})$$

with $t \in U \setminus C$ a fresh argument (not occurring in F). Clearly H is given over $U \setminus C$. We show that $S = E \cup \{t\}$ is a complete extension of $F \cup H$. As $E \in com_C(F)$ it is defended against all attackers from C. Moreover, by the construction of H and $t \in S$ we have that S is also defended against attackers from $A(F \cup H) \setminus C$, i.e., S is admissible in $F \cup H$. As in $F \cup H$ all arguments from $A(F \cup H) \setminus (C \cup E \cup \{t\})$ are self-attacking, none of them can be defended by S. Suppose there is an argument $c \in C \setminus E$ that is defended by S in $F \cup H$. Then, for each a with $(a, c) \in R$ we either have that (i) t attacks a in H or (ii) E attacks a in F, i.e., $a \in E^+_F$. In the former case, by construction of H, $a \in E^-_F \setminus C$. That is all attackers a of c are contained in either E^+_F or $E^-_F \setminus C$ and thus, by $\mathcal{F}_{F,C,E}(E) = E$, already $c \in E$. It follows that $S \in com(F \cup H)$.

$\Leftarrow$: Consider $T \in com(F \cup H)$ for some H over $U \setminus C$ and define $E = T \cap A(F)$. We have to show that $E \in com_C(F)$. (i) E is conflict-free in F, as T is conflict-free in $F \cup H$. (ii) Towards a contradiction assume $E \neq \mathcal{F}_{F,C,E}(E)$. This can be either due to (a) there is an $a \in E$ which is not defended against attacker $c \in C$ in F or (b) there is a $c \in C \setminus E$, such that for all $a \in A(F)$ attacking c in F, $a \in E^+_F \cup (E^-_F \setminus C)$. In case (a) argument a is not defended by T in $F \cup H$, thus $T \notin com(F \cup H)$. For case (b), let a be an attacker of c in F. If $a \in E^+_F$, we also have $a \in T^+_{F \cup H}$; otherwise a attacks some $b \in E$ and since $E \subseteq T$ and $T \in com(F \cup H)$, there must exist some $(t, a) \in R(H)$ with $t \in T$. Hence, also in this case $a \in T^+_{F \cup H}$. Hence, all attackers of c in F are attacked by T in $F \cup H$, i.e., c is defended by T in $F \cup H$. Again, we observe that $T \notin com(F \cup H)$, thus in both cases we have a contradiction to $T \in com(F \cup H)$. $\square$

4.1.5 C-restricted Grounded Semantics

Our C-restricted version of grounded semantics also makes use of the C-restricted characteristic functions. That is, the C-restricted grounded extensions are characterized as least conflict-free fixed-points of the C-restricted characteristic function. Notice that, in contrast to standard grounded semantics, grd_C is not a unique status semantics.

Definition 10. *Let F be an AF, $C \subseteq U$ and $E \subseteq A(F)$. We define*

- $E \in grd_C(F)$ *if* $E \in cf(F)$ *and* $E = \mathcal{F}^{\infty}_{F,C,E}(\emptyset)$.

Example 7. *For the AF F^G_{abc} from the introduction as well as for the AF F from Example 1, the C-restricted complete and the C-restricted grounded extensions coincide, i.e., $grd_C(F^G_{abc}) = grd_C(F) = \{\emptyset, \{d, b\}\}$. In case, we add an attack (b, a) to those frameworks, $\{b\}$ would become a C-restricted complete extension, but $\{b\}$ is not C-restricted grounded.*

We next show that $grd_C(F)$ exactly characterizes the sets of arguments that can be extended to a grounded extension of some expansion $F \cup H$ of F with H over $U \setminus C$. With some abuse of notation, we occasionally shall use $grd(F)$ to denote the unique grounded extension of F.

Lemma 5. *Let F be an AF, $C \subseteq U$, and $E \subseteq A(F)$. It holds that $E \in grd_C(F)$ iff there exists an AF H over $U \setminus C$ such that $grd(F \cup H) \cap A(F) = E$.*

Proof. $\Rightarrow$: We use the same construction as in the proof of Lemma 4. Let $B = A(F) \setminus (E \cup C)$ and consider

$$H = (\{t\} \cup B, \{(t, b) \mid b \in E^-_F \setminus C\} \cup \{(b, b) \mid b \in B\})$$

with $t \in U \setminus C$ a fresh argument (not occurring in F). Clearly H is given over $U \setminus C$. Let $S = E \cup \{t\}$. We show that $grd(F \cup H) = S$.

- $S \subseteq grd(F \cup H)$: $t \in grd(F \cup H)$ as t is not attacked at all. For the remaining arguments we show that $a \in \mathcal{F}^i_{F,C,E}(\emptyset)$ implies $a \in \mathcal{F}^{i+1}_{F \cup H}(\emptyset)$. As base case let $i = 1$. Then a is either not attacked or all attackers of a are not contained in C. In the first case, a remains unattacked in $F \cup H$ and thus $a \in \mathcal{F}^1_{F \cup H}(\emptyset)$; otherwise $a \in \mathcal{F}^2_{F \cup H}(\emptyset)$, since all attackers are attacked by t and $t \in \mathcal{F}^1_{F \cup H}(\emptyset)$ by definition of H. Assume now $a \in \mathcal{F}^j_{F,C,E}(\emptyset)$ implies $a \in \mathcal{F}^{j+1}_{F \cup H}(\emptyset)$ holds for $j < i$ and let $a \in \mathcal{F}^i_{F,C,E}(\emptyset)$. Hence, all attackers of a from C are attacked by $\mathcal{F}^{i-1}_{F,C,E}(\emptyset)$. By induction hypothesis, all attackers of a are attacked by $\mathcal{F}^i_{F \cup H}(\emptyset)$. Hence, $a \in \mathcal{F}^{i+1}_{F \cup H}(\emptyset)$.

- We show $S \supseteq grd(F \cup H)$ by induction. For the induction base we show $\mathcal{F}^1_{F \cup H}(\emptyset) \subseteq S$. Let $a \in \mathcal{F}^1_{F \cup H}(\emptyset)$ then a has no attacker in $F \cup H$ (and thus in F) and therefore $a \in S$. For the induction step assume $\mathcal{F}^i_{F \cup H}(\emptyset) \subseteq S$. Consider $a \in \mathcal{F}^{i+1}_{F \cup H}(\emptyset)$. If $a \in A(F) \setminus C$ then $a \in E$ as all the other arguments are self-attacking in $F \cup H$. Thus consider $a \in C \cap A(F)$. By the definition of $\mathcal{F}_{F \cup H}$ the argument a is defended against all attackers by $\mathcal{F}^i_{F \cup H}(\emptyset) \subseteq S$ in $F \cup H$. Thus a is also defended by E in $F \cup H$ (since there is no attack (t, a) in $F \cup H$) and moreover $a \in \mathcal{F}_{F,C,E}(E)$. Now, as $E \in grd_C(F)$ and $a \in C$ we have $a \in E$.

$\Leftarrow$: To this end let $S = grd(F \cup H)$ and $E = grd(F \cup H) \cap A(F)$. First of all as S is conflict-free also E is conflict-free. We show that $E = \mathcal{F}^{\infty}_{F,C,E}(\emptyset)$.

- $E \supseteq \mathcal{F}^{\infty}_{F,C,E}(\emptyset)$: By definition of $\mathcal{F}^{\infty}_{F,C,E}$ only arguments in C and E are added to the set. Moreover arguments in C are only added iff they are defended by arguments in E when assuming that all arguments in $E^-_F \setminus C$ are attacked. As H does not add any additional attacks against arguments in C, and S is admissible in $F \cup H$ and thus attacks each argument in $E^-_F \setminus C$, such an argument is also defended in $F \cup H$ and thus contained in S and E respectively.

- $E \subseteq \mathcal{F}^{\infty}_{F,C,E}(\emptyset)$: By induction. Clearly $\mathcal{F}^{0}_{F\cup H}(\emptyset) \cap A(F) \subseteq \mathcal{F}^{\infty}_{F,C,S}(\emptyset)$. Now assume $\mathcal{F}^{i-1}_{F\cup H}(\emptyset) \cap A(F) \subseteq \mathcal{F}^{\infty}_{F,C,S}(\emptyset)$ and consider $a \in \mathcal{F}^{i}_{F\cup H}(\emptyset) \cap A(F)$. As $a \in E$ it is defended against all attackers from C in $F \cup H$ and this can be only because of arguments in $\mathcal{F}^{i-1}_{F\cup H}(\emptyset) \cap A(F)$ (H does not add any additional attacks against arguments in C). Thus a is also defended against C by $\mathcal{F}^{\infty}_{F,C,E}(\emptyset)$ in F and hence $a \in \mathcal{F}^{\infty}_{F,C,E}(\emptyset)$.

□

4.2 Properties of C-restricted semantics

In this section we will first give results on properties of C-restricted semantics and summarize properties all the C-restricted semantics have in common. Then, we give the first necessary conditions for two AFs to be C-relativized equivalent which happen to be the same for all semantics. This necessary conditions will later also appear as part of our full equivalence characterizations of all semantics under consideration.

Notice that in case C contains all arguments of an AF, C-restricted semantics as given in Defs. 6 – 10 reduce to the original semantics, while for empty C they reduce to conflict-free sets.

Observation 2. *Let $\sigma \in \{stb, adm, prf, com, grd\}$.*

- *For any set $C \subseteq U$ and AF F with $A(F) \subseteq C$, we have $\sigma_C(F) = \sigma(F)$.*
- *For any set $C \subseteq U$ and AF F with $A(F) \cap C = \emptyset$, we have $\sigma_C(F) = cf(F)$.*

Another crucial feature of C-restricted semantics is that $\sigma_C(F)$ returns all the argument sets that are projections of σ-extensions in some $F \cup H$ with H defined over $U \setminus C$.

Proposition 2. *Let F be an AF, $\sigma \in \{stb, adm, com, grd, prf\}$, $C \subseteq U$, and $E \subseteq A(F)$. Then, $E \in \sigma_C(F)$ iff there exists an AF H over $U \setminus C$ and $T \in \sigma(F \cup H)$ such that $T \cap A(F) = E$.*

Proof. Immediate by Lemmas 1 – 5. □

The proposition above establishes a close relationship between C-restricted semantics and the *enforcing problem* [7]. More precisely, the C-restricted σ-extensions E are exactly the sets enforceable without touching the core arguments, i.e., for any E there exist an C-neutral modification of the initial AF s.t. E becomes a subset of a σ-extension in the resulting framework. Moreover, with Proposition 2 we can show the C-restricted semantics relate to each other as in the standard case.

Proposition 3. *Let F be an AF and $C \subseteq U$. Then, the following relations hold: $stb_C(F) \subseteq prf_C(F) \subseteq com_C(F) \subseteq adm_C(F)$; $grd_C(F) \subseteq com_C(F)$.*

Proof. Let $\sigma, \theta \in \{stb, adm, com, grd, prf\}$ with $\sigma(F) \subseteq \theta(F)$ for any AF F and $E \in \sigma_C(F)$. We show $E \in \theta_C(F)$. By Proposition 2, there exists an AF H over $U \setminus C$ and $T \in \sigma(F \cup H)$ such that $T \cap A(F) = E$. By the assumption that $T \in \theta(F \cup H)$ and applying Proposition 2 in the other direction, $T \cap A(F) = E \in \theta_C(F)$. □

Next we consider properties that will appear in the C-relativized equivalence characterizations of all semantics $\sigma \in \{stb, adm, com, grd, prf\}$.

Proposition 4. *Let* $\sigma \in \{stb, adm, com, grd, prf\}$. *If* $F \equiv_C^\sigma G$ *then* $A(F) \setminus C = A(G) \setminus C$ *or* $\sigma_C(F) = \sigma_C(G) = \emptyset$.

Proof. We proof this separately for each of the considered semantics. Notice that for $\sigma \neq stb$ the statement simplifies to $F \equiv_C^\sigma G$ implies $A(F) \setminus C = A(G) \setminus C$.

Stable Semantics: If $stb_C(F) = \emptyset$, $stb(F \cup H) = \emptyset$ for all H over $U \setminus C$ by Lemma 1. Thus, by $F \equiv_C^{stb} G$ and Lemma 1, also $stb_C(G) = \emptyset$. Now suppose $\emptyset \neq stb_C(F)$ (and thus $\emptyset \neq stb_C(G)$) and $A(F) \setminus C \neq A(G) \setminus C$. W.l.o.g. there is an $a \in A(F) \setminus C$ and $a \notin A(G)$. We show $F \not\equiv_C^{stb} G$. First, assume there is $E \in stb_C(F)$ with $a \in E$. Then by Lemma 1 we can give an AF H such that there is a $T \in stb(F \cup H)$ with $T \cap A(F) = E$. Inspecting the proof of Lemma 1 shows, that H can be given without arguments from E. Thus $a \notin A(G \cup H)$ and hence $T \notin stb(G \cup H)$, i.e. $F \not\equiv_C^{stb} G$.

Thus assume there is no $E \in stb_C(F)$ with $a \in E$ and let $E \in stb_C(G)$. By Lemma 1, there is an AF H such that there is a $T \in stb(G \cup H)$ with $T \cap A(G) = E$ and we can build this H such that it does not contain argument a. Now we have that $T \cup \{a\} \in stb(G \cup H \cup (\{a\}, \{\}))$. For $F \cup H \cup (\{a\}, \{\}) = F \cup H$, we observe that it cannot be that $T \cup \{a\} \in stb(F \cup H)$ as this, by Lemma 1, would give rise to an $E \in stb_C(F)$ with $a \in E$; a contradiction to the assumption that no $E \in stb_C(F)$ with $a \in E$.

Admissible Semantics: Suppose $A(F) \setminus C \neq A(G) \setminus C$ and w.l.o.g. let $a \in A(F) \setminus C$ such that $a \notin A(G)$. First, if there is an $E \in adm_C(F)$ with $a \in E$ then by Lemma 2 we can give an H such that there is a $T \in adm(F \cup H)$ with $T \cap A(F) = E$. Notice that the H constructed in the proof of Lemma 2 does not contain arguments from E. Thus $a \notin A(G \cup H)$ and thus $T \notin adm(G \cup H)$, yielding $F \not\equiv_C^{adm} G$.

Thus for the remainder of the proof we can assume there is no $E \in adm_C(F)$ with $a \in E$. Consider $E \in adm_C(G)$ and let $B = A(G) \setminus (E \cup C)$. We construct

$$H = (\{a, t\} \cup B,\ \{(t, b) \mid b \in E_G^- \setminus C\} \cup \{(b, b) \mid b \in B\})$$

with $t \in U \setminus C$ a fresh argument (not occurring in F or G). Clearly H is given over $U \setminus C$. By similar arguments to Lemma 2, it can be shown that $E \cup \{a, t\}$ is an admissible set of $G \cup H$ (note that a has no relation to other arguments in $G \cup H$). On the other hand, we have to show that $E \cup \{a, t\}$ is not admissible in $F \cup H$. Towards a contradiction, suppose $E \cup \{a, t\}$ is admissible in $F \cup H$. Then, by Lemma 2, $D = (E \cup \{a, t\}) \cap A(F) \in adm_C(F)$ which is in contradiction to our observation that there is no $D \in adm_C(F)$ with $a \in D$ (recall that $a \in A(F)$).

Preferred Semantics: Suppose $A(F) \setminus C \neq A(G) \setminus C$. W.l.o.g. let $a \in A(F) \setminus C$ and $a \notin A(G)$. First, if there is an $E \in prf_C(F)$ with $a \in E$ then by Lemma 3 we can give an AF H such that there is a $T \in prf(F \cup H)$ with $T \cap A(F) = E$. Notice that H as constructed in the proof of Lemma 3 does not contain arguments from E. Thus $a \notin A(G \cup H)$ and $T \notin adm(G \cup H)$, hence $F \not\equiv_C^{prf} G$.

Thus assume there is no $E \in prf_C(F)$ with $a \in E$. Let $E \in prf_C(G)$ be subset-maximal in $prf_C(G)$ Let $B = A(G) \setminus (E \cup C)$ and consider

$$H = (\{a\} \cup B, \{(a, b) \mid b \in B\}).$$

Clearly, H is given over $U \setminus C$. By the above $E \cup \{a\} \notin \mathit{prf}_C(F)$ and thus, by Lemma 3, $E \cup \{a\} \notin \mathit{prf}(F \cup H)$. However, $E \cup \{a\} \in \mathit{adm}(G \cup H)$. It remains to show $E \cup \{a\} \in \mathit{prf}(G \cup H)$. Suppose there exists a $D \supset (E \cup \{a\})$. By Lemma 3, $D' = D \setminus \{a\} \in \mathit{prf}_C(G)$. Since $D' \supset E$ this is in contradiction to the assumption that E is subset-maximal in $\mathit{prf}_C(G)$.

Complete Semantics: Suppose $A(F) \setminus C \neq A(G) \setminus C$ and w.l.o.g. let $a \in A(F) \setminus C$ such that $a \notin A(G)$. First, if there is an $E \in \mathit{com}_C(F)$ with $a \in E$ then by Lemma 4 we can give an H such that there is a $T \in \mathit{com}(F \cup H)$ with $T \cap A(F) = E$. Notice that the H constructed in the proof of Lemma 4 does not contain arguments from E. Thus $a \notin A(G \cup H)$ and thus $T \notin \mathit{com}(G \cup H)$. We conclude $F \not\equiv_C^{com} G$.

Thus, assume there is no $E \in \mathit{com}_C(F)$ with $a \in E$. Now consider $E \in \mathit{com}_C(G)$ and let $B = A(G) \setminus (E \cup C)$. We construct

$$H = (\{t, a\} \cup B,\ \{(t, b) \mid b \in E_G^- \setminus C\} \cup \{(b, b) \mid b \in B\})$$

with $t \in U \setminus C$ a fresh argument (not occurring in F or G). Clearly H is given over $U \setminus C$. By similar arguments to Lemma 4, it can be shown that $E \cup \{a, t\}$ is a complete extension of $G \cup H$ (note that a has no relation to other arguments in $G \cup H$). On the other hand, we have to show that $E \cup \{a, t\}$ is not a complete extension of $F \cup H$. Towards a contradiction, suppose $E \cup \{a, t\}$ is a complete extension of $F \cup H$ then by Lemma 4, $D = (E \cup \{a, t\}) \cap A(F) \in \mathit{com}_C(F)$ which is in contradiction to our observation that there is no $D \in \mathit{com}_C(F)$ with $a \in D$ (recall that $a \in A(F)$).

Grounded Semantics: Suppose $A(F) \setminus C \neq A(G) \setminus C$. W.l.o.g. let $a \in A(F) \setminus C$ and $a \notin A(G)$. First, if there is an $E \in \mathit{grd}_C(F)$ with $a \in E$ then by Lemma 5 we can give an H such that $E \subseteq \mathit{grd}(F \cup H)$. Notice that the H constructed in the proof of Lemma 5 does not contain arguments from E. Thus $a \notin A(G \cup H)$ and $a \notin \mathit{grd}(G \cup H)$, yielding $F \not\equiv_C^{grd} G$.

Now, assume there is no $E \in \mathit{grd}_C(F)$ with $a \in E$. Let $E \in \mathit{grd}_C(F)$, $B = A(G) \setminus (E \cup C)$ and consider

$$H = (\{t, a\} \cup B,\ \{(t, b) \mid b \in E_G^- \setminus C\} \cup \{(b, b) \mid b \in B\})$$

with $t \in U \setminus C$ a fresh argument (not occurring in F or G). Clearly H is given over $U \setminus C$. First, the argument a is not in the grounded extension of $F \cup H$, as, by Lemma 5, this would imply that there is a $D \in \mathit{grd}_C(F)$ with $a \in D$. However, one can easily show that a is in the grounded extension of $G \cup H$ (note that a has no relation to other arguments in $G \cup H$). □

Next we obtain that two AFs can only be C-relativized equivalent w.r.t. one of our semantics σ if the σ_C semantics coincides on the two AFs.

Proposition 5. *If $F \equiv_C^\sigma G$ then $\sigma_C(F) = \sigma_C(G)$.*

Proof. Suppose $F \equiv_C^\sigma G$ holds. We show $\sigma_C(F) \subseteq \sigma_C(G)$; $\sigma_C(F) \supseteq \sigma_C(G)$ follows by symmetry.

By Proposition 2, for each $E \in \sigma_C(F)$ there is an H over $U \setminus C$ and $T \in \sigma(F \cup H)$, such that $T \cap A(F) = E$. By assumption $T \in \sigma(G \cup H)$ and, by Proposition 2, $E' = T \cap A(G) \in \sigma_C(G)$. As $A(H) \cap C = \emptyset$, we have that $T \cap C = E \cap C = E' \cap C$, and, by Proposition 4, $E \setminus C = E' \setminus C$. Thus, $E = E' \in \sigma_C(G)$. □

4.3 Characterizations

In the following we give the characterizations for all semantics under consideration. We already have seen that two AFs can only be C-relativized equivalent w.r.t. σ if $A(F) \setminus C = A(G) \setminus C$ (or $\sigma_C(F) = \emptyset$) and $\sigma_C(F) = \sigma_C(G)$. Now depending on the concrete semantics we have to appoint additional conditions for the sets $E \in \sigma_C(F)$ to ensure that they appear in the same expansions of F and G.

4.3.1 Stable Semantics

For stable semantics we require for each $E \in stb_C(F)$ that the range of E coincides in F and G outside of C. That is, the arguments that have to be attacked by H to make E stable in $F \cup H$ coincide with the arguments that have to be attacked by H to make E stable in $G \cup H$.

Theorem 1. *Let F, G be AFs and $C \subseteq U$. Then, $F \equiv_C^{stb} G$ iff the following conditions jointly hold:*

(1) if $stb_C(F) \neq \emptyset$, $A(F) \setminus C = A(G) \setminus C$;

(2) $stb_C(F) = stb_C(G)$; and

(3) for all $E \in stb_C(F)$, $E_F^+ \setminus C = E_G^+ \setminus C$.

Proof Sketch. $\Rightarrow$: The conditions (1) and (2) are immediate by Proposition 4 and Proposition 5. Now let $stb_C(F) = stb_C(G) \neq \emptyset$ and $A(F) \setminus C = A(G) \setminus C$, and assume, towards a contradiction that there is an $E \in stb_C(F)$ such that $E_F^+ \setminus C \neq E_G^+ \setminus C$. W.l.o.g. let $a \in E_G^+ \setminus C$ such that $a \notin E_F^+ \setminus C$. Hence $a \in A(G)$, and since $A(F) \setminus C = A(G) \setminus C$, $a \in A(F)$. Moreover, since $E \in stb_C(G)$, E is conflict-free in G, and thus $a \notin E$. Consequently, $a \in A(F) \setminus E_F^{\oplus}$ and $a \notin A(G) \setminus E_G^{\oplus}$. Let

$$H = (\{t\} \cup A(G) \setminus E_G^{\oplus}, \{(t, b) \mid b \in A(G) \setminus E_G^{\oplus})\})$$

where t is fresh argument from $U \setminus C$ not occurring in F or G. Observe that H does not contain arguments from C since $E \in stb_C(G)$ and thus each $a \in C$ occurring in G is attacked by E. We show $E \cup \{t\} \in stb(G \cup H)$. As is easily verified $E \cup \{t\}$ is conflict-free in $G \cup H$ (E is conflict-free in G, since $E \in stb_C(G)$; t is only linked to arguments not in $E_G^{\oplus}$); moreover each argument a from $G \cup H$ that is different from $E \cup \{t\}$ is attacked either by E or t by construction. On the other hand, $E \cup \{t\} \notin stb(F \cup H)$, since neither t attacks a in $F \cup H$, nor E attacks a in $F \cup H$. Thus, we have a contradiction to $F \equiv_C^{stb} G$.

$\Leftarrow$: Suppose $F \not\equiv_C^{stb} G$. W.l.o.g. there is an AF H over $U \setminus C$ and a set S such that $S \in stb(F \cup H)$ but $S \notin stb(G \cup H)$. By Lemma 1, $E = S \cap A(F) \in stb_C(F)$. If now $E \notin stb_C(G)$ or $A(F) \setminus C \neq A(G) \setminus C$, we are done, i.e. condition (1) or (2) is already violated. So suppose $E \in stb_C(G)$, and $A(F) \setminus C = A(G) \setminus C$. We have to show $E_F^+ \setminus C \neq E_G^+ \setminus C$. Recall that $S \notin stb(G \cup H)$. Since $E \in stb_C(G)$ there exists an $a \in A(F \cup H) \setminus C$ not attacked by S in $G \cup H$, thus in particular $a \notin E_G^+$. Since S does not attack a via H and $S \in stb(F \cup H)$ we conclude that either $a \in E_F^+$ or $a \notin A(F)$. However, since $a \notin C$ and $A(F) \setminus C = A(G) \setminus C$, it follows that $a \in E_F^+$. Hence, condition (3) is violated. $\square$

Example 8. *Recall F from Example 1 and let $F' = F^G_{ab} \cup (\{a, d, e\}, \{(a, e), (e, e), (e, d)\})$, i.e. instead of the cycle through a, b, c present in F, we have just two arguments a, b where a attacks itself and b. For $C = \{a, b, c\}$, it is easily checked that F and F' satisfy all three conditions, i.e., we have $F \equiv^{stb}_C F'$. In fact, even for the AF $F'' = (\{a, b, d, e\}, \{(a, a), (a, e), (e, e), (e, d)\})$, i.e., F' without the attack from a to b, $F \equiv^{stb}_C F''$ holds.*

If we had $C = \{a, b\}$, condition (1) would be violated; indeed $F \not\equiv^{stb}_C F'$ is then witnessed by adding $H = (\{c, e, t\}, \{(t, e)\})$, as $stb(F \cup H) = \{\{t, d, b\}\}$ and $stb(F' \cup H) = \{\{t, d, b, c\}\}$. On the other hand, for $C = \{a, b, c\}$, the role of b and c is indeed different: if we use in F' argument c instead of b, we have $stb_C(F') = \{\{d, c\}\}$; thus condition (2) would be violated. Finally, consider F''' given by F' plus an additional attack (b, e). Note that we still have $stb_C(F''') = \{\{d, b\}\}$, but now $E^+_F \setminus C \neq E^+_{F'''} \setminus C$, hence condition (3) is violated here. Even without expanding the AFs, we obtain different stable extensions, i.e., $stb(F) = \emptyset$ while $stb(F''') = \{\{d, b\}\}$. $\Diamond$

Remark 1. *When considering $C = \emptyset$ the above characterization boils down to (1) $A(F) = A(G)$, (2) $cf(F) = cf(G)$ and (3) for all $E \in cf(F)$, $E^+_F = E^+_G$. That is the two AFs F and G have to coincide except for attacks from self-attacking arguments, i.e., we end up with the concept of stable kernels from [23], which characterize strong equivalence for stb.*

For $C = A(F \cup G)$, only condition (2) remains which, in this case, is equivalent to $stb(F) = stb(G)$ (Observation 2), i.e. we obtain standard equivalence as expected.

4.3.2 Admissible Semantics

For adm semantics we have the additional condition that for each $E \in adm_C(F)$ the attackers of E that are not already attacked by E coincide in F and G.

Theorem 2. *Let F, G be AFs and $C \subseteq U$. Then, $F \equiv^{adm}_C G$ iff the following conditions jointly hold:*

(1) $A(F) \setminus C = A(G) \setminus C$;

(2) $adm_C(F) = adm_C(G)$; and

(3) for all $E \in adm_C(F)$, (3a) $E^+_F \setminus C = E^+_G \setminus C$ and (3b) $E^-_F \setminus E^+_F = E^-_G \setminus E^+_G$.

Proof. $\Rightarrow$: The conditions (1) and (2) are immediate by Proposition 4 and Proposition 5. Let now $A(F) \setminus C = A(G) \setminus C$ and $adm_C(F) = adm_C(G)$. Towards a contradiction suppose that there is an $E \in adm_C(F)$ such that either (3a) $E^+_F \setminus C \neq E^+_G \setminus C$ or (3b) $E^-_F \setminus E^+_F \neq E^-_G \setminus E^+_G$.

- Suppose $E^+_F \setminus C \neq E^+_G \setminus C$. W.l.o.g. let $a \in E^+_G \setminus C$ such that $a \notin E^+_F \setminus C$. Hence $a \in A(G)$, and since $A(F) \setminus C = A(G) \setminus C$, $a \in A(F)$. Moreover, since E is conflict-free in G (due to $E \in adm_C(G)$), $a \notin E$. We use the abbreviations $A'_F = A(F) \setminus (E^\oplus_F \cup C)$ and $A'_G = A(G) \setminus (E^\oplus_G \cup C)$. Hence, $a \in A'_F$ and $a \notin A'_G$. In particular, we are ensured that $a \in E^\oplus_G$. Let
$$H = (\{s, t, a\} \cup A'_G, \{(t, b) \mid b \in A'_G\} \cup \{(a, s)\})$$
with $s, t \in U \setminus C$ be fresh arguments. Note that H is given over $U \setminus C$, by definition of A'_G and since $a \in A'_F$. We show that $E \cup \{t, s\} \in adm(G \cup H)$ while $E \cup \{t, s\} \notin adm(F \cup H)$:

- $E \cup \{t, s\} \in adm(G \cup H)$: conflict-freeness is obvious; t is unattacked in $G \cup H$; each attacker $b \in C$ of E is attacked by E in G (since $E \in adm_C(G)$); each other attacker of E is attacked by t in $G \cup H$; finally, the sole attacker a of s is attacked by E (by assumption $a \in E_G^{\oplus}$).
- $E \cup \{t, s\} \notin adm(F \cup H)$, since s is attacked by a which itself is neither attacked by E nor by s or t in $F \cup H$.

That is we have a contradiction to $F \equiv_C^{adm} G$.

- Suppose $E_F^- \setminus E_F^+ \neq E_G^- \setminus E_G^+$. W.l.o.g., let $a \in E_F^- \setminus E_F^+$ and $a \notin E_G^- \setminus E_G^+$. Consider

$$H = (\{t\} \cup E_G^- \setminus E_G^+, \{(t, b) \mid b \in E_G^- \setminus E_G^+\})$$

where t is a fresh argument from $U \setminus C$. Notice that as $E \in adm_C(G)$ the set $E_G^- \setminus E_G^+$ is disjoint from C and thus all arguments of H are drawn from $U \setminus C$. We then have that $E \cup \{t\} \in adm(G \cup H)$ while in $F \cup H$ the argument $a \in E_F^-$ is neither attacked by E nor by t and thus $E \cup \{t\} \notin adm(F \cup H)$. Hence, we have the desired contradiction to $F \equiv_C^{adm} G$.

$\Leftarrow$: Suppose $F \not\equiv_C^{adm} G$. W.l.o.g. there is an AF H over $U \setminus C$ and a set S such that $S \in adm(F \cup H)$ but $S \notin adm(G \cup H)$. Note that this implies $S \neq \emptyset$. By Lemma 2, $E = S \cap A(F) \in adm_C(F)$.

If now $E \notin adm_C(G)$ or $A(F) \setminus C \neq A(G) \setminus C$, we are done, i.e. condition (1) or (2) is already violated. So suppose $E \in adm_C(G)$, and $A(F) \setminus C = A(G) \setminus C$. We prove that (3) is violated, by showing that either (a) $E_F^+ \setminus C \neq E_G^+ \setminus C$ or (b) $E_F^- \setminus E_F^+ \neq E_G^- \setminus E_G^+$. We have that $S \notin adm(G \cup H)$ but, as $E \in adm_C(G)$ and $S \in adm(F \cup H)$, we have $S \in cf(G \cup H)$. Thus there exists an argument $b \in S_{G \cup H}^-$ such that $b \notin S_{G \cup H}^+$. As $S \in adm(F \cup H)$ we have that $b \notin A(H) \setminus A(F)$ and as moreover $E \in adm_C(F)$ we know that $b \in A(F) \setminus C$ and thus $b \in A(G) \setminus C$ by assumption that (2) already holds. Further, as $S \in adm(F \cup H)$, we have that either (i) $b \in S_{F \cup H}^+$ or (ii) $b \notin S_{F \cup H}^-$.

- In the former case: From $b \in S_{F \cup H}^+$ and $b \notin S_{G \cup H}^+$ we deduce that $b \in E_F^+$ and $b \notin E_G^+$. That is we satisfy (i) as b is contained in $E_F^+ \setminus C$ but not in $E_G^+ \setminus C$.
- In the latter case: From $b \notin S_{F \cup H}^-$ and $b \in S_{G \cup H}^-$ we deduce $b \notin E_F^-$ and $b \in E_G^-$. Moreover from $b \notin S_{G \cup H}^+$ we also have $b \notin E_G^+$. Thus we satisfy (ii) as b is not contained in $E_F^- \setminus E_F^+$ but in $E_G^- \setminus E_G^+$.

Hence we arrive at either (a) or (b) as desired. □

Example 9. *Let us first consider F, F' and F'' from Example 8, again with $C = \{a, b, c\}$. For F and F' it can be shown that all three conditions hold, i.e., $F \equiv_C^{adm} F'$. However, F'' is a too drastic simplification for admissible semantics, since $\{b\} \in adm_C(F'')$ but $\{b\} \notin adm_C(F)$.*

To show the role of condition (3b), consider the AFs $F_1 = F \cup (\{g\}, \{(g, g)\})$ and $F_2 = F \cup (\{g, b\}, \{(g, g), (g, b)\})$; conditions (1), (2), and (3a) are fulfilled. However, for $E = \{d, b\} \in adm_C(F_1)$, we have $E_{F_1}^- \setminus E_{F_1}^+ = \{e\}$, while $E_{F_2}^- \setminus E_{F_2}^+ = \{e, g\}$. Hence condition (3b) is violated, witnessed by the expansion $H = (\{t, e\}, \{(t, e)\})$, which yields $\{t, d, b\} \in adm(F_1 \cup H)$, but $\{t, d, b\} \notin adm(F_2 \cup H)$. ◊

Remark 2. *When considering $C = \emptyset$ the characterization of Theorem 2 simplifies to the characterization of strong equivalence as follows. We have that the C-restricted admissible sets (for $C = \emptyset$) are just the conflict-free sets. If we consider a singleton $\{a\}$ we thus have that either (i) a is self-attacking in both F and G (and thus not restricted admissible) or (ii) by (3a) it has exactly the same outgoing attacks in both F and G. By (3b) the attackers of an argument a can only differ by attacks that are counter-attacked by a. That is we can drop an attack only if it is either between two self-attacking arguments or from a self-attack against an argument that attacks back. That is, we get exactly the characterization of strong equivalence from [23].*

For $C = A(F \cup G)$, only conditions (2) and (3b) remain. In this case, (2) is equivalent to $adm(F) = adm(G)$ (cf. Observation 2) and (3b) trivially holds since each admissible extension defends itself, i.e., $E_F^- \setminus E_F^+ = E_G^- \setminus E_G^+ = \emptyset$ for all $E \in adm(F)$. We thus obtain standard equivalence as expected.

4.3.3 Preferred Semantics

The characterization for prf is very much like for adm, the only difference being that one considers $prf_C(\cdot)$ instead of $adm_C(\cdot)$. This similarity reflects the fact that $F \equiv_C^{prf} G$ whenever $F \equiv_C^{adm} G$.

Theorem 3. *Let F, G be AFs and $C \subseteq U$. Then, $F \equiv_C^{prf} G$ iff the following conditions jointly hold:*

(1) $A(F) \setminus C = A(G) \setminus C$;

(2) $prf_C(F) = prf_C(G)$; and

(3) for all $E \in prf_C(F)$, (3a) $E_F^+ \setminus C = E_G^+ \setminus C$ and (3b) $E_F^- \setminus E_F^+ = E_G^- \setminus E_G^+$.

Proof. $\Rightarrow$: The conditions (1) and (2) are immediate by Proposition 4 and Proposition 5. Let now $A(F) \setminus C = A(G) \setminus C$ and $prf_C(F) = prf_C(G)$, and towards a contradiction assume that there is an $E \in prf_C(F)$ such that either (a) $E_F^+ \setminus C \neq E_G^+ \setminus C$ or (b) $E_F^- \setminus E_F^+ \neq E_G^- \setminus E_G^+$.

- For $E_F^+ \setminus C \neq E_G^+ \setminus C$ we use, as in the proof of Theorem 2, $A'_F = A(F) \setminus (E_F^\oplus \cup C)$, $A'_G = A(G) \setminus (E_G^\oplus \cup C)$, and assume $a \in A'_F$ and $a \notin A'_G$. By assumption, we are ensured that $a \in E_G^\oplus$. Let

  $$\begin{aligned} H = &(\{s, t, a\} \cup A(F) \setminus (E \cup C), \\ &\{(t, b) \mid b \in E_G^- \setminus E_G^+\} \cup \{(b, t) \mid b \in E_G^+ \setminus C\} \cup \\ &\{(b, b) \mid b \in A(F) \setminus (E \cup C)\} \cup \{(a, s)\}) \end{aligned}$$

  with $s, t \in U \setminus C$ being fresh arguments. Note that H is given over $U \setminus C$, in particular since $a \in A'_F$ and since $E \in prf_C(G)$, $(E_G^- \setminus E_G^+) \cap C = \emptyset$.

  $E \cup \{t, s\}$ is admissible in $G \cup H$: conflict-freeness is obvious; t is only attacked by E_G^+ in $G \cup H$; each attacker $b \in C$ of E is attacked by E in G (since $E \in adm_C(G)$); each other attacker of E is attacked by t in $G \cup H$; finally, the sole attacker a of s is attacked by E (by assumption $a \in E_G^\oplus$).

Now consider the maximality of $E \cup \{t, s\}$. Towards a contradiction assume there is a preferred extension T of $G \cup H$ with $E \cup \{t, s\} \subset T$. By the construction we get that $E \cup \{t, s\} \setminus C = T \setminus C$ and thus also $E \cap C \subset T \cap C$. Let $D = T \cap A(G)$. As T defends t we have that $E_G^+ \setminus C \subseteq D_G^+ \setminus C$ and as T defends D also $E_G^- \setminus E_G^+ \supseteq D_G^- \setminus D_G^+$. Finally, as D is a C-restricted admissible set of G, we obtain a contradiction to the assumption that E is C-restricted preferred in G.

$E \cup \{t, s\}$ cannot be admissible in $F \cup H$, since s is attacked by a which itself is neither attacked by E nor by s or t in $F \cup H$. We are in contradiction to $F \equiv_C^{prf} G$.

- Suppose $E_F^- \setminus E_F^+ \neq E_G^- \setminus E_G^+$. W.l.o.g., let $a \in E_F^- \setminus E_F^+$ and $a \notin E_G^- \setminus E_G^+$. Consider

$$\begin{aligned} H = &(\{t\} \cup A(G) \setminus (E \cup C), \\ &\{(t, b) \mid b \in E_G^- \setminus E_G^+\} \cup \{(b, t) \mid b \in E_G^+ \setminus C\} \cup \\ &\{(b, b) \mid b \in A(G) \setminus (E \cup C)\}) \end{aligned}$$

where t is a fresh argument from $U \setminus C$. By similar observations as before,H is given over $U \setminus C$. We then have that $E \cup \{t\} \in adm(G \cup H)$ and we next show that also $E \cup \{t\} \in prf(G \cup H)$. Towards a contradiction assume there is a preferred extension T of $G \cup H$ with $E \cup \{t\} \subset T$. By the construction we get that $(E \cup \{t\}) \setminus C = T \setminus C$ and thus also $E \cap C \subset T \cap C$. Let $D = T \cap A(G)$. We have $E \cap C \subset D \cap C$. As T defends t we have that $E_G^+ \setminus C \subseteq D_G^+ \setminus C$ and as T defends D also $E_G^- \setminus E_G^+ \supseteq D_G^- \setminus D_G^+$. Finally, as D is a C-restricted admissible set of G, we obtain a contradiction to the assumption that E is C-restricted preferred in G. In $F \cup H$ the argument $a \in E_F^-$ is neither attacked by E nor by t and thus $E \cup \{t\} \notin adm(F \cup H)$. Hence, we have the desired contradiction to $F \equiv_C^{prf} G$.

$\Leftarrow$: Suppose $F \not\equiv_C^{prf} G$. Then there is an AF H over $U \setminus C$ such that $prf(F \cup H) \neq prf(G \cup H)$. W.l.o.g. there is a $S \in prf(F \cup H)$ such that $S \notin adm(G \cup H)$.[2] Note that this implies $S \neq \emptyset$. By Lemma 3, $E = S \cap A(F) \in prf_C(F)$.

If now $E \notin prf_C(G)$ or $A(F) \setminus C \neq A(G) \setminus C$, we are done, i.e. condition (1) or (2) is already violated. So suppose $E \in prf_C(G)$, and $A(F) \setminus C = A(G) \setminus C$. We show that (3) is violated as well, i.e. either (a) $E_F^+ \setminus C \neq E_G^+ \setminus C$ or (b) $E_F^- \setminus E_F^+ \neq E_G^- \setminus E_G^+$ holds. We have that $S \notin adm(G \cup H)$ but, as $E \in adm_C(G)$ and $S \in adm(F \cup H)$, we have $S \in cf(G \cup H)$. Thus there exists an argument $b \in S_{G \cup H}^-$ such that $b \notin S_{G \cup H}^+$. As $S \in prf(F \cup H)$ we have that $b \notin A(H) \setminus A(F)$ and as moreover $E \in prf_C(G)$ we know that $b \in A(F) \setminus C$. Further, as $S \in prf(F \cup H)$, we have that either (i) $b \in S_{F \cup H}^+$ or (ii) $b \notin S_{F \cup H}^-$.

- In the former case: From $b \in S_{F \cup H}^+$ and $b \notin S_{G \cup H}^+$ we deduce that $b \in E_F^+$ and $b \notin E_G^+$. That is we satisfy (i) as b is contained in $E_F^+ \setminus C$ but not in $E_G^+ \setminus C$).
- In the latter case: From $b \notin S_{F \cup H}^-$ and $b \in S_{G \cup H}^-$ we deduce $b \notin E_F^-$ and $b \in E_G^-$. Moreover from $b \notin S_{G \cup H}^+$ we also have $b \notin E_G^+$. Thus we satisfy (ii) as b is not contained in $E_F^- \setminus E_F^+$ but in $E_G^- \setminus E_G^+$.

[2] If S would be admissible in $G \cup H$ one would consider a preferred extension E in $G \cup H$ containing S, which then cannot be admissible in $F \cup H$.

Hence we either satisfy (a) or (b) as desired. $\square$

Remark 3. *When considering $C = \emptyset$ condition (2) of Theorem 3 reduces to $adm_C(F) = adm_C(G)$ (see Definition 8), i.e. we exactly have the same characterization than in Theorem 2. However, this is as expected since strong equivalence for admissible and preferred semantics coincide [23].*

For $C = A(F \cup G)$, only conditions (2) and (3b) remain. In this case, (2) is equivalent to $prf(F) = prf(G)$ (cf. Observation 2) and (3b) trivially holds since each preferred extension defends itself, i.e. $E_F^- \setminus E_F^+ = E_G^- \setminus E_G^+ = \emptyset$ for all $E \in prf(F)$. We obtain standard equivalence as expected.

4.3.4 Complete Semantics

For complete semantics we have all the conditions we had for admissible semantics, but also the additional condition (3c) that ensures that the same arguments are defended in $F \cup H$ and $G \cup H$, for all AFs H over $U \setminus C$.

Theorem 4. *Let F, G be AFs and $C \subseteq U$. Then, $F \equiv_C^{com} G$ iff the following conditions jointly hold:*

(1) $A(F) \setminus C = A(G) \setminus C$;

(2) $com_C(F) = com_C(G)$; and

(3) for all $E \in com_C(F)$,

  (3a) $E_F^+ \setminus C = E_G^+ \setminus C$,
  (3b) $E_F^- \setminus E_F^+ = E_G^- \setminus E_G^+$, and
  (3c) for all S with $E_F^- \setminus E_F^+ \subseteq S \subseteq A(F) \setminus (C \cup E)$, if $\mathcal{F}_{F \setminus S}(E) \cap C = E \cap C$ or $\mathcal{F}_{G \setminus S}(E) \cap C = E \cap C$ then $\mathcal{F}_{F \setminus S}(E) = \mathcal{F}_{G \setminus S}(E)$.

Proof. $\Rightarrow$: The conditions (1) and (2) are immediate by Proposition 4 and Proposition 5. Now towards a contradiction let us assume that either (3a), (3b), or (3c) is violated.

- If (3a) is violated then w.l.o.g. there is a set $E \in com_C(F)$ and an argument with $y \in E_F^+ \setminus C$ and $y \notin E_G^+ \setminus C$. Consider the following AF $H = (A_H, R_H)$ with

  $$A_H = \{t, x\} \cup A(F) \setminus (C \cup E)$$
  $$R_H = \{(t, s) \mid s \in E_F^- \setminus E_F^+\} \cup \{(a, a) \mid a \in A(F) \setminus (C \cup E)\} \cup \{(y, x)\}$$

  where t and x are fresh arguments. H is indeed given over $U \setminus C$, in particular since $E \in com_C(F)$ (and thus $E_F^- \setminus E_F^+$ does not contain arguments from C). Let $T = S \cup \{t, x\}$. We will show that (i) $T \in com(F \cup H)$ but (ii) $T \notin com(G \cup H)$ which is in contradiction to $F \equiv_C^{com} G$.

(i) $T \in com(F \cup H)$: First, T is admissible in $F \cup H$, as t is not attacked at all, y the only attacker of x is in E^+_F, and the remaining arguments are defended by the fact that t attacks all arguments in $E^-_F \setminus E^+_F$. An argument $a \in A(F) \setminus (C \cup E)$ can not be defended as it is self-attacking by construction. If an argument $c \in C$ is defended by T in $F \cup H$ then E defends it against attackers from C and all the other attackers are in $E^-_F \setminus C$. Thus, as $E \in com_C(F)$, also $c \in E$.

(ii) $T \notin com(G \cup H)$: The argument $x \in T$ is attacked by $y \in A(G) \setminus C$. As $y \in E^+_F$, by construction of H, we have $y \notin \{t\}^+_{F \cup H}$ and as $y \notin E^+_G \setminus C$ we have that y is not attacked by T in $G \cup H$. Hence T is not even admissible in $G \cup H$.

- If (3b) is violated then w.l.o.g.there is a set $E \in com_C(F)$ and an argument $a \in E^-_F \setminus E^+_F$ such that $a \notin E^-_G \setminus E^+_G$. Consider the following AF $H = (A_H, R_H)$ over $U \setminus C$ (t is a fresh argument):

$$A_H = \{t\} \cup A(F) \setminus (C \cup E)$$
$$R_H = \{(t, s) \mid s \in E^-_G \setminus E^+_G\} \cup \{(a, a) \mid a \in A(F) \setminus (C \cup E)\}$$

Let $T = E \cup \{t\}$. We will show that (i) $T \notin com(F \cup H)$ but (ii) $T \in com(G \cup H)$, which is in contradiction to $F \equiv^{com}_C G$.

(i) $T \notin com(F \cup H)$: T is not admissible in F$\cup H$ as it does not attack a but $a \in E^-_F$.

(ii) $T \in com(G \cup H)$: First, T is admissible in $G \cup H$, as t is not attacked at all and the remaining arguments are defended by the fact that t attacks all arguments in $E^-_G \setminus E^+_G$. Now consider an argument in $a \in A(G) \setminus C$. An argument $a \in A(G) \setminus C$ can only be defended by T if it is in E, as all the other arguments are self-attacking. If an argument $c \in C$ is defended by T in $G \cup H$ then E defends it against attackers from C and all the other attackers are in $E^-_G \setminus C$. Thus, as $E \in com_C(G)$, also $c \in E$.

- If (3c) is violated then there are sets E, S with $E \in com_C(F)$ and $E^-_F \setminus E^+_F \subseteq S \subseteq A(F) \setminus (E \cup C)$ such that $\mathcal{F}_{F \setminus S}(E) \cap C = E \cap C$ and $\mathcal{F}_{F \setminus S}(E) \neq \mathcal{F}_{G \setminus S}(E)$.

First consider the case where $\mathcal{F}_{G \setminus S}(E) \cap C \neq E \cap C$. Consider the following AF $H = (A_H, R_H)$ over $U \setminus C$ with

$$A_H = \{t\} \cup A(F) \setminus (C \cup E)$$
$$R_H = \{(t, s) \mid s \in S\} \cup \{(a, a) \mid a \in A(F) \setminus (C \cup E)\}$$

where t is a fresh argument from $U \setminus C$. Let $T = E \cup \{t\}$. We will show that (i) $T \in com(F \cup H)$ but (ii) $T \notin com(G \cup H)$, which is in contradiction to $F \equiv^{com}_C G$.

(i) $T \in com(F \cup H)$: First, T is admissible in $F \cup H$, as t is not attacked at all and the remaining arguments are defended because t attacks all arguments in S and $S \supseteq E^-_F \setminus E^+_F$. If an argument $c \in C$ is defended by T in $F \cup H$ then E defends it against attackers from C and all attackers that are not in E^+_F are contained in S and thus attacked by t. Thus, $\mathcal{F}_{F \setminus S}(E) \cap C = E \cap C$. Moreover, an argument $a \in A(F) \setminus (C \cup E \cup \{t\})$ cannot be defended in $F \cup H$ as it is self-attacking. Hence T is a complete extension of $F \cup H$.

(ii) $T \notin com(G \cup H)$: As $E \in com_C(G)$ we have that $E \subseteq \mathcal{F}_{G \setminus S}(E)$. Thus, there is an argument $x \in \mathcal{F}_{G \setminus S}(E) \cap C$, such that $x \notin E$. But as $S \subseteq \{t\}^+_{G \cup H}$, the argument x is also defended by T in $G \cup H$ and thus T is not complete.

For the remainder we can assume that $\mathcal{F}_{F \setminus S}(E) \cap C = \mathcal{F}_{G \setminus S}(E) \cap C = E \cap C$. W.l.o.g. there is an argument $x \in A(F) \setminus E$ such that x is defended by E in $F \setminus S$ but not in $G \setminus S$. By assumption thus $x \notin C$. Consider the following AF $H = (A_H, R_H)$ with

$$
\begin{aligned}
A_H =& \{t\} \cup A(F) \setminus (C \cup E) \\
R_H =& \{(t, s) \mid s \in S\} \cup \{(a, a) \mid a \in A(F) \setminus (C \cup E \cup \{x\})\}.
\end{aligned}
$$

The AF is identical to H from before, but removes the self-attack from x. Let $T = E \cup \{t\}$. We will show that (i) $T \notin com(F \cup H)$ but (ii) $T \in com(G \cup H)$, which is in contradiction to $F \equiv^{com}_C G$.

(i) $T \notin com(F \cup H)$: We argue that the argument $x \notin T$ is defended by T. As x is defended in $F \setminus S$ all attackers are either in E^+_F or in S. As $E \subset T$ also $E^+_F \subseteq T^+_{F \cup H}$ and as $t \in T$ attacks all arguments in S also $S \subseteq T^+_{F \cup H}$. That is T defends $x \notin T$ and is thus not complete in $F \cup H$.

(ii) $T \in com(G \cup H)$: First, T is admissible in $G \cup H$, as t is not attacked at all and the remaining arguments are defended because t attacks all arguments in S and $S \supseteq E^-_G \setminus E^+_G$ (we can assume $E^-_F \setminus E^+_F = E^-_G \setminus E^+_G$, otherwise violation of (3b) applies, which we have already dealt with). If an argument $c \in C$ is defended by T in $G \cup H$ then E defends it against attackers from C and all the other attackers are in S. Thus, $c \in \mathcal{F}_{G \setminus S}(E) \cap C = E \cap C$. Moreover an argument $a \in A(F) \setminus (C \cup E \cup \{x\})$ cannot be defended in $G \cup H$ as it is self-attacking. Finally consider the argument x. As x is not defended in $G \setminus S$ there is an argument $y \in A(G) \setminus S$ attacking x and $y \notin E^+_G$. But then also $y \notin T^+_{G \cup H}$ and thus x is not defended by T in $G \cup H$. Hence T is a complete extension of $G \cup H$.

Thus in all three cases we have a contradiction to our initial assumption that $F \equiv^{com}_C G$.

$\Leftarrow$: Towards a contradiction assume that $F \not\equiv^{com}_C G$. Then there exists an AF H such that $com(F \cup H) \neq com(G \cup H)$. W.l.o.g. there is an $T \in com(F \cup H)$ with $T \notin com(G \cup H)$. By Lemma 4 we have $E = T \cap A(F) \in com_C(F)$. Now if (1) or (2) are violated we are done. Thus let us for the remainder of this proof assume $com_C(F) = com_C(G)$ and $A(F) \setminus C = A(G) \setminus C$.

As $T \notin com(G \cup H)$ either (i) there is an argument $a \in T$ that is not defended by T in $G \cup H$ or (ii) there is an argument $a \in A(G \cup H) \setminus T$ that is defended by T in $G \cup H$.

(i) If $a \in T \setminus E$ then $\{a\}^-_{F \cup H} = \{a\}^-_{G \cup H} = \{a\}^-_H$ and $\{a\}^-_H \cap C = \emptyset$. Now as $T \in com(F \cup H)$ we have that $\{a\}^-_H \subseteq T^+_{F \cup H}$ and as $T \notin com(G \cup H)$ there is an $x \in \{a\}^-_H$ with $x \notin T^+_{F \cup H}$. That is $x \in E^+_F \setminus C$ but $x \notin E^+_G \setminus C$, i.e., condition (3a) is violated.

Now consider $a \in E$. There is an argument b attacking a with $b \notin T^+_{G \cup H}$. We obtain that also $b \notin E^+_G$ and $b \notin (T \cap A(H))^+_H$. Notice, that as T is C-restricted complete in both F and G

the argument b cannot be contained in C. As T is defended in $F \cup H$ either $b \notin T^-_{F\cup H}$ or $b \in T^+_{F\cup H}$. In the former case we get $b \notin E^-_F$ and from the above we have $b \in \{a\}^-_G \setminus E^+_G$, i.e., $E^-_F \setminus E^+_F \neq E^-_G \setminus E^+_G$ and thus condition (3b) is violated. In the latter case we have $b \in T^+_{F\cup H}$ and as $b \notin (T \cap A(H))^+_H$ we get $b \in E^+_F$. Hence, as $b \notin C$ we have $E^+_F \setminus C \neq E^+_G \setminus C$ and thus condition (3a) is violated.

(ii) If $a \in A(H) \setminus A(F)$ then $\{a\}^-_{F\cup H} = \{a\}^-_{G\cup H} = \{a\}^-_H$ and $\{a\}^-_H \cap C = \emptyset$. Thus if a is not defended in $F \cup H$ but in $G \cup H$ there is a $b \in E^+_G$ which is not contained in E^+_F and $b \notin C$. Thus we violate condition (3a).

Hence, let us assume $a \in A(F)$ and let $S = (T \cap H)^+_H$. We show that $a \notin \mathcal{F}_{F\setminus S}(E) = E$ while $a \in \mathcal{F}_{G\setminus S}(E)$ and thus (3c) is violated.

- $a \notin \mathcal{F}_{F\setminus S}(E) = E$: First E is admissible in $F \setminus S$ as each attacker in $F \setminus S$ is also an attacker in $F \cup H$ and thus in E^+_F. Towards a contradiction assume there is an argument $b \in A(F \setminus S) \setminus E$ that is defended by E in $F \setminus S$. This argument b is then also defended by T in $F \cup H$ while $b \notin T$, a contradiction to T being complete. That is $\mathcal{F}_{F\setminus S}(E) = E$ and as by assumption $a \notin T \supset E$ the claim follows.
- $a \in \mathcal{F}_{G\setminus S}(E)$: Consider an argument $b \in A(G) \setminus S$ attacking a in $G \setminus S$. As b also attacks a on $G \cup H$ we have $b \in T^+_{G\cup H}$. Now as $b \notin S = (T \cap H)^+_H$ we have that $b \in E^+_{G\setminus S}$. That is, a is defended by E in $G \setminus S$.

□

Remark 4. *When considering $C = \emptyset$ then condition (2) of Theorem 4 reduces to $cf(F) = cf(G)$, which means that both AFs have (a) the same self-attacking arguments, and (b) the same conflicts between two arguments (with potentially different direction), except between self-attacking arguments. By (3a) if an argument a is not self-attacking then it has the same outgoing attacks in F and G (consider $E = \{a\}$). Now consider (3c), $E = \emptyset$, and an argument a that is not self-attacking such that $\{a\}^-_F \neq \{a\}^-_G$. W.l.o.g. there is a $b \in \{a\}^-_G$ with $b \notin \{a\}^-_F$. For $S = \{a\}^-_F$ we then have that $a \in \mathcal{F}_{F\setminus S}(E)$ but $a \notin \mathcal{F}_{G\setminus S}(E)$, a contradiction to (3c). Thus, if an argument is not self-attacking it has the same attackers in F and G. Hence, we can drop an attack only if it is between two self-attacking arguments. That is, we get exactly the characterization of strong equivalence from [23].*

For $C = A(F \cup G)$, only conditions (2), (3b) and (3c) remain. As before, (2) is equivalent to $com(F) = com(G)$ and (3b) trivially holds since each complete extension defends itself. (3c) only applies to $S = \emptyset$ where $\mathcal{F}_{F\setminus S}(E) = \mathcal{F}_{G\setminus S}(E)$ clearly holds in the light of $com(F) = com(G)$. Hence, we obtain standard equivalence for complete semantics.

4.3.5 Grounded Semantics

For the characterization of the grounded semantics we make use of the following variant of the characteristic function,

$$\mathcal{F}_{F,E}(S) = \{a \in E \mid S \text{ defends } a \text{ in } F\} \text{ for } E \subseteq A(F)$$

which allows to restrict the set of arguments that are tested for being defended.

When considering expansions $F \cup H$ and $G \cup H$ the crucial impacts on the grounded extensions are (a) the arguments attacked by new arguments that happen to be in the grounded extension of the expansion and (b) the arguments that are excluded from being in the grounded extensions by H, for instance, via self-attacks. To deal with (a), in our third condition, we test all sets S that might be attacked by H and then perform all tests for all C-restricted grounded sets E and compatible set S. To deal with (b) we use the $\mathcal{F}_{F,E}$ variant of the characteristic function that does not allow to add arguments that are not in E or C.

We then first check condition (3a), that, when assuming that arguments S are already disabled from outside and only arguments in $E \cup C$ can be defended, tests whether the modified AFs propose the same grounded extension. In case the proposed extension coincides with the tested C-restricted grounded set E we perform two further checks: (3b) Similar to the other semantics we check whether the range is the same in both frameworks, but excluding both C and the S from the range; and with condition (3c) we test whether the set E defends the same arguments in both modified AFs, again assuming that arguments S are attacked by H.

Theorem 5. *Let F, G be AFs and $C \subseteq U$. Then, $F \equiv_C^{grd} G$ iff the following holds:*

(1) $A(F) \setminus C = A(G) \setminus C$;

(2) $grd_C(F) = grd_C(G)$; *and*

(3) for all $E \in grd_C(F)$ and all $S \subseteq A(F) \setminus (C \cup E)$

(3a) $\mathcal{F}^{\infty}_{F \setminus S, E \cup C}(\emptyset) = \mathcal{F}^{\infty}_{G \setminus S, E \cup C}(\emptyset)$,

(3b) if $\mathcal{F}^{\infty}_{F \setminus S, E \cup C}(\emptyset) = E$ then $E^+_F \setminus (C \cup S) = E^+_G \setminus (C \cup S)$, and

(3c) if $\mathcal{F}^{\infty}_{F \setminus S, E \cup C}(\emptyset) = E$ then $\mathcal{F}_{F \setminus S}(E) = \mathcal{F}_{G \setminus S}(E)$.

Proof. $\Rightarrow$: The conditions (1) and (2) are immediate by Proposition 4 and Proposition 5. It remains to show (3a), (3b), and (3c):

(3a) Let $E \in grd_C(F)$, $S \subseteq A(F) \setminus (C \cup E)$, $B = A(G) \setminus (E \cup C)$ and consider the following AF

$$H = (\{t\} \cup B, \{(t, s) \mid s \in S\} \cup \{(b, b) \mid b \in B\})$$

with $t \in U \setminus C$ a fresh argument (not occurring in F or G). Clearly H is given over $U \setminus C$. By construction we have that (i) $\{t\} \cup \mathcal{F}^{\infty}_{F \setminus S, E \cup C}(\emptyset) = grd(F \cup H)$ and (ii) $\{t\} \cup \mathcal{F}^{\infty}_{G \setminus S, E \cup C}(\emptyset) = grd(G \cup H)$. By $F \equiv_C^{grd} G$ we have $grd(F \cup H) = grd(G \cup H)$ and obtain (3a).

(3b) Consider $E \in grd_C(F)$, $S \subseteq A(F) \setminus (C \cup E)$ with $\mathcal{F}^{\infty}_{F \setminus S, E \cup C}(\emptyset) = E$ and towards a contradiction suppose there is an $x \in E^+_F \setminus (C \cup S)$ with $x \notin E^+_G \setminus (C \cup S)$. Let $B = A(G) \setminus (E \cup C)$, and consider the following AF

$$H = (\{t, y\} \cup B, \{(t, s) \mid s \in S\} \cup \{(b, b) \mid b \in B\} \cup \{x, y\})$$

with $t, y \in U \setminus C$ being fresh arguments (not occurring in F or G). Clearly H is given over $U \setminus C$. Now it is easy to check that (i) $\{t\} \cup E \neq grd(F \cup H)$ (as y is defended by E), and (ii) $\{t\} \cup E = grd(G \cup H)$. This is in contradiction to $F \equiv_C^{grd} G$.

(3c) Consider $E \in grd_C(F)$, $S \subseteq A(F) \setminus (C \cup E)$ with $\mathcal{F}^{\infty}_{F \setminus S, E \cup C}(\emptyset) = E$ and towards a contradiction suppose there is an $x \in \mathcal{F}_{F \setminus S}(E)$ with $x \notin \mathcal{F}_{G \setminus S}(E)$. As $E \in grd_C(F) = grd_C(G)$ we have that $x \in A(F) \setminus (C \cup E)$.

Let $B = A(G) \setminus (E \cup C)$ and consider the following AF

$$H = (\{t\} \cup B, \{(t, s) \mid s \in S\} \cup \{(b, b) \mid b \in B \setminus \{x\}\})$$

with $t \in U \setminus C$ a fresh argument (not occurring in F or G). Clearly H is given over $U \setminus C$. We again have that (i) $\{t\} \cup E \neq grd(F \cup H)$ (as x is defended by E), and (ii) $\{t\} \cup E = grd(G \cup H)$. This is in contradiction to $F \equiv_C^{grd} G$.

$\Leftarrow$: Assume that $F \not\equiv_C^{grd} G$. Then there exists an H such that $grd(F \cup H) \neq grd(G \cup H)$. By Lemma 5 we have $E = grd(F \cup H) \cap A(F) \in grd_C(F)$ and $D = grd(G \cup H) \cap A(G) \in grd_C(G)$. Now if $E \notin grd_C(G)$, $D \notin grd_C(F)$ or $A(F) \setminus C \neq A(G) \setminus C$ we are done. Thus let us assume (1) and (2) hold.

It is well known that the grounded extension can be computed iteratively as follows. Start from the empty set, in each iteration add an arbitrary argument defended by the current set but not yet in the set, and stop when all defended arguments are in the set. In order to make the above algorithm deterministic we put a total order on the arguments, with the arguments in C being the smallest ones and the arguments $A(F) \setminus C$ the largest ones. In each step then the smallest argument that is defended (and not in the set) is added to the set.

We run the above algorithm on both $F \cup H$ and $G \cup H$. As the first step of the algorithm we consider iteratively adding all arguments $c \in C$ defended by the current set. Later steps of the algorithm first add an argument $a \in A(F \cup H) \setminus C$ and then iteratively add all arguments $c \in C$ defended by the set. We consider the first step where the run of the algorithm on $F \cup H$ disagrees with the run on $G \cup H$. As $grd(F \cup H) \neq grd(G \cup H)$ this has to happen at some point.

Let T be the set of arguments before this step. We first deal with the special case where the two runs of the algorithm diverge even in the first step:

- Let the run on $F \cup H$ return a set $T_1 \subset C$ and the run on $G \cup H$ return a set $T_2 \subset C$ such that $T_1 \neq T_2$. We then have that $T_1 \in grd_C(F)$ and $T_2 \in grd_C(G)$. For $E = T_1$ and $S = \emptyset$ we get $\mathcal{F}^{\infty}_{F, T_1 \cup C}(\emptyset) = \mathcal{F}^{\infty}_{F,C}(\emptyset) = T_1$ and $\mathcal{F}^{\infty}_{G, T_1 \cup C}(\emptyset) = \mathcal{F}^{\infty}_{G,C}(\emptyset) = T_2$, which contradicts (3a).

Now consider $H' = (A_{H'}, R_{H'})$ with $A_{H'} = A(F \cup H) \setminus (C \cup E)$ and $R_{H'} = \{(a, a) \mid a \in A_{H'} \setminus T\}$. It is easy to check that $T = grd(F \cup H \cup H') = grd(G \cup H \cup H')$ and thus, by Lemma 5, we have that $D = T \cap A(F)$ is a C-restricted grounded set of F as well as of G. Moreover, for $S = T_H^+$ we have $\mathcal{F}^{\infty}_{F \setminus S, D \cup C}(\emptyset) = \mathcal{F}^{\infty}_{G \setminus S, D \cup C}(\emptyset) = D$. Now let us consider the following cases:

- The run on $F \cup H$ adds an argument $a \in A(F) \setminus C$ together with some arguments from C resulting in a set T', while the run on $G \cup H$ either terminates or adds an argument $b \in A(F) \setminus C$ with $a < b$. Then T defends a in $F \cup H$ but not in $G \cup H$. That is, there is a

$b \in A(G)$ that attacks a in $G \cup H$ but $b \notin T^+_{G\cup H}$. We consider two cases: (i) If $(b, a) \notin R(G)$ then $(b, a) \in R(H)$ and thus $b \in T^+_{F\cup H}$, i.e., $b \in T^+_F$ while $b \notin T^+_G$. A contradiction to (3b). (ii) If $(b, a) \in R(G)$ then $a \notin \mathcal{F}_{G\setminus S}(D)$. Now consider an attacker c of a in $F \cup H$. Such c is attacked by T either via an attack in H, i.e., $c \in T^+_H = S$ or an attack in F, i.e., $c \in D^+_F$. Thus $a \in \mathcal{F}_{F\setminus S}(D)$ and we have a contradiction to (3c).

- The run on $F \cup H$ adds an argument $a \in A(H) \setminus A(F)$ together with some arguments from C while the run on $G \cup H$ either terminates or adds an argument b with $a < b$. Then T defends a in $F \cup H$ but not in $G \cup H$. We then have that there is $b \in \{a\}^-_{G\cup H} \setminus T^+_{G\cup H}$ (thus $b \notin S$) and, as $a \in A(H) \setminus A(F)$, we have $(b, a) \in R(H)$. Now as T defends a in $F \cup H$ we have $b \in T^+_{F\cup H}$, but from the above $b \notin T^+_H$ and $b \notin T^+_G$. Thus we have $b \in D^+_F \setminus (C \cup S)$ and $b \notin D^+_G \setminus (C \cup S)$, a contradiction to (3b).

- Both the run on $F \cup H$ and the run on $G \cup H$ add the same argument $a \in A(F \cup H) \setminus C$, but they add different subsets of C to the set. Let T_1, T_2 be the sets returned by the first and second run. By Lemma 5 we have that $T_1 \cap A(F)$ is C-restricted grounded in F and $T_2 \cap A(F)$ is C-restricted grounded in G.

  For $E = T_1$ and $S = (T_1)^+_H = (T_2)^+_H$ we get $\mathcal{F}^\infty_{F\setminus S,T_1\cup C}(\emptyset) = T_1$ and $\mathcal{F}^\infty_{G\setminus S,T_1\cup C}(\emptyset) = T_2$, which contradicts (3a). Notice that $T_1 \cup C = T_2 \cup C$.

The remaining case are by symmetry in F and G and thus the claim follows. □

Remark 5. *When considering $C = \emptyset$ then condition (2) of Theorem 5 reduces to $cf(F) = cf(G)$, which means that both AFs have (a) the same self-attacking arguments, and (b) the same conflicts between two arguments (with potentially different direction), except between self-attacking arguments. Now consider an argument a that is not self-attacking and $E = \emptyset$. Towards a contradiction suppose $\{a\}^-_F \neq \{a\}^-_G$. W.l.o.g. there is a $b \in \{a\}^-_G$ with $b \notin \{a\}^-_F$. For $S = \{a\}^-_F$ we then have that $a \in \mathcal{F}_{F\setminus S}(E)$ but $a \notin \mathcal{F}_{G\setminus S}(E)$, a contradiction to (3c). Thus, if an argument is not self-attacking it has the same attackers in F and G. Now suppose $\{a\}^+_F \neq \{a\}^+_G$. W.l.o.g. there is a $b \in \{a\}^+_F$ with $b \notin \{a\}^+_G$. Then for $S = \{a\}^-_F$ and $E = \{a\}$ we have $\mathcal{F}^\infty_{F\setminus S,E\cup C}(\emptyset) = E$. In order to satisfy (3b) we must have $b \in S$. Thus, for an argument a that is not self-attacking an outgoing attack can only be dropped if the attacked argument b also attacks a. Hence, we can drop an attack only if it is between two self-attacking arguments or it is from an argument a to a self-attacking argument, where this self-attacker counter-attacks a. That is, we get exactly the characterization of strong equivalence from [23].*

For $C = A(F \cup G)$, only conditions (2) and (3) for $S = \emptyset$ remain. (2) is equivalent to $grd(F) = grd(G)$; (3a) and (3c) are is easily verified to hold whenever $grd(F) = grd(G)$; (3b) holds trivially. Hence, we obtain standard equivalence for grounded semantics.

4.3.6 Conflict-free and Naive Semantics

Notice that two AFs possess the same conflict-free sets iff they possess the same naive extensions and thus $\equiv^{cf}_C$ and $\equiv^{naive}_C$ coincide. Moreover, the C-restricted semantics of cf is just cf itself.

Table 1: Complexity of Equivalence Testing.

σ	$naive$	grd	stb	adm	com	prf
$F \equiv^{\sigma}_{S} G$	L	L	L	L	L	L
$F \equiv^{\sigma} G$	L	P-c	coNP-c.	coNP-c.	coNP-c.	Π_2^{P}-c.
$F \equiv^{\sigma}_{C} G$	L	coNP-c.	coNP-c.	coNP-c.	coNP-c.	Π_2^{P}-c.

Theorem 6. *Let F, G be AFs and $C \subseteq U$. Then, $F \equiv^{cf}_{C} G$ ($F \equiv^{naive}_{C} G$) iff the following conditions jointly hold: (1) $cf(F) = cf(G)$ and (2) $A(F) \setminus C = A(G) \setminus C$.*

Proof. Since $naive(F) = naive(G)$ iff $cf(F) = cf(G)$ for any two AFs F, G, we have $F \equiv^{naive}_{C} G$ iff $F \equiv^{cf}_{C} G$. Hence, it suffices to show the assertion for conflict-free sets.

$\Rightarrow$: Given $F \equiv^{cf}_{C} G$. Due to Observation 1 we immediately have $cf(F) = cf(G)$. Towards a contradiction let us assume $A(F) \setminus C \neq A(G) \setminus C$. W.l.o.g. let $a \in A(F) \setminus C$ but $a \notin A(G) \setminus C$. Since $cf(F) = cf(G)$ is already known we derive $(a, a) \in R(F)$. Thus, considering $H = (\{a\}, \emptyset)$ yields $\{a\} \in cf(G \cup H)$ but $\{a\} \notin cf(F \cup H)$, implying $F \not\equiv^{cf}_{C} G$.

$\Leftarrow$: Suppose $F \not\equiv^{cf}_{C} G$. W.l.o.g. there is an AF H over $U \setminus C$ and a set S such that $S \in cf(F \cup H)$ but $S \notin cf(G \cup H)$. That is S either contains an argument a that is not in $G \cup H$ or in G there is an attack (a, b) with $a, b \in E = S \cap A(G)$. In the former case $\{a\} \in cf(F)$ while $\{a\} \notin cf(G)$ and thus (1) is violated. In the latter case we have to distinguish whether $a, b \in A(F)$ or not. If $a, b \in A(F)$ then $\{a, b\} \in cf(F)$ while $\{a, b\} \notin cf(G)$ and thus (1) is violated. Otherwise w.l.o.g. $a \notin A(F)$ and thus $a \in A(H)$. Hence $a \notin C$ and we have $a \notin A(F) \setminus C$ but $a \in A(G) \setminus C$, i.e., (2) is violated. $\square$

5 Computational Properties

While strong equivalence can be efficiently decided (cf. [23]), i.e., even in logarithmic space (L), testing standard equivalence is coNP-hard for $\sigma \in \{stb, adm, prf, com\}$ as it generalizes the problem of deciding whether an AF has a (non-empty) extension [17]. These hardness results extend to C-relativized equivalence. Upper bounds of complexity are given by the characterizations presented in Section 4.3. Our complexity results are summarized in Table 1 ($\mathcal{C}$-c. stands for $\mathcal{C}$-complete). Grounded semantics has a special behavior: while both standard and strong equivalence are tractable[3], C-relativized equivalence is coNP-complete as we show next.

Theorem 7. *Deciding $F \equiv^{grd}_{C} G$ is* coNP*-complete.*

Proof. The membership in coNP can be shown via the characterization in Theorem 5 using the complement problem. For the coNP-hardness consider the problem of deciding whether two CNF formulas are equivalent. Hence, let φ and ψ be two CNF formulas over atoms X and let C_{φ} be

[3] Testing standard equivalence for grounded semantics is P-complete due to the P-completeness result for verifying the grounded extension [19].

the set of clauses of φ and C_ψ be the set of clauses of ψ. Moreover we add clauses $\{x, \neg x\}$ for all $x \in X$ to both formulas to ensure that there are no partial models.

For a CNF formula ϕ with clauses C_ϕ we define the corresponding AF $F_\phi = (A, R)$ with $A = X \cup \bar{X} \cup C_\phi \cup \{t\}$ and $R = \{(c,t),(c,c) \mid c \in C_\phi\} \cup \{(x,\bar{x}),(\bar{x},x) \mid x \in X\} \cup \{(x,c) \mid x \in c \in C_\phi\} \cup \{(\bar{x},c) \mid \neg x \in c \in C_\phi\}$.

To complete the proof we show that $\varphi \equiv \psi$ iff $F_\varphi \equiv_C^{grd} F_\psi$ for $C = C_\varphi \cup C_\psi \cup \{t\}$.

$\Leftarrow$: If there is a model $M \subseteq X$ of φ that is not a model of ψ then consider $H = (\{s\} \cup X \cup \bar{X}, \{(s,x) \mid x \in X \setminus M\} \cup \{(t,\bar{x}) \mid x \in M\})$. Now it is easy to verify that $\{s,t\} \cup M \cup \overline{X \setminus M}$ is the grounded extensions of $F_\varphi \cup H$ but the grounded extension of $G_\varphi \cup H$ is $\{s\} \cup M \cup \overline{X \setminus M}$. That is $F_\varphi \not\equiv_C^{grd} F_\psi$.

$\Rightarrow$: If $F_\varphi \not\equiv_C^{grd} F_\psi$ then there exists an H over $U \setminus C$ such that $grd(F_\varphi \cup H) \neq grd(F_\psi \cup H)$. By construction of F_φ and F_ψ the two AFs are identical except for arguments in C and moreover C has no outgoing attacks; i.e., there is no attack in F_φ or F_ψ from arguments in C to arguments in $X \cup \bar{X}$. Thus we have $grd(F_\varphi \cup H) \setminus C = grd(F_\psi \cup H) \setminus C$. As t is the only argument in C that is not self-attacking we have w.l.o.g. $t \in grd(F_\varphi \cup H)$ but $t \notin grd(F_\psi \cup H)$. Now consider $M = X \cap grd(F_\varphi \cup H)$. On can easily show that M is a model of φ but not of ψ. That is $\varphi \not\equiv \psi$. $\square$

It remains to show Π_2^{P}-hardness of $F \equiv_C^{prf} G$. We prove the result for $F \equiv^{prf} G$ by reduction from the Π_2^{P}-complete problem of deciding whether an AF F is coherent [16], i.e., whether $stb(F) = prf(F)$.

Theorem 8. *Deciding $F \equiv_C^{prf} G$ or $F \equiv^{prf} G$ is Π_2^{P}-complete.*

Proof. Membership in Π_2^{P} follows from Theorem 3. We show hardness for testing $prf(F) = prf(G)$. It is well known that testing whether an AF F is coherent, i.e., whether $stb(F) = prf(F)$ is Π_2^{P}-complete [16]. Moreover, we can assume that $\emptyset \notin prf(F)$. When can then transform F to an AF $F' = (A', R')$ with $A' = A(F) \cup \{t\}$ and $R' = R(F) \cup \{(t,a),(a,t) \mid a \in A(F)\}$. It is easy to show that $stb(F') = stb(F) \cup \{\{t\}\}$ and $prf(F') = prf(F) \cup \{\{t\}\}$. That is we have that F' is coherent iff F is coherent but we have $stb(F') \neq \emptyset$. Now we can apply Translation 4 from [19] which maps the F' to an AF G such that $stb(F') = prf(G)$ [4] and can be efficiently computed. That is we have that $stb(F) = prf(F)$ iff $prf(F') = prf(G)$. $\square$

Recall that for $C = \emptyset$, testing $\equiv_C^\sigma$ equivalence is computationally easy, while it is hard in the general case. Thus, one promising approach towards practical feasible algorithms is to consider characterizations whose performance depends on the set C. In other words, given AFs F and G to be compared under $\equiv_C^\sigma$, we aim to restrict the comparison of the C-restricted extensions (which is indeed the most expensive test in all characterizations). In order to give a first result into that direction for stable semantics we define the *stable reduct* of F w.r.t. E and B:

Definition 11. *Let F be an AF and $B, E \subseteq U$. The* (stable) reduct of F w.r.t. E and B *is defined as the AF $F^*_{B,E} = (A^*, R^*)$ with*

$$A^* = A(F) \setminus E_F^\oplus$$
$$R^* = \{(a,b) \in R(F) \mid a, b \in A^*\} \cup \{(a,a) \mid a \in A^* \cap B\}.$$

[4]Notice that this only holds when $stb(F') \neq \emptyset$.

Theorem 9. *Let F, G be AFs, $C \subseteq A(F \cup G)$, and $B = C^{\oplus}_{F\cup G} \cup C^{-}_{F\cup G}$. Then, $F \equiv^{stb}_{C} G$ iff the following conditions jointly hold (1) if $stb_C(F \cap B) \neq \emptyset$, $A(F) \setminus C = A(G) \setminus C$; (2) $stb_C(F \cap B) = stb_C(G \cap B)$; and (3) for all $E \in stb_C(F \cap B)$, $F^*_{B,E} \equiv^{stb}_{\mathcal{S}} G^*_{B,E}$.*

In the above characterization the number of C-restricted sets we have to consider in (1) and (2) does not depend on the number of total arguments but only on the number of arguments that are either in $C \cap A(F \cup G)$ or neighbors of such arguments. Moreover, the strong equivalence in (3) can be tested in polynomial time.

Proof. Consider $E \subseteq B$. Clearly $E \in cf(F \cap B)$ iff $E \in cf(F)$. Moreover, all the attacks from E are maintained by $F \cap B$ and thus $A(F) \cap C \subseteq F^{+}_{F\cap B}$ iff $A(F) \cap C \subseteq F^{+}_{F}$. Thus we obtain the following.

Observation: $E \in stb_C(F \cap B)$ iff $E \in stb_C(F)$, for each $E \subseteq B$.

$\Rightarrow$: (1) and (2) follow immediately from Theorem 1 and the above observation. It remains to show that if $F \equiv^{stb}_{C} G$ then for all $E \in stb_C(F \cap B)$, $F^*_{B,E} \equiv^{stb}_{\mathcal{S}} G^*_{B,E}$. To this end suppose $stb_C(F \cap B) = stb_C(G \cap B)$, $A(F) \setminus C = A(G) \setminus C$, and that there is an $E \in stb_C(F \cap B)$ such that $F^*_E \not\equiv^{stb}_{\mathcal{S}} G^*_E$, i.e. there exists a H such that $F^*_E \cup H \not\equiv^{stb} G^*_E \cup H$.

- Let us first suppose $A(F^*_{B,E}) = A(G^*_{B,E})$. Then, we can assume that $A(H)$ is disjoint from $E^{\oplus}_{F}$ and $E^{\oplus}_{G}$, and moreover, that H is given over $U \setminus C$. W.l.o.g. let $S \in stb(F^*_{B,E} \cup H)$ such that $S \notin stb(G^*_{B,E} \cup H)$ and recall that by assumption $E \in stb_C(F \cap B) = stb_C(G \cap B)$, E is thus conflict-free in $F \cap B$ and also in $G \cap B$. Notice that $F^*_{B,E}$, $G^*_{B,E}$ are constructed such that all arguments in conflict with E are either removed, if they are already attacked by E, or self-attacking in the modified AFs. That is $E \cup S \in cf(F \cup H)$. Moreover all arguments not attacked by E in F are still present in $F^*_{B,E}$ and thus attacked by S, i.e., $E \cup S \in stb(F \cup H)$. As $S \notin stb(G^*_{B,E} \cup H)$ there is either (a) a conflict between two arguments in S or (b) an argument $a \in A(G^*_{B,E} \cup H) \setminus S$ not attacked by S. In the former case this conflict is either present in $G \cup H$ or was introduced in the construction of $G^*_{B,E}$, because an argument in S is in conflict with an argument of E. Thus, in this case $E \cup S \notin stb(G \cup H)$. In the latter case the argument is also not attacked by $E \cup S$ in $G \cup H$ and thus also $E \cup S \notin stb(G \cup H)$. A contradiction to the assumption $F \equiv^{stb}_{C} G$.

- Now suppose $A(F^*_{B,E}) \neq A(G^*_{B,E})$. W.l.o.g., let $a \in A(F^*_{B,E})$ but $a \notin A(G^*_{B,E})$. We observe that $a \notin C$ and thus $a \in A(G)$ (due to $A(F) \setminus C = A(G) \setminus C$). It follows that $a \in E^{+}_{G}$. Let

$$H = (\{t\} \cup A(G^*_{B,E}), \{(t, b) \mid b \in A(G^*_E)\})$$

  where t is fresh argument from $U \setminus C$. Observe that H does not contain arguments from C since $A(G^*_E) \cap C = \emptyset$. Since $E \in stb_C(G)$, E is conflict-free in G, and moreover, we have that $\{t\}$ is stable in $G^*_E \cup H$; now it can be easily checked that also $E \cup \{t\} \in stb(G \cup H)$. On the other hand, $E \cup \{t\} \notin stb(F \cup H)$, since neither t attacks a in $F \cup H$, nor E attacks any argument from F^*_E in $F \cup H$, in particular E does not attack a. Thus, we have a contradiction to $F \equiv^{stb}_{C} G$.

$\Leftarrow$: Suppose $F \not\equiv_C^{stb} G$. W.l.o.g. there is an AF H over $U \backslash C$ and a set S such that $S \in stb(F \cup H)$ but $S \notin stb(G \cup H)$. By Lemma 1 we have $S \cap A(F) \in stb_C(F)$ and it is easy to verify that also $E = S \cap A(F) \cap B \in stb_C(F \cap B)$.

If now $E \notin stb_C(G \cap B)$ or $A(F) \backslash C \neq A(G) \backslash C$, we are done, i.e. condition (1) or (2) is already violated. So suppose $E \in stb_C(G \cap B)$, and $A(F) \setminus C = A(G) \setminus C$. We show $F^*_{B,E} \not\equiv_{\mathcal{S}}^{stb} G^*_{B,E}$. This holds in case $A(F^*_E) \neq A(G^*_E)$, by known results [23]. Hence, suppose $A(F^*_E) = A(G^*_E)$ and let $H' = H \setminus E^{\oplus}_F = H \setminus E^{\oplus}_G$.[5] We have $S \in stb(F \cup H')$ (conflict-freeness is obvious; if an argument $a \notin S$ has been attacked by an argument in $H \setminus H'$ then a remains attacked by S via F). It is easy to verify that $(S \setminus E) \in stb(F^*_E \cup H')$.

We need to show $(S \setminus E) \notin stb(G^*_E \cup H')$. Again this readily holds, if $S \notin stb(G \cup H')$. From $S \notin stb(G \cup H)$ we obtain that either (a) S is not conflict-free in $G \cup H$ or (b) there is an argument $a \in A(G \cup H)$ with $a \notin S^{\oplus}_{G \cup H}$. In case (a) from the fact that H does not contain conflicts between arguments in S we obtain that S has a conflict in G and thus $S \notin stb(G \cup H')$. In case (b) from the fact that $R(H') \subseteq R(H)$ we obtain that $a \notin S^{\oplus}_{G \cup H'}$ and thus $S \notin stb(G \cup H')$. Hence we have $F^*_{B,E} \not\equiv_{\mathcal{S}}^{stb} G^*_{B,E}$. □

6 Simplifications

We come back to the issue of simplification raised in the introduction. We begin by defining the notion of replacement.

Definition 12. *Given AFs F, F', G such that $A(F') \subseteq A(F) \cup (U \setminus A(G))$ and F is a sub-AF of G (i.e., $A(F) \subseteq A(G)$ and $R(F) = R(G) \cap (A(F) \times A(F))$), let $A = (A(G) \setminus A(F)) \cup A(F')$. The* replacement of F by F' in G *is defined as* $G[F/F'] = (A, ((R(G) \setminus R(F)) \cap (A \times A)) \cup R(F'))$.

As it turns out, faithfulness of the replacement of a sub-AF by another within a larger AF G follows from C-relativized equivalence of the the sub-AFs conjoined with their immediate neighborhood in G.

Proposition 6. *For AFs F, F', G and $C \subseteq U$ such that $A(F) \cup A(F') \subseteq C$, $(A(G) \setminus A(F)) \cap C = \emptyset$, and F is a sub-AF of G, let $B = (A(F))^{\oplus}_G \cup (A(F))^{-}_G$ and $F^G = (B, R(G) \cap (B \times B))$. Then, $F^G \equiv^{\sigma}_C F^G[F/F']$ implies $G \equiv^{\sigma} G[F/F']$.*

Proof. By assumption F is an sub-AF of G and thus there is an AF H_G over $U \setminus C$ such that $F^G \cup H_G = G$. Further, by construction $G[F/F'] = F^G[F/F'] \cup H_G$. As $F^G \equiv^{\sigma}_C F^G[F/F']$, we have that $\sigma(F^G \cup H) = \sigma(F^G[F/F'] \cup H)$ for each AF H over $U \setminus C$, in particular for $H = H_G$. That is $\sigma(G) = \sigma(G[F/F'])$ and thus $G \equiv^{\sigma} G[F/F']$. □

A key feature of Def. 12 is that the attacks connecting the AFs F and F' to G are not changed, unless the involved argument in F is removed in F' (then the attack is also removed). Therefore the condition for C-relativized equivalence boils down to $stb_C(F^G) = stb_C(F^G[F/F'])$, since the

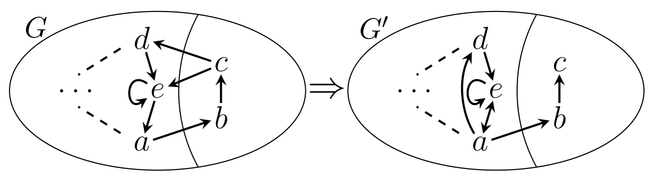
[5]Notice that, as $A(F) \setminus C = A(G) \setminus C$ and $A(F^*_E) = A(G^*_E)$, the sets $E^{\oplus}_F$ and $E^{\oplus}_G$ can only differ on arguments in C, and by definition H does not contain arguments from C.

other conditions from Theorem 1 are trivially satisfied (similar observations can be given for the other semantics).

Example 10. *Recalling the introductory example, faithfulness of replacing F_{abc} by F_{ab} in an arbitrary larger AF G being connected to F_{abc} by an attack (d, a) (cf. Figure 1), is then verified by $stb_C(F^G_{abc}) = \{\{d, b\}\} = stb_C(F^G_{ab})$. In other words we have that cycles of length 3 can be simplified under the stable semantics to two arguments, whenever the cycle has exactly one incoming attack. This kind of simplification can be generalized to arbitrary odd-length cycles in C, allowing for potential deletion of several arguments.* $\diamondsuit$

The replacement of sub-AFs with fixed connections to the rest-AF is a particular application of the results of Section 4.3. The notion of C-relativized equivalence is, however, more general and gives rise to simplifications of the following kind.

Example 11. *Consider the AFs G and G' depicted below.*



Note the single strongly connected component in G is split into three (smaller) components in G'. Let F, F' be the sub-AFs of G, G' with arguments $\{a, b, c, d, e\}$. To prove $G \equiv^{stb} G'$ we show $F \equiv^{stb}_C F'$ for $C = \{b, c\}$: (1) $A(F) \setminus C = \{a, d, e\} = A(F') \setminus C$, (2) $stb_C(F) = \{\{a, c\}, \{b\}, \{b, d\}\} = stb_C(F')$, and (3) $\{a, c\}^+_F \setminus C = \{d, e\} = \{a, c\}^+_{F'} \setminus C$, $\{b\}^+_F \setminus C = \emptyset = \{b\}^+_{F'} \setminus C$, $\{b, d\}^+_F \setminus C = \{e\} = \{b, d\}^+_{F'} \setminus C$. Again, this result can be generalized to arbitrary even-length paths among arguments in C. $\diamondsuit$

7 Related Work

Strong equivalence as well as further related notions have been thoroughly studied in the literature (cf. [6, 8, 12]). Almost all of these notions are somehow disappointing regarding their potential for simplification. In fact, for most of these notions no arguments are redundant and deletions of attacks rely on the presence of self-loops. In particular, in case of self-loop-free AFs nothing can be simplified.

The concept of restricted admissible and stable semantics has been considered in dynamic programming algorithms based on tree-decompositions [18, 14]. An investigation on the amount of neighborhood (in a graph-theoretical sense) needed to verify acceptability for the different semantics was conducted in [11].

The issue of local evaluation of AFs was also tackled in the work on input/output AFs [1, 21]. There the behavior of AFs (with dedicated input- and output-arguments) is described by the possible valuations of the output-arguments for each possible input. For the most prominent semantics it is shown whether having the same I/O behavior is sufficient for replacing one AF by another without affecting the evaluation of the entire AF. Our work differs to this concept, as we do not explicitly model I/O arguments and are more focused on finding exact conditions for the faithfulness of replacements.

The work on splitting [4] and division-based semantics [3] allow for local evaluations of strongly connected components (SCCs) but require that SCCs are considered in a specific order. Baumann et al. [9] relaxed these conditions for stable semantics.

The concept of relativized equivalence was also studied for other nonmonotonic formalisms, in particular for Answer-Set Programming, see e.g. [20]. As well, simplification strategies have been suggested on basis of equivalence notions. Such replacements are typically defined as an exchange of rules in a logic program. This already indicates the main difference to our work, since replacing sub-graphs in AFs provides some subtle issues to be taken into consideration (cf. Section 6). This also might explain why in abstract argumentation the relation between equivalence notions and simplifications has been underexplored so far.

8 Discussion

In this paper, we introduced a general notion of equivalence for AFs and studied their characterizations and complexity.

There are several ways to pursue the presented research. First, an inclusion of other extension-based and labelling-based semantics is an immediate objective. Another direction to consider are weaker versions of C-relativized equivalence, for instance in analogy to normal expansion equivalence [5], altering Def. 5 such that attacks between the original arguments of F and G cannot be changed. This situation is typical in the instantiation-based context (where AFs are constructed from an underlying knowledge base) since usually one can decide whether there is a conflict between arguments by solely considering these arguments.

On the practical side, we plan to employ our notion of equivalence for a systematic investigation of possible simplifications and to implement these findings in a preprocessing tool for abstract argumentation systems. That is, to first use our equivalence notion to identify certain patterns of small AFs together with C-equivalent simpler AFs and then to design a preprocessing system which checks a large AF for these patterns and replaces them with the corresponding simpler ones.

Finally, we plan to study restricted equivalence in the general setting of graph problems (as it was already done for strong equivalence by Lonc and Truszczyński [22]) which might yield interesting results that go beyond the field of argumentation.

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