A proof of the irrationality of $\sqrt{2}$

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Abstract

Printable version of a sample proof that uses Lamport's proof style [1], illustrating how structured proofs can be converted to HTML pages via LaTeX2HTML enriched with extensions for Lamport's proof style.

Theorem There does not exist r in $\mathbf{Q}$ such that $r^2 = 2$.

PROOF SKETCH: We assume $r^2 = 2$ for $r \in \mathbf{Q}$ and obtain a contradiction. Writing $r = m/n$, where m and n have no common divisors (step $\langle 1 \rangle 1$), we deduce from $(m/n)^2 = 2$ and the lemma that both m and n must be divisible by 2 ($\langle 1 \rangle 2$ and $\langle 1 \rangle 3$).

ASSUME: 1. $r \in \mathbf{Q}$
2. $r^2 = 2$

PROVE: False

$\langle 1 \rangle 1$. Choose m, n in $\mathbf{Z}$ such that
1. $\gcd(m, n) = 1$
2. $r = (m/n)$

$\langle 2 \rangle 1$. Choose p, q in $\mathbf{Z}$ such that $q \neq 0$ and $r = p/q$.
PROOF: By assumption $\langle 0 \rangle$:1.

LET: $m \triangleq p/\gcd(p, q)$
$n \triangleq q/\gcd(p, q)$

$\langle 2 \rangle 2$. $m, n \in \mathbf{Z}$
PROOF: $\langle 2 \rangle 1$ and definition of m and n.

$\langle 2 \rangle 3$. $r = m/n$

PROOF:
$$\begin{aligned} m/n &= \frac{p/\gcd(p,q)}{q/\gcd(p,q)} && \text{[Definition of } m \text{ and } n\text{]} \\ &= p/q && \text{[Simple algebra]} \\ &= r && \text{[By } \langle 2 \rangle 1\text{]} \end{aligned}$$

$\langle 2 \rangle 4$. $\gcd(m, n) = 1$
PROOF: By the definition of the gcd, it suffices to:

ASSUME: 1. s divides m
2. s divides n

PROVE: $s = \pm 1$

$\langle 3 \rangle 1$. $s \cdot \gcd(p, q)$ divides p.

PROOF: $\langle 2\rangle$:1 and the definition of m.

$\langle 3\rangle 2$. $s \cdot \gcd(p, q)$ divides q.

PROOF: $\langle 2\rangle$:2 and definition of n.

$\langle 3\rangle 3$. Q.E.D.

PROOF: $\langle 3\rangle 1$, $\langle 3\rangle 2$, and the definition of gcd.

$\langle 2\rangle 5$. Q.E.D.

$\langle 1\rangle 2$. 2 divides m.

$\langle 2\rangle 1$. $m^2 = 2n^2$

PROOF: $\langle 1\rangle 1.1$ implies $(m/n)^2 = 2$.

$\langle 2\rangle 2$. Q.E.D.

PROOF: By $\langle 2\rangle 1$ and the lemma.

$\langle 1\rangle 3$. 2 divides n.

$\langle 2\rangle 1$. Choose p in $\mathbf{Z}$ such that $m = 2p$.

PROOF: By $\langle 1\rangle 2$.

$\langle 2\rangle 2$. $n^2 = 2p^2$

PROOF:
$$\begin{aligned} 2 &= (m/n)^2 && [\langle 1\rangle 1.2 \text{ and } \langle 0\rangle{:}2] \\ &= (2p/n)^2 && [\langle 2\rangle 1] \\ &= 4p^2/n^2 && [\text{Algebra}] \end{aligned}$$
from which the result follows easily by algebra.

$\langle 2\rangle 3$. Q.E.D.

PROOF: By $\langle 2\rangle 2$ and the lemma.

$\langle 1\rangle 4$. Q.E.D.

PROOF: $\langle 1\rangle 1.1$, $\langle 1\rangle 2$, $\langle 1\rangle 3$, and definition of gcd.

References

[1] Leslie Lamport, 1993, How to write a proof. In *Global Analysis of Modern Mathematics*, pp. 311–321. Publish or Perish, Houston, Texas, February 1993. A symposium in honor of Richard Palais' sixtieth birthday (also published as SRC Research Report 94). http://research.microsoft.com/users/lamport/proofs/src94.ps.Z