Belief merging within fragments of propositional logic

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Recently, belief change within the framework of fragments of propositional logic has gained increasing attention. Previous research focused on belief contraction and belief revision on the Horn fragment. However, the problem of belief merging within fragments of propositional logic has been neglected so far. We present a general approach to define new merging operators derived from existing ones such that the result of merging remains in the fragment under consideration. Our approach is not limited to the case of Horn fragment but applicable to any fragment of propositional logic characterized by a closure property on the sets of models of its formulæ. We study the logical properties of the proposed operators regarding satisfaction of merging postulates, considering in particular distance-based merging operators for Horn and Krom fragments.

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1. INTRODUCTION

Belief merging consists in achieving a synthesis between pieces of information provided by different sources. Although these sources are individually consistent, they may mutually conflict. The aim of merging is to provide a consistent set of information, making maximum use of the information provided by the sources while not favoring any of them. Belief merging is an important issue in many fields of Artificial Intelligence (AI) [?] and symbolic approaches to multi-source fusion gave rise to increasing interest within the AI community since the 1990s [?; ?; ?; ?; ?]. One of today's major approaches is belief merging under (integrity) constraints, which generalizes both merging (without constraints) and revision (of old information by a new piece of information). For the latter the constraints then play the role of the new piece of information. Postulates characterizing the rational behavior of such merging operators, known as IC postulates, have been proposed by ?] and improved by ?] in the same spirit as the seminal AGM [?] postulates for revision. Concrete merging operators have been proposed according to either semantic (model-based) or syntactic (formula-based) points of view in a classical logic setting. We focus here on the model-based approach of distance-based merging operators [?; ?; ?]. These operators are parameterized by a distance which represents the closeness between interpretations and an aggregation function which captures the merging strategy and takes the origin of beliefs into account.

Belief change operations within the framework of fragments of classical logic constitute a vivid research branch. In particular, contraction [?; ?; ?; ?] and revision [?; ?; ?; ?] have been thoroughly analyzed in the literature. The motivation for such a research is twofold:

— In many applications, the language is restricted a priori. For instance, a rule-based formalization of expert's knowledge is much easier to understand and manipulate for standard users. If users revise or merge some sets of rules, they indeed expect that the outcome is still in the easy-to-read format they are used to.

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— Many fragments of propositional logic allow for efficient reasoning methods. Suppose an agent has to take a decision according to a group of experts' beliefs. Since this should be efficiently doable, the expert's beliefs are stored as formulæ known to be in a tractable class. For making a decision, it is desired that the result of the change operation yields a set of formulæ in the same fragment. Hence, the agent still can use the dedicated solving method she is equipped with for this fragment.

Most of previous research has focused on the Horn fragment except that of **?**] that studied revision in several fragments of propositional logic. However, as far as we know, the problem of *belief merging* within *fragments of propositional logic* has been neglected so far.

The main obstacle is that for language fragment $\mathcal{L}'$, given n belief bases $K_1, \ldots, K_n \in 2^{\mathcal{L}'}$ and a constraint $\mu \in \mathcal{L}'$, it is not guaranteed that the outcome of the merging, $\Delta_\mu(\{K_1, \ldots, K_n\})$, remains in $\mathcal{L}'$ as well. Let for example, $K_1 = \{a\}$, $K_2 = \{b\}$ and $\mu = \neg a \vee \neg b$ be two sets of formulæ and a formula expressed in the Horn fragment. Merging with a family of typical distance-based operators proposed by **?**] does not remain in the Horn language fragment since the result of merging is equivalent to $(a \vee b) \wedge (\neg a \vee \neg b)$, which is not equivalent to any Horn formula [**?**]. This example per se does not rule out the possibility of merging operators that remain within the specified fragment. But it shows that there is indeed a need for identifying and studying the class of merging operators that possess this property.

One line of research tackles this problem by modifying the existing set of postulates in such a way that they classify merging operators within a specific language fragment. This has been done within Horn logic for revision [**?**], contraction [**?**; **?**] and more recently for merging [**?**]. A more general logic, namely the first-order conjunctive logic, which subsumes both Horn and Krom fragments, has also been investigated for entrenchment-based contraction [**?**].

Here we propose the concept of *refinement* of merging operators to overcome the problems mentioned above. Refinements have been proposed for revision [**?**] and capture the intuition of adapting a given operator (defined for full classical logic) to become applicable within a fragment. The basic properties of a refinement are thus

(1) to guarantee the result of the change operation to be in the same fragment as the belief change scenario given and
(2) to keep the behavior of the original operator unchanged if it delivers a result which already fits in the fragment.

Refinements are interesting from different points of view. Several fragments can be treated in a uniform way and a general characterization of refinements is provided for any fragment. Defining and studying refinements of merging operators is not a straightforward extension of the revision case. It is more complex due to the nature of the merging operators. Even if the constraints play the role of the new piece of information in revision, model-based merging deals with multi-sets of models. Moreover applying this approach to different distance-based merging operators, each parameterized by a distance and an aggregation function, reveals that all the different parameters matter, thus showing a rich variety of behaviors for refined merging operators.

Our main contributions are the following:

— We propose to adapt known belief merging operators to make them applicable in fragments of propositional logic. We provide natural criteria which refined operators should satisfy. We characterize refined operators in a constructive way.
— This characterization allows us to study their properties regarding satisfaction of the IC postulates [**?**]. On the one hand we prove that the basic postulates (IC0–IC3) are preserved for any refinement for any fragment. On the other hand we show that the situation is more complex for the remaining postulates. We provide detailed results for the Horn and the Krom fragment where we study two kinds of distance-based merging operators and three approaches for refinements.

This article is an extension of the paper [**?**]. Besides providing full proofs for all results, we add here also a discussion of the Majority Postulate (Section **??**) as well as a generalization towards merging scenarios where either the belief bases or the integrity constraint is not in the fragment (Section **??**).

2. PRELIMINARIES

Propositional Logic. We consider $\mathcal{L}$ as the language of propositional logic over some fixed alphabet $\mathcal{U}$ of propositional atoms. We use standard connectives $\vee$, $\wedge$, $\neg$, $\oplus$, and constants $\top$, $\bot$. A clause is a disjunction of literals. A clause is called (i) *Horn* if at most one of its literals is positive; (ii) *dual Horn* if at most one of its literals is negative; (iii) *Krom* if it consists of at most two literals. A $\oplus$-clause is defined like a clause but using exclusive– instead of standard– disjunction. We identify the following subsets of $\mathcal{L}$:

— $\mathcal{L}_{Horn}$ is the set of all formulæ in $\mathcal{L}$ being conjunctions of Horn clauses
— $\mathcal{L}_{DHorn}$ is the set of all formulæ in $\mathcal{L}$ being conjunctions of dual Horn clauses
— $\mathcal{L}_{Krom}$ is the set of all formulæ in $\mathcal{L}$ being conjunctions of Krom clauses
— $\mathcal{L}_{Affine}$ is the set of all formulæ in $\mathcal{L}$ being conjunctions of $\oplus$-clauses

In what follows we sometimes just talk about arbitrary fragments $\mathcal{L}' \subseteq \mathcal{L}$. Hereby, we tacitly assume that any such fragment $\mathcal{L}' \subseteq \mathcal{L}$ contains at least the formula $\top$.

An interpretation is represented either by a set $\omega \subseteq \mathcal{U}$ of atoms (corresponding to the variables set to true) or by its corresponding characteristic bit-vector of length $|\mathcal{U}|$. For instance if we consider $\mathcal{U} = \{x_1, \ldots, x_6\}$, the interpretation $x_1 = x_3 = x_6 = 1$ and $x_2 = x_4 = x_5 = 0$ will be represented either by $\{x_1, x_3, x_6\}$ or by $(1, 0, 1, 0, 0, 1)$. As usual, if an interpretation ω satisfies a formula ϕ, we call ω a model of ϕ. By $\mathrm{Mod}(\phi)$ we denote the set of all models (over $\mathcal{U}$) of ϕ. Moreover, $\psi \models \phi$ if $\mathrm{Mod}(\psi) \subseteq \mathrm{Mod}(\phi)$ and $\psi \equiv \phi$ (ϕ and ψ are equivalent) if $\mathrm{Mod}(\psi) = \mathrm{Mod}(\phi)$.

A *base* K is a finite set of propositional formulæ $\{\phi_1, \ldots, \phi_n\}$. We shall often identify K via $\bigwedge K$, the conjunction of formulæ of K, i.e., $\bigwedge K = \phi_1 \wedge \cdots \wedge \phi_n$. Thus, a base K is said to be consistent if $\bigwedge K$ is consistent, $\mathrm{Mod}(K)$ is a shortcut for $\mathrm{Mod}(\bigwedge K)$, $K \models \phi$ stands for $\bigwedge K \models \phi$, etc. Given $\mathcal{L}' \subseteq \mathcal{L}$ we denote by $\mathcal{K}_{\mathcal{L}'}$ the set of bases restricted to formulæ from $\mathcal{L}'$. For fragments $\mathcal{L}' \subseteq \mathcal{L}$, we also use $T_{\mathcal{L}'}(K) = \{\phi \in \mathcal{L}' \mid K \models \phi\}$.

A *profile* E is a non-empty finite multiset of consistent bases $E = \{K_1, \ldots, K_n\}$ and represents a group of n agents having different beliefs. Given $\mathcal{L}' \subseteq \mathcal{L}$, we denote by $\mathcal{E}_{\mathcal{L}'}$ the set of profiles restricted to the use of formulæ from $\mathcal{L}'$. We denote $\bigwedge K_1 \wedge \ldots \wedge \bigwedge K_n$ by $\bigwedge E$. The profile is said to be consistent if $\bigwedge E$ is consistent. By abuse of notation we write $K \sqcup E$ to denote the multi-set union $\{K\} \sqcup E$. The multi-set consisting of the sets of models of the bases in a profile is denoted $\mathcal{M}\mathrm{od}(E) = \{\mathrm{Mod}(K_1), \ldots, \mathrm{Mod}(K_n)\}$. Two profiles E_1 and E_2 are equivalent, denoted by $E_1 \equiv E_2$ if $\mathcal{M}\mathrm{od}(E_1) = \mathcal{M}\mathrm{od}(E_2)$. Finally, for a set of interpretations $\mathcal{M}$ and a profile $E = \{K_1, \ldots, K_n\}$ we define $\#(\mathcal{M}, E) = |\{i : \mathcal{M} \cap \mathrm{Mod}(K_i) \neq \emptyset\}|$.

Characterizable Fragments of Propositional Logic. Let $\mathcal{B}$ denote the set of all Boolean functions $\beta\colon \{0,1\}^k \to \{0,1\}$ that have the following two properties[1]:

— *symmetry*, i.e., for all permutations σ, $\beta(x_1, \ldots, x_k) = \beta(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$ and
— 0- and 1-*reproduction*, i.e., for all $x \in \{0,1\}$, $\beta(x, \ldots, x) = x$.

Examples are

— the binary AND function denoted by $\wedge$;
— the binary OR function denoted by $\vee$;
— the ternary MAJORITY function, $\mathrm{maj}_3(x, y, z) = 1$ if at least two of the variables x, y, and z are set to 1;
— the ternary XOR function $\oplus_3(x, y, z) = x \oplus y \oplus z$.

[1] These two properties are also known as anonymity and unanimity.

We extend Boolean functions to interpretations by applying coordinate-wise the original function (recall that we consider interpretations also as bit-vectors). So, if we have $M_1, \ldots, M_k \in \{0,1\}^n$, then $\beta(M_1, \ldots, M_k)$ is defined by

$$\beta(M_1, \ldots, M_k) = (\beta(M_1[1], \ldots, M_k[1]), \ldots, \beta(M_1[n], \ldots, M_k[n])),$$

where $M[i]$ is the i-th coordinate of the interpretation M.

Definition 2.1. Given a set $\mathcal{M} \subseteq 2^{\mathcal{U}}$ of interpretations and $\beta \in \mathcal{B}$, we define $Cl_\beta(\mathcal{M})$, the *closure* of $\mathcal{M}$ under β, as the smallest set of interpretations that contains $\mathcal{M}$ and that is closed under β, i.e., if $M_1, \ldots, M_k \in Cl_\beta(\mathcal{M})$, then also $\beta(M_1, \ldots, M_k) \in Cl_\beta(\mathcal{M})$.

Let us mention some easy properties of such a closure:

— If $\mathcal{M} \subseteq \mathcal{N}$, then $Cl_\beta(\mathcal{M}) \subseteq Cl_\beta(\mathcal{N})$ (monotonicity);
— If $|\mathcal{M}| = 1$, then $Cl_\beta(\mathcal{M}) = \mathcal{M}$ (because by assumption β is 0- and 1-reproducing);
— $Cl_\beta(\emptyset) = \emptyset$.

Definition 2.2. Let $\beta \in \mathcal{B}$. A set $\mathcal{L}' \subseteq \mathcal{L}$ of propositional formulæ is a *β-fragment* (or *characterizable fragment*) if:

(1) for all $\psi \in \mathcal{L}'$, $\mathrm{Mod}(\psi) = Cl_\beta(\mathrm{Mod}(\psi))$
(2) for all $\mathcal{M} \subseteq 2^{\mathcal{U}}$ with $\mathcal{M} = Cl_\beta(\mathcal{M})$ there exists a $\psi \in \mathcal{L}'$ with $\mathrm{Mod}(\psi) = \mathcal{M}$
(3) if $\phi, \psi \in \mathcal{L}'$ then $\phi \wedge \psi \in \mathcal{L}'$.

It is well known that $\mathcal{L}_{Horn}$ is an $\wedge$-fragment, $\mathcal{L}_{DHorn}$ is an $\vee$-fragment, $\mathcal{L}_{Krom}$ is a maj_3-fragment, and $\mathcal{L}_{Affine}$ is a $\oplus_3$-fragment [**?**].

Let us mention at this point that, for instance, $\mathcal{L}_{Horn}$ as defined above is not the only $\wedge$-fragment. This is due to the syntactic nature in the definition of $\mathcal{L}_{Horn}$ while to be an $\wedge$-fragment is a semantical concept. Indeed, any set of propositional formulae equivalent to $\mathcal{L}_{Horn}$ is also an $\wedge$-fragment, for instance the set of all conjunctions of implications where the antecedents are a conjunction of atoms and the consequence is either an atom or $\bot$. Analogous observations indeed apply to other fragments.

As suggested by their names the Horn fragment and the dual Horn fragment are dual in the following sense: a formula ϕ is Horn if and only if the formula $dual(\phi)$ obtained from ϕ in negating each literal is dual Horn. Moreover the set of models of ϕ is in one-to-one correspondence with the set of models of $dual(\phi)$: $M \in \mathrm{Mod}(\phi)$ if and only if $\overline{M}$, which denotes the complement of M, is a model of $dual(\phi)$. The Horn fragment is an $\wedge$-fragment whereas the dual Horn fragment is an $\vee$-fragment. As a consequence from now on we will omit discussions about the dual Horn fragment. All the results stated in the following for the Horn fragment also hold for the dual Horn fragment in replacing the function $\wedge$ by the function $\vee$.

Logical Merging Operators. Belief merging aims at combining several pieces of information coming from different sources. Merging operators we consider are functions from the set of profiles and the set of propositional formulæ to the set of bases, i.e., $\Delta\colon \mathcal{E}_\mathcal{L} \times \mathcal{L} \to \mathcal{K}_\mathcal{L}$. For $E \in \mathcal{E}_\mathcal{L}$ and $\mu \in \mathcal{L}$ we will write $\Delta_\mu(E)$ instead of $\Delta(E, \mu)$. The formula μ is called the *integrity constraint* (IC) and restricts the result of the merging.

As for belief revision some logical properties that one could expect from any reasonable merging operator have been stated and discussed in detail [**?**]. Intuitively $\Delta_\mu(E)$ is the "closest" belief base to the profile E satisfying the integrity constraint μ. This is what the following postulates try to capture.

(IC0). $\Delta_\mu(E) \models \mu$
(IC1). If μ is consistent, then $\Delta_\mu(E)$ is consistent
(IC2). If $\bigwedge E$ is consistent with μ, then $\Delta_\mu(E) = \bigwedge E \wedge \mu$
(IC3). If $E_1 \equiv E_2$ and $\mu_1 \equiv \mu_2$, then $\Delta_{\mu_1}(E_1) \equiv \Delta_{\mu_2}(E_2)$

(IC4). If $K_1 \models \mu$ and $K_2 \models \mu,$ then $\Delta_\mu(\{K_1, K_2\}) \wedge K_1$ is consistent if and only if $\Delta_\mu(\{K_1, K_2\}) \wedge K_2$ is consistent
(IC5). $\Delta_\mu(E_1) \wedge \Delta_\mu(E_2) \models \Delta_\mu(E_1 \sqcup E_2)$
(IC6). If $\Delta_\mu(E_1) \wedge \Delta_\mu(E_2)$ is consistent, then $\Delta_\mu(E_1 \sqcup E_2) \models \Delta_\mu(E_1) \wedge \Delta_\mu(E_2)$
(IC7). $\Delta_{\mu_1}(E) \wedge \mu_2 \models \Delta_{\mu_1 \wedge \mu_2}(E)$
(IC8). If $\Delta_{\mu_1}(E) \wedge \mu_2$ is consistent, then $\Delta_{\mu_1 \wedge \mu_2}(E) \models \Delta_{\mu_1}(E) \wedge \mu_2$

The meaning of the postulates is the following: (IC0) assures that the result of the merging satisfies the integrity constraint. (IC1) states that if the integrity constraint is consistent, so is the result of the merging. (IC2) states that the result of the merging is exactly the conjunction of the belief bases with the integrity constraint, whenever this conjunction is consistent. (IC3) expresses the principle of irrelevance of the syntax. (IC4) is the fairness postulate, it says that when we merge two belief bases no preference should be given to one of them. (IC5) and (IC6) together express that if two subgroups agree on at least one alternative, then the result of the merging is exactly those alternatives the two subgroups agree on. (IC7) and (IC8) together express conditions on conjunctions of integrity constraints.

Similarly to belief revision, a representation theorem [**?**] shows that a merging operator corresponds to a family of total preorders over interpretations satisfying certain conditions. More formally, for $E \in \mathcal{E}_\mathcal{L}$, $\mu \in \mathcal{L}$ and $\leq_E$ a total preorder over interpretations, a model-based operator is defined by $\mathrm{Mod}(\Delta_\mu(E)) = min(\mathrm{Mod}(\mu), \leq_E)$. The model-based merging operators select interpretations that are the "closest" to the original belief bases.

Note that belief revision is a special case of belief merging in which there is only one belief base, the integrity contraint represents then the new information, i.e., $\Delta_\mu(\{K\}) = K \circ \mu$. As a consequence some IC postulates for belief merging are direct generalization of KM postulates proposed for belief revision [**?**]. Namely (IC0)–(IC3) correspond to (R1)–(R4), and (IC7) and (IC8) correspond respectively to (R5) and (R6). In contrast the postulates (IC4)–(IC6) take into account the profiles and are specific to merging.

Distance-based operators where closeness is calculated based on the definition of a distance (or a pseudo-distance[2]) between interpretations and from an aggregation function have been proposed [**?**; **?**]. More formally, let $E = \{K_1, \ldots, K_n\} \in \mathcal{E}_\mathcal{L}$, $\mu \in \mathcal{L}$, d be a pseudo-distance and f be an aggregation function, we consider the family of $\Delta_\mu^{d,f}$ merging operators defined by $\mathrm{Mod}(\Delta_\mu^{d,f}(E)) = min(\mathrm{Mod}(\mu), \leq_E)$ where $\leq_E$ is a total preorder over the set $2^\mathcal{U}$ of interpretations defined as follows:

— $d(\omega, K_i) = min_{\omega' \models K_i} d(\omega, \omega')$,
— $d(\omega, E) = f(d(\omega, K_1), \ldots, d(\omega, K_n))$, and
— $\omega \leq_E \omega'$ if $d(\omega, E) \leq d(\omega', E)$.

Several distance-based merging operators have been proposed according to the chosen pseudo-distance and the selected aggregation function. We first recall some known pseudo-distances.

Definition 2.3. A *counting distance* between interpretations is a function $d : 2^\mathcal{U} \times 2^\mathcal{U} \to \mathbb{R}^+$ defined for every pair of interpretations (ω, ω') by

$$d(\omega, \omega') = g(|(\omega \setminus \omega') \cup (\omega' \setminus \omega)|),$$

where $g : \mathbb{N} \to \mathbb{R}^+$ is a nondecreasing function such that $g(n) = 0$ if and only if $n = 0$. If $g(n) = g(1)$ for every $n \neq 0$, we call d a *drastic distance* and denote it via d_D. If $g(n) = n$ for all n, we call d the *Hamming distance* and denote it via d_H. If for every interpretations ω, ω' and ω'' we have $d(\omega, \omega') \leq d(w, w'') + d(w'', w')$, then we say that the distance d satisfies the triangular inequality.

[2]Let $\omega, \omega' \subseteq \mathcal{U}$, a pseudo-distance d is such that $d(\omega, \omega') = d(\omega', \omega)$ and $d(\omega, \omega') = 0$ if and only if $\omega = \omega'$.

Observe that a counting distance is indeed a pseudo-distance, and that both the Hamming distance and the drastic distance satisfy the triangular inequality.

As aggregation functions, we consider here Σ, the sum aggregation function, and the aggregation function GMax defined as follows. Let $E = \{K_1, \ldots, K_n\} \in \mathcal{E}_\mathcal{L}$ and ω, ω' be two interpretations. Let $(d_1^\omega, \ldots, d_n^\omega)$, where $d_j^\omega = d_H(\omega, K_j)$, be the vector of distances between ω and the n belief bases in E. Let L_ω^E be the vector obtained from $(d_1^\omega, \ldots, d_n^\omega)$ by ranking it in decreasing order. The aggregation function GMax is defined by $\mathrm{GMax}(d_1^\omega, \ldots, d_n^\omega) = L_\omega^E$, with $\mathrm{GMax}(d_1^\omega, \ldots, d_n^\omega) \leq \mathrm{GMax}(d_1^{\omega'}, \ldots, d_n^{\omega'})$ if $L_\omega^E \leq_{lex} L_{\omega'}^E$, where $\leq_{lex}$ denotes the lexicographical ordering.

We focus on the $\Delta^{d,\Sigma}$ and $\Delta^{d,\mathrm{GMax}}$ operators where d is an arbitrary counting distance. These operators are known to satisfy the postulates (IC0)–(IC8) [**?**], generalizing previous and more specific results [**?**; **?**]. Interestingly, these two operators coincide for the drastic distance. The operator $\Delta^{d,\Sigma}$ is a majority merging operator, whereas $\Delta^{d,\mathrm{GMax}}$ is an arbitration merging operator. Majority and Arbitration are expressed by the following additional postulates [**?**]:

(Maj). $\exists n \ \Delta_\mu(E_1 \sqcup E_2^n) \models \Delta_\mu(E_2)$
(Arb). If $\Delta_{\mu_1}(K_1) \equiv \Delta_{\mu_2}(K_2), \Delta_{\mu_1 \leftrightarrow \neg\mu_2}\{K_1, K_2\} \equiv (\mu_1 \leftrightarrow \neg\mu_2), \mu_1 \not\models \mu_2,$ and $\mu_2 \not\models \mu_1,$ then $\Delta_{\mu_1 \vee \mu_2}\{K_1, K_2\} \equiv \Delta_{\mu_1}(K_1)$.

The (Maj) postulate expresses the fact that if a subset of belief bases is repeated sufficiently many times then this subset will prevail. The (Arb) postulate says that if a set of alternatives preferred among one set of integrity constraints μ_1 for a belief base K_1 corresponds to the set of alternatives preferred among another set of integrity constraints μ_2 for a belief base K_2, and if the alternatives that belong to a set of integrity constraints but not to the other are equally preferred for the whole group $\{K_1, K_2\}$, then the subset of preferred alternatives among the disjunction of integrity constraints will coincide with the preferred alternatives of each belief base among their respective integrity constraints. We refer to [**?**] for further explanations.

We are interested here in merging operators which are tailored for certain fragments. The following definition thus serves our purposes and is very general. Of course, we later shall consider merging operators which satisfy several criteria and postulates.

Definition 2.4. A merging operator for $\mathcal{L}' \subseteq \mathcal{L}$ is any function $\Delta : \mathcal{E}_{\mathcal{L}'} \times \mathcal{L}' \to \mathcal{K}_{\mathcal{L}'}$. We say that Δ *satisfies* an (IC) postulate (IC_i) $(i \in \{0, \ldots, 8\})$ *in* $\mathcal{L}'$ if the respective postulate holds when restricted to formulæ from $\mathcal{L}'$.

Postulate (Arb) as stated above requires that the formula $\mu_1 \leftrightarrow \neg\mu_2$ is expressible in the considered language fragment. Since we cannot guarantee this to be the case, we restrict the scope of the paper to postulates (IC0)–(IC8) and (Maj).

3. REFINED OPERATORS

Let us reconsider the example from Section **??** to illustrate the problem of standard operators when applied within a β-fragment.

Example 3.1. Let $\mathcal{U} = \{a, b\}$, $E = \{K_1, K_2\} \in \mathcal{E}_{\mathcal{L}_{Horn}}$ and $\mu \in \mathcal{L}_{Horn}$ such that $\mathrm{Mod}(K_1) = \{\{a\}, \{a, b\}\}$, $\mathrm{Mod}(K_2) = \{\{b\}, \{a, b\}\}$, and $\mathrm{Mod}(\mu) = \{\emptyset, \{a\}, \{b\}\}$. Consider the distance-based merging operators, $\Delta^{d_H,\Sigma}$ and $\Delta^{d_H,\mathrm{GMax}}$. Table **??** gives the distances between the models of μ and the belief bases, and the result of the aggregation functions Σ and GMax. The table should be read as follows: The first column contains all possible models of μ. The cell contents of the second column depict the minimal distance between the corresponding model of μ and any model of K_1. Analogously, the third column contains the minimal distances between μ and the models of K_2. The column with the header Σ shows the outcome of aggregating the second and third column with the Σ function and the last column depicts the outcome of aggregating with the GMax function.

Hence, we have $\mathrm{Mod}(\Delta_\mu^{d_H,\Sigma}(E)) = \mathrm{Mod}(\Delta_\mu^{d_H,\mathrm{GMax}}(E)) = \{\{a\}, \{b\}\}$. Thus, for instance, we can return $\phi = (a \vee b) \wedge (\neg a \vee \neg b)$ as the merging result for both operators. However, there is no

Table I. Example Merging Scenario.

	K_1	K_2	Σ	GMax
$\emptyset$	1	1	2	$(1,1)$
$\{a\}$	0	1	1	$(1,0)$
$\{b\}$	1	0	1	$(1,0)$

$\psi \in \mathcal{L}_{Horn}$ with $\mathrm{Mod}(\psi) = \{\{a\},\{b\}\}$ (each $\psi \in \mathcal{L}_{Horn}$ satisfies the following closure property in terms of its set of models: for every $\omega, \omega' \in \mathrm{Mod}(\psi)$, also $\omega \cap \omega' \in \mathrm{Mod}(\psi)$)). Thus, the result of the operator has to be *refined*, such that it fits into the Horn fragment. On the other hand, it holds that $\mu \in \mathcal{L}_{Krom}$, $E \in \mathcal{E}_{\mathcal{L}_{Krom}}$ and also the result ϕ is in Krom. This shows that different fragments behave differently on certain instances. Nonetheless, we aim for a uniform approach for refining merging operators.

We are interested in the following: Given a known merging operator Δ and a fragment $\mathcal{L}'$ of propositional logic, how can we adapt Δ to a new merging operator $\Delta^\star$ such that, for each $E \in \mathcal{E}_{\mathcal{L}'}$ and $\mu \in \mathcal{L}'$, $\Delta^\star_\mu(E) \in \mathcal{K}_{\mathcal{L}'}$? Let us define a few natural desiderata for $\Delta^\star$ inspired by the research on belief revision [**?**].

Definition 3.2. Let $\mathcal{L}'$ be a fragment of classical logic and Δ a merging operator. We call an operator $\Delta^\star\colon \mathcal{E}_{\mathcal{L}'} \times \mathcal{L}' \to \mathcal{K}_{\mathcal{L}'}$ a *Δ-refinement for $\mathcal{L}'$* if it satisfies the following properties, for each $E, E_1, E_2 \in \mathcal{E}_{\mathcal{L}'}$ and $\mu, \mu_1, \mu_2 \in \mathcal{L}'$.

(1) consistency: $\Delta_\mu(E)$ is consistent if and only if $\Delta^\star_\mu(E)$ is consistent
(2) equivalence: if $E_1 \equiv E_2$ and $\Delta_{\mu_1}(E_1) \equiv \Delta_{\mu_2}(E_2)$ then $\Delta^\star_{\mu_1}(E_1) \equiv \Delta^\star_{\mu_2}(E_2)$
(3) containment: $T_{\mathcal{L}'}(\Delta_\mu(E)) \subseteq T_{\mathcal{L}'}(\Delta^\star_\mu(E))$
(4) invariance: If $\Delta_\mu(E) \in \mathcal{K}_{\langle \mathcal{L}' \rangle}$, then $T_{\mathcal{L}'}(\Delta^\star_\mu(E)) \subseteq T_{\mathcal{L}'}(\Delta_\mu(E))$, where $\langle \mathcal{L}' \rangle$ denotes the set of formulæ in $\mathcal{L}$ for which there exists an equivalent formula in $\mathcal{L}'$.

Let us briefly discuss these properties. *Containment* ensures that $\Delta^\star$ can be seen as a form of approximation of Δ when applied in the $\mathcal{L}'$ fragment. On the other hand, *invariance* states that if Δ behaves as expected (i.e., the result of the merging is equivalent to a base contained in $\mathcal{K}_{\mathcal{L}'}$) there is no need for $\Delta^\star$ to do more. Containment and invariance jointly imply that for each $E \in \mathcal{E}_{\mathcal{L}'}$ and $\mu \in \mathcal{L}'$ such that $\Delta_\mu(E) \in \mathcal{K}_{\langle \mathcal{L}' \rangle}$, $\Delta_\mu(E) \equiv \Delta^\star_\mu(E)$ holds. The first two conditions are rather independent from $\mathcal{L}'$, but relate the refined operator $\Delta^\star$ to the original merging operator Δ in certain ways. To be more precise, *consistency* states that the refined operator $\Delta^\star$ should yield a consistent merging exactly if the original operator Δ does so. *Equivalence* means that for two equivalent profiles, the definition of the $\Delta^\star$-operator should not be syntax-dependent: mergings which are equivalent w.r.t Δ are also equivalent w.r.t. $\Delta^\star$. Observe that these properties are a generalization of the ones proposed for refinements in the context of belief revision [**?**]. On the one hand properties (1), (3) and (4) are direct generalizations of the corresponding ones for belief revision. On the other hand property (2), which reflects the syntax-independence, has to take the belief profiles into account in order to distinguish the merging strategies.

One can show that a Δ-refinement $\Delta^\star$ for a β-fragment satisfies the properties:

(1) $\mathrm{Mod}(\Delta^\star_\mu(E)) \subseteq Cl_\beta(\mathrm{Mod}(\Delta_\mu(E)))$ and
(2) $\mathrm{Mod}(\Delta^\star_\mu(E)) = \mathrm{Mod}(\Delta_\mu(E))$ in case $\mathrm{Mod}(\Delta_\mu(E))$ is closed under β.

This motivates the following candidates for such refinements.

Definition 3.3. Let Δ be a merging operator and $\beta \in \mathcal{B}$. We define the *Cl_β-based* refined operator Δ^{Cl_β} as:

$$\mathrm{Mod}(\Delta^{Cl_\beta}_\mu(E)) = Cl_\beta(\mathcal{M}).$$

where $\mathcal{M} = \mathrm{Mod}(\Delta_\mu(E))$.

We define the Min-*based* refined operator Δ^{Min} as[3]:

$$\mathrm{Mod}(\Delta_\mu^{\mathrm{Min}}(E)) = \begin{cases} \mathcal{M} & \text{if } Cl_\beta(\mathcal{M}) = \mathcal{M}, \\ \{\mathrm{Min}(\mathcal{M})\} & \text{otherwise,} \end{cases}$$

where Min is a function that selects the smallest interpretation from any set of interpretations, the order on the interpretations being given and fixed.

We define the Min/Cl_β-*based* refined operator $\Delta^{\mathrm{Min}/Cl_\beta}$ as:

$$\Delta_\mu^{\mathrm{Min}/Cl_\beta}(E) = \begin{cases} \Delta_\mu^{\mathrm{Min}}(E) & \text{if } \#(\mathcal{M}, E) = 0 \\ \Delta_\mu^{Cl_\beta}(E) & \text{otherwise.} \end{cases}$$

The first two refinements above are inspired from the ones proposed for revision [**?**]. The last refinement takes the origin of beliefs into account. In words, this refinement selects the smallest interpretation of a given and fixed order if $\Delta_\mu(E)$ shares no model with the bases of E. Otherwise, it switches to the Cl_β-based refinement. The intuition behind this is to ensure that the fairness postulate (IC4) is satisfied. We will come back to this issue in the next section.

PROPOSITION 3.4. *For any merging operator $\Delta : \mathcal{E}_\mathcal{L} \times \mathcal{L} \to \mathcal{K}_\mathcal{L}$, $\beta \in \mathcal{B}$ and $\mathcal{L}' \subseteq \mathcal{L}$ a β-fragment, the operators Δ^{Cl_β}, Δ^{Min} and $\Delta_\mu^{\mathrm{Min}/Cl_\beta}$ are Δ-refinements for $\mathcal{L}'$.*

PROOF. Let $\mu \in \mathcal{L}'$, $E \in \mathcal{E}_{\mathcal{L}'}$ and $\beta \in \mathcal{B}$. We show that each operator yields a base from $\mathcal{K}_{\mathcal{L}'}$ and moreover satisfies consistency, equivalence, containment and invariance, cf. Definition **??**.

Δ^{Cl_β}: $\Delta_\mu^{Cl_\beta}(E) \in \mathcal{L}'$ since by assumption $\mathcal{L}'$ is a β-fragment and thus closed under β. Consistency holds since $\mathrm{Mod}(\Delta_\mu^{Cl_\beta}(E)) = Cl_\beta(\mathrm{Mod}(\Delta_\mu(E)))$ and $Cl_\beta(\mathcal{M}) = \emptyset$ if and only if $\mathcal{M} = \emptyset$. Equivalence holds since $\mathrm{Mod}(\Delta_{\mu_1}(E_1)) = \mathrm{Mod}(\Delta_{\mu_2}(E_2))$ implies $Cl_\beta(\mathrm{Mod}(\Delta_{\mu_1}(E_1))) = Cl_\beta(\mathrm{Mod}(\Delta_{\mu_2}(E_2)))$. Containment: let $\phi \in T_{\mathcal{L}'}(\Delta_\mu(E))$, i.e., $\phi \in \mathcal{L}'$ and $\mathrm{Mod}(\Delta_\mu(E)) \subseteq \mathrm{Mod}(\phi)$. By monotonicity of Cl_β, we have $Cl_\beta(\mathrm{Mod}(\Delta_\mu(E))) \subseteq Cl_\beta(\mathrm{Mod}(\phi))$. Since $\phi \in \mathcal{L}'$, it holds that $Cl_\beta(\mathrm{Mod}(\Delta_\mu(E))) \subseteq \mathrm{Mod}(\phi)$, and therefore $\phi \in T_{\mathcal{L}'}(\Delta_\mu^{Cl_\beta}(E))$. Invariance: let $\phi \in T_{\mathcal{L}'}(\Delta_\mu^{Cl_\beta}(E))$, i.e., $\phi \in \mathcal{L}'$ and $Cl_\beta(\mathrm{Mod}(\Delta_\mu(E))) \subseteq \mathrm{Mod}(\phi)$. By hypothesis $Cl_\beta(\mathrm{Mod}(\Delta_\mu(E))) \supseteq \mathrm{Mod}(\Delta_\mu(E))$, therefore $\phi \in T_{\mathcal{L}'}(\Delta_\mu(E))$.

Δ^{Min}: if $\mathrm{Mod}(\Delta_\mu^{\mathrm{Min}}(E))) = Cl_\beta(\mathrm{Mod}(\Delta_\mu(E)))$ (i.e., $\Delta_\mu(E) \in \mathcal{K}_{\langle\mathcal{L}'\rangle}$) then Δ^{Min} satisfies all the required properties as shown above; otherwise consistency, equivalence and containment hold since $\mathrm{Mod}(\Delta_\mu^{\mathrm{Min}}(E))) = \{\mathrm{Min}(\mathrm{Mod}(\Delta_\mu(E)))\}$. Moreover, by definition each fragment contains a formula ϕ with $\mathrm{Mod}(\phi) = \{\omega\}$ where ω is an arbitrary interpretation. Thus, $\Delta_\mu^{\mathrm{Min}}(E) \in \mathcal{L}'$ also holds in this case.

$\Delta^{\mathrm{Min}/Cl_\beta}$: satisfies the required properties since Δ^{Cl_β} and Δ^{Min} satisfy them. $\square$

Example 3.5. Consider the profile E, the integrity constraint μ given in Example **??**, the distance-based merging operator $\Delta^{d_H,\Sigma}$, and let β be the binary AND function. Let us have the following order over the set of interpretations on $\{a,b\}$: $\emptyset < \{a\} < \{b\} < \{a,b\}$. The result of merging is $\mathrm{Mod}(\Delta_\mu^{d_H,\Sigma}(E)) = \{\{a\},\{b\}\}$. The Min-based $\Delta^{d_H,\Sigma}$-refined operator, denoted by Δ^{Min}, is such that $\mathrm{Mod}(\Delta_\mu^{\mathrm{Min}}(E)) = \{\{a\}\}$. The Cl_β-based $\Delta^{d_H,\Sigma}$-refined operator, denoted by Δ^{Cl_β}, is such that $\mathrm{Mod}(\Delta_\mu^{Cl_\beta}(E)) = \{\{a\},\{b\},\emptyset\}$. The same result is achieved by the Min/Cl_β-based $\Delta^{d_H,\Sigma}$-refined operator since $\#(\mathrm{Mod}(\Delta_\mu^{d_H,\Sigma}(E)), E) = 2$.

In what follows we show how to capture not only a particular refined operator but characterize the class of *all* refined operators. Towards a more general approach to define merging operators

[3]Note that the Min-based refinement depends on β since it involves a check to determine if the set of models is closed under β. But in order to stress that selecting the smallest interpretation is independent of β, we will use Δ^{Min} instead of $\Delta^{\mathrm{Min}_\beta}$.

fitting into fragments that are obtained by refining existing operators we define a concept of mapping, which has to satisfy some basic properties which we give in the forthcoming definition. The intuition behind these mappings is that they model a post-processing computation after the original merging operator was applied. Hence, these mappings take as argument the set of models which the original merging operator returned, and provide as a result again a set of models. To capture profile-dependent refinements such as the Min/Cl_β-based refined operator, we provide these mappings with a second argument, which corresponds to the models of the profile. The mappings which we define here differ from the ones used for revision, since those only needed one argument [**?**]. They aim at producing refined merging operators that have different merging strategies and that take into account the different sources of information. Therefore it is natural that in case of merging the mappings take also into account the belief profile. However, we shall identify mappings for which this second argument plays no role and call them profile-independent.

Definition 3.6. Given $\beta \in \mathcal{B}$, we define a β-*mapping*, f_β, as an application which to every set of models $\mathcal{M}$ and every multi-set of sets of models $\mathcal{X}$ associates a set of models $f_\beta(\mathcal{M}, \mathcal{X})$ such that:

(1) $Cl_\beta(f_\beta(\mathcal{M}, \mathcal{X})) = f_\beta(\mathcal{M}, \mathcal{X})$ (i.e., $f_\beta(\mathcal{M}, \mathcal{X})$ is closed under β)
(2) $f_\beta(\mathcal{M}, \mathcal{X}) \subseteq Cl_\beta(\mathcal{M})$
(3) if $\mathcal{M} = Cl_\beta(\mathcal{M})$, then $f_\beta(\mathcal{M}, \mathcal{X}) = \mathcal{M}$
(4) If $\mathcal{M} \neq \emptyset$, then $f_\beta(\mathcal{M}, \mathcal{X}) \neq \emptyset$.

A β-mapping f_β is profile-independent if for all set of models $\mathcal{M}$ and all multisets of sets of models $\mathcal{X}$ and $\mathcal{X}'$, we have $f_\beta(\mathcal{M}, \mathcal{X}) = f_\beta(\mathcal{M}, \mathcal{X}')$.

For example, the Cl_β-based refinement can be expressed through a profile-independent β-mapping. Indeed, for $\mathcal{M} = \mathrm{Mod}(\Delta_\mu(E))$ and $\mathcal{X} = \mathcal{M}\mathrm{od}(E)$ we can define $\mathrm{Mod}(\Delta_\mu^{Cl_\beta}(E)) = f_\beta(\mathcal{M}, \mathcal{X})$ where $f_\beta(\mathcal{M}, \mathcal{X}) = Cl_\beta(\mathcal{M})$. Similarly the Min-based refinement can be expressed through a profile-independent β-mapping: $\mathrm{Mod}(\Delta_\mu^{\mathrm{Min}}(E)) = f_\beta(\mathcal{M}, \mathcal{X})$ where

$$f_\beta(\mathcal{M}, \mathcal{X}) = \begin{cases} \mathcal{M} & \text{if } Cl_\beta(\mathcal{M}) = \mathcal{M}, \\ \{\mathrm{Min}(\mathcal{M})\} & \text{otherwise,} \end{cases}$$

In contrast the Min/Cl_β-based refinement can be expressed through β-mapping that is profile-dependent.

The concept of mappings allows us to define a family of refined operators for fragments of classical logic that captures the examples given before.

Definition 3.7. Let $\Delta : \mathcal{E}_\mathcal{L} \times \mathcal{L} \to \mathcal{K}_\mathcal{L}$ be a merging operator and $\mathcal{L}' \subseteq \mathcal{L}$ be a β-fragment of classical logic with $\beta \in \mathcal{B}$. For a β-mapping f_β we denote with $\Delta^{f_\beta} : \mathcal{E}_{\mathcal{L}'} \times \mathcal{L}' \to \mathcal{K}_{\mathcal{L}'}$ the operator for $\mathcal{L}'$ defined as

$$\mathrm{Mod}(\Delta_\mu^{f_\beta}(E)) = f_\beta(\mathrm{Mod}(\Delta_\mu(E)), \mathcal{M}\mathrm{od}(E)).$$

The class $[\Delta, \mathcal{L}']$ contains all operators Δ^{f_β} where f_β is a β-mapping and $\beta \in \mathcal{B}$ such that $\mathcal{L}'$ is a β-fragment.

The next proposition is central in reflecting that the class $[\Delta, \mathcal{L}']$ captures all refined operators we had in mind, cf. Definition **??**.

PROPOSITION 3.8. *Let $\Delta : \mathcal{E}_\mathcal{L} \times \mathcal{L} \to \mathcal{K}_\mathcal{L}$ be a merging operator and $\mathcal{L}' \subseteq \mathcal{L}$ a characterizable fragment of classical logic. Then, $[\Delta, \mathcal{L}']$ is the set of all Δ-refinements for $\mathcal{L}'$.*

PROOF. Since $\mathcal{L}'$ is a characterizable fragment it is also a β-fragment for some $\beta \in \mathcal{B}$. Let $\Delta^\star \in [\Delta, \mathcal{L}']$. We show that $\Delta^\star$ is a Δ-refinement for $\mathcal{L}'$. Since $\Delta^\star \in [\Delta, \mathcal{L}']$ there exists a β-mapping f_β, such that $\mathrm{Mod}(\Delta_\mu^\star(E)) = f_\beta(\mathrm{Mod}(\Delta_\mu(E)), \mathcal{M}\mathrm{od}(E))$ for all $\mu \in \mathcal{L}'$ and $E \in \mathcal{E}_{\mathcal{L}'}$. Since f_β satisfies Property **??** in Definition **??** and $\mathcal{L}'$ is a β-fragment, $\Delta_\mu^\star(E)$

is indeed in $\mathcal{K}_{\mathcal{L}'}$. Consistency for $\Delta^\star$: Let $\mu \in \mathcal{L}'$ and $E \in \mathcal{E}_{\mathcal{L}'}$. If $\mathrm{Mod}(\Delta_\mu(E)) \neq \emptyset$ then $\mathrm{Mod}(\Delta^\star_\mu(E)) = f_\beta(\mathrm{Mod}(\Delta_\mu(E)), \mathcal{M}\mathrm{od}(E)) \neq \emptyset$ by Property **??** in Definition **??**. If $\mathrm{Mod}(\Delta_\mu(E)) = \emptyset$, we make use of the fact that $Cl_\beta(\emptyset) = \emptyset$ holds for all $\beta \in \mathcal{B}$. By Property **??** in Definition **??**, we get $\mathrm{Mod}(\Delta^\star_\mu(E)) = f_\beta(\mathrm{Mod}(\Delta_\mu(E)), \mathcal{M}\mathrm{od}(E)) \subseteq Cl_\beta(\mathrm{Mod}(\Delta_\mu(E))) = \emptyset$. Equivalence for $\Delta^\star$ is clear by definition and since f_β is defined on sets of models. To show containment for $\Delta^\star$, let $\phi \in T_{\mathcal{L}'}(\Delta_\mu(E))$, i.e., $\phi \in \mathcal{L}'$ and $\mathrm{Mod}(\Delta_\mu(E)) \subseteq \mathrm{Mod}(\phi)$. We have $Cl_\beta(\mathrm{Mod}(\Delta_\mu(E))) \subseteq Cl_\beta(\mathrm{Mod}(\phi))$ by monotonicity of Cl_β. By Property **??** of Definition **??**, $\mathrm{Mod}(\Delta^\star_\mu(E)) \subseteq Cl_\beta(\mathrm{Mod}(\Delta_\mu(E)))$. Since $\phi \in \mathcal{L}'$ we have $Cl_\beta(\mathrm{Mod}(\phi)) = \mathrm{Mod}(\phi)$. Thus, $\mathrm{Mod}(\Delta^\star_\mu(E)) \subseteq \mathrm{Mod}(\phi)$, i.e., $\phi \in T_{\mathcal{L}'}(\Delta^\star_\mu(E))$. Finally, we require invariance for $\Delta^\star$: If $\Delta_\mu(E) \in \mathcal{K}_{\langle \mathcal{L}' \rangle}$, we have $Cl_\beta(\mathrm{Mod}(\Delta_\mu(E))) = \mathrm{Mod}(\Delta_\mu(E))$ since $\mathcal{L}'$ is a β-fragment. By Property **??** in Definition **??**, we have $\mathrm{Mod}(\Delta^\star_\mu(E)) = f_\beta(\mathrm{Mod}(\Delta_\mu(E)), \mathcal{M}\mathrm{od}(E)) = \mathrm{Mod}(\Delta_\mu(E))$. Thus $T_{\mathcal{L}'}(\Delta^\star_\mu(E)) \subseteq T_{\mathcal{L}'}(\Delta_\mu(E))$ as required.

Let $\Delta^\star$ be a Δ-refinement for $\mathcal{L}'$. We show that $\Delta^\star \in [\Delta, \mathcal{L}']$. Let f be defined as follows for any set $\mathcal{M}$ of interpretations and $\mathcal{X}$ a multi-set of sets of interpretations: $f(\emptyset, \mathcal{X}) = \emptyset$. For $\mathcal{M} \neq \emptyset$, if $Cl_\beta(\mathcal{M}) = \mathcal{M}$ then $f(\mathcal{M}, \mathcal{X}) = \mathcal{M}$, otherwise if there exists a pair $(E, \mu) \in (\mathcal{E}_{\mathcal{L}'}, \mathcal{L}')$ such that $\mathcal{M}\mathrm{od}(E) = \mathcal{X}$ and $\mathrm{Mod}(\Delta_\mu(E)) = \mathcal{M}$, then we define $f(\mathcal{M}, \mathcal{X}) = \mathrm{Mod}(\Delta^\star_\mu(E))$. If there is no such a pair (E, μ) then we arbitrarily define $f(\mathcal{M}, \mathcal{X}) = Cl_\beta(\mathcal{M})$. Thus the refined operator $\Delta^\star$ behaves like the operator Δ^f.

We show that such a mapping f is a β-mapping. Since $\Delta^\star$ is a Δ-refinement for $\mathcal{L}'$, it satisfies the property of equivalence, thus the actual choice of the pair (E, μ) is not relevant, i.e., given $(\mathcal{M}, \mathcal{X})$, and pairs (E, μ) and (E', μ') such that $\mathcal{M}\mathrm{od}(E) = \mathcal{M}\mathrm{od}(E') = \mathcal{X}$ and $\Delta_\mu(E) = \Delta_{\mu'}(E') = \mathcal{M}$, we have that $\Delta^\star_\mu(E)$ is equivalent to $\Delta^\star_{\mu'}(E')$. Thus f is well-defined.

We continue to show that the four properties in Definition **??** hold for f. Property **??** is ensured since for every pair $(\mathcal{M}, \mathcal{X})$, $f(\mathcal{M}, \mathcal{X})$ is closed under β. Indeed, either $f(\mathcal{M}, \mathcal{X}) = \mathcal{M}$ if $\mathcal{M}$ is closed under β, or $f(\mathcal{M}, \mathcal{X}) = \mathrm{Mod}(\Delta^\star_\mu(E))$ and since $\Delta^\star_\mu(E) \in \mathcal{K}_{\mathcal{L}'}$ its set of models is closed under β, or $f(\mathcal{M}, \mathcal{X}) = Cl_\beta(\mathcal{M})$, and thus is also closed under β. Let us show Property **??**, i.e., $f(\mathcal{M}, \mathcal{X}) \subseteq Cl_\beta(\mathcal{M})$ for any pair $(\mathcal{M}, \mathcal{X})$. It is obvious when $\mathcal{M} = \emptyset$ (then $f(\mathcal{M}, \mathcal{X}) = \emptyset$), as well as when $f(\mathcal{M}, \mathcal{X}) = Cl_\beta(\mathcal{M})$ and when $\mathcal{M}$ is closed and thus $f(\mathcal{M}, \mathcal{X}) = \mathcal{M}$. Otherwise $f(\mathcal{M}, \mathcal{X}) = \mathrm{Mod}(\Delta^\star_\mu(E))$ and since $\Delta^\star$ satisfies containment $\mathrm{Mod}(\Delta^\star_\mu(E)) \subseteq Cl_\beta(\mathrm{Mod}(\Delta_\mu(E))$. Therefore in any case we have $f(\mathcal{M}, \mathcal{X}) \subseteq Cl_\beta(\mathcal{M})$. Property **??** follows trivially from the definition of $f(\mathcal{M}, \mathcal{X})$ when $\mathcal{M}$ is closed under β. Property **??** is ensured by consistency of $\Delta^\star$. □

An easy consequence of this characterization of refined operators is the following. Let $\Delta\colon \mathcal{E}_{\mathcal{L}} \times \mathcal{L} \to \mathcal{K}_{\mathcal{L}}$ be a merging operator, let $\mathcal{L}' \subseteq \mathcal{L}$ be a β-fragment of classical logic with $\beta \in \mathcal{B}$, and let $\Delta^\star\colon \mathcal{E}_{\mathcal{L}'} \times \mathcal{L}' \to \mathcal{K}_{\mathcal{L}'}$ be a Δ-refinement for $\mathcal{L}'$. Then for each $E \in \mathcal{E}_{\mathcal{L}'}$, $\mathrm{Mod}(\Delta^\star(E)) \subseteq Cl_\beta(\mathrm{Mod}(\Delta(E)))$.

4. PRESERVATION OF POSTULATES

The aim of this section is to study whether refinements of merging operators preserve the IC postulates. We first show that if the initial operator satisfies the most basic postulates ((IC0)–(IC3)), then so does any of its refinements. It turns out that this result cannot be extended to the remaining postulates. For (IC4) we characterize a subclass of refinements for which this postulate is preserved. For the four remaining postulates we study two representative kinds of distance-based merging operators. We show that postulates (IC5) and (IC7) are violated for all of our proposed examples of refined operators with the exception of the Min-based refinement. For (IC6) and (IC8) the situation is even worse in the sense that no refinement of our proposed examples of merging operators can satisfy them neither for $\mathcal{L}_{Horn}$ nor for $\mathcal{L}_{Krom}$. Finally, we study the preservation of the majority postulate (Maj). Table **??** gives an overview of most of the results of this section.

Table II. Overview of some results for (IC0)–(IC8) for refinements together with restrictions under which the results hold.

Postulate	Satisfied	Restriction
IC0 - IC3	yes	no restriction (Prop. **??**)
IC4	yes	any fair refinement (Prop. **??**) Δ^{Cl_β} with $\Delta^{d,\Sigma}$, counting distance d satisfying triangular inequality (Prop. **??**)
	no	Δ^{Min} for $\mathcal{L}_{Horn}$ and $\mathcal{L}_{Krom}$ with $\Delta \in \{\Delta^{d,\Sigma}, \Delta^{d,\mathrm{GMax}}\}$, counting distance d (Prop. **??**) Δ^{Min} with $\Delta \in \{\Delta^{d_D,\Sigma}, \Delta^{d_D,\mathrm{GMax}}\}$ (Prop. **??**) Δ^{Cl_β} for $\mathcal{L}_{Horn}$ and $\mathcal{L}_{Krom}$ with $\Delta^{d,\mathrm{GMax}}$, non-drastic counting distance d (Prop. **??**)
IC5, IC7	yes	Δ^{Min} (Prop. **??**)
	no	any fair refinement for $\mathcal{L}_{Horn}$ with $\Delta \in \{\Delta^{d,\Sigma}, \Delta^{d,\mathrm{GMax}}\}$, counting distance d (Prop. **??**) Δ^{Cl_β} and $\Delta^{\mathrm{Min}/Cl_\beta}$ for $\mathcal{L}_{Horn}$ and $\mathcal{L}_{Krom}$ with $\Delta \in \{\Delta^{d,\Sigma}, \Delta^{d,\mathrm{GMax}}\}$, counting distance d (Prop. **??**)
IC6, IC8	no	for $\mathcal{L}_{Horn}$ and $\mathcal{L}_{Krom}$ with $\Delta \in \{\Delta^{d,\Sigma}, \Delta^{d,\mathrm{GMax}}\}$, counting distance d (Prop. **??**)
Maj	yes	Δ^{Cl_β} with $\Delta^{d,\Sigma}$, counting distance d (Prop. **??**)
	no	Δ^{f_β} s.t. $f_\beta(\mathcal{M}, \mathcal{X}) \neq Cl_\beta(\mathcal{M})$ for non-closed sets $\mathcal{M}$; for $\mathcal{L}_{Horn}$ and $\mathcal{L}_{Krom}$ with $\Delta^{d,\Sigma}$, counting distance d (Prop. **??**) $\Delta^{\mathrm{Min}/Cl_\beta}$ for $\mathcal{L}_{Horn}$ and $\mathcal{L}_{Krom}$ with $\Delta^{d,\Sigma}$, non-drastic counting distance d (Prop. **??**)

In the following, within a characterizable fragment, it is implicit that any β-mapping we refer to uses the β which characterizes the fragment. This means that within $\mathcal{L}_{Horn}$ (resp. $\mathcal{L}_{Krom}$) a β-mapping is a $\wedge$-mapping (resp., maj_3-mapping).

4.1. Basic Postulates IC 0 – IC 3

We first prove that, as in the framework of belief revision, the refined merging operatores preserve the basic postulates.

PROPOSITION 4.1. *Let Δ be a merging operator satisfying postulates* (IC0)–(IC3)*, and $\mathcal{L}' \subseteq \mathcal{L}$ a characterizable fragment. Then each Δ-refinement for $\mathcal{L}'$ satisfies* (IC0)–(IC3) *in $\mathcal{L}'$ as well.*

PROOF. Since $\mathcal{L}'$ is characterizable there exists a $\beta \in \mathcal{B}$, such that $\mathcal{L}'$ is a β-fragment. Let $\Delta^\star$ be a Δ-refinement for $\mathcal{L}'$. According to Proposition **??** we can assume that $\Delta^\star \in [\Delta, \mathcal{L}']$ is an operator of form Δ^{f_β} where f_β is a suitable β-mapping. In what follows, we can restrict ourselves to $E \in \mathcal{E}_{\mathcal{L}'}$ and to $\mu \in \mathcal{L}'$ since we have to show that Δ^{f_β} satisfies (IC0)–(IC3) in $\mathcal{L}'$.

(IC0): Since Δ satisfies (IC0), $\mathrm{Mod}(\Delta_\mu(E)) \subseteq \mathrm{Mod}(\mu)$. Thus, by monotonicity of the closure $Cl_\beta(\mathrm{Mod}(\Delta_\mu(E))) \subseteq Cl_\beta(\mathrm{Mod}(\mu))$. This yields together with $\mu \in \mathcal{L}'$ and the fact that $\mathcal{L}'$ is a β-fragment, that $Cl_\beta(\mathrm{Mod}(\Delta_\mu(E))) \subseteq \mathrm{Mod}(\mu)$. According to Property **??** in Definition **??** we have $f_\beta(\mathrm{Mod}(\Delta_\mu(E)), \mathcal{M}\mathrm{od}(E)) \subseteq Cl_\beta(\mathrm{Mod}(\Delta_\mu(E)))$, and therefore by definition of $\Delta^\star$, $\mathrm{Mod}(\Delta^\star_\mu(E)) \subseteq \mathrm{Mod}(\mu)$, which proves that $\Delta^\star_\mu(E) \models \mu$.

(IC1): Suppose μ satisfiable. Since Δ satisfies (IC1), $\Delta_\mu(E)$ is satisfiable. Since Δ^{f_β} is a Δ-refinement (Proposition **??**), $\Delta^{f_\beta}_\mu(E)$ is also satisfiable by the property of consistency (see Definition **??**).

(IC2): Suppose $\bigwedge E$ is consistent with μ. Since Δ satisfies (IC2), $\Delta_\mu(E) = \bigwedge E \wedge \mu$. We have $\mathrm{Mod}(\Delta^\star_\mu(E)) = f_\beta(\mathrm{Mod}(\Delta_\mu(E)), \mathcal{M}\mathrm{od}(E)) = f_\beta(\mathrm{Mod}(\bigwedge E \wedge \mu), \mathcal{M}\mathrm{od}(E))$. Since $\bigwedge E \wedge \mu \in \mathcal{L}'$ by Property **??** of Definition **??** we have $\mathrm{Mod}(\Delta^\star_\mu(E)) = \bigwedge E \wedge \mu$.

(IC3): Let $E_1, E_2 \in \mathcal{E}_{\mathcal{L}'}$ and $\mu_1, \mu_2 \in \mathcal{L}'$ with $E_1 \equiv E_2$ and $\mu_1 \equiv \mu_2$. Since Δ satisfies (IC3), $\Delta_{\mu_1}(E_1) \equiv \Delta_{\mu_2}(E_2)$. By the property of equivalence in Definition **??** we have $\Delta^\star_{\mu_1}(E_1) \equiv \Delta^\star_{\mu_2}(E_2)$. □

4.2. Fairness Postulate IC 4

A natural question is whether refined operators for characterizable fragments in their full generality preserve other postulates, and if not whether one can nevertheless find some refined operators that satisfy some of the remaining postulates.

First we show that one cannot expect to extend Proposition **??** to (IC4). Indeed, in the two following propositions we present merging operators which satisfy all postulates, whereas some of their refinements violate (IC4) in certain fragments. The proof of these propositions together with other missing proofs can be found in the appendix.

PROPOSITION 4.2. *Let Δ be a merging operator with $\Delta \in \{\Delta^{d,\Sigma}, \Delta^{d,\mathrm{GMax}}\}$, where d is an arbitrary counting distance. Then the* Min*-based refined operator Δ^{Min} violates postulate* (IC4) *in $\mathcal{L}_{Horn}$ and $\mathcal{L}_{Krom}$. In case d is the drastic distance, Δ^{Min} violates postulate* (IC4) *in every characterizable fragment $\mathcal{L}' \subset \mathcal{L}$.*

PROPOSITION 4.3. *Let $\Delta = \Delta^{d,\mathrm{GMax}}$ be a merging operator where d is an arbitrary non-drastic counting distance. Then the closure-based refined operator Δ^{Cl_β} violates* (IC4) *in $\mathcal{L}_{Horn}$ and $\mathcal{L}_{Krom}$.*

To identify a class of refinements which satisfy (IC4), we now introduce the concept of fairness for Δ-refinements.

Definition 4.4. Let $\mathcal{L}'$ be a fragment of classical logic. A Δ-refinement for $\mathcal{L}'$, $\Delta^\star$, is *fair* if it satisfies the following property for each $E \in \mathcal{E}_{\mathcal{L}'}$, $\mu \in \mathcal{L}'$: If $\#(\Delta_\mu(E), E) \neq 1$ then $\#(\Delta^\star_\mu(E), E) \neq 1$.

As we will see in the following the fairness of a refined operator depends on both the initial operator and the β-mapping used to refine it.

PROPOSITION 4.5. *Let $\mathcal{L}'$ be a characterizable fragment. Then the Min/Cl_β-based refinement of any merging operator for $\mathcal{L}'$ is fair.*

To show fairness for the Cl_β-based refinement, we first prove that it coincides for certain operators with the Min/Cl_β-based refinement.

PROPOSITION 4.6. *Let $\mathcal{L}'$ be a characterizable fragment and let Δ be a merging operator with $\Delta \in \{\Delta^{d_D,\Sigma}, \Delta^{d_D,\mathrm{GMax}}\}$. Then the Cl_β-based refinement and the Min/Cl_β-based refinement coincide.*

From Proposition **??** and Proposition **??** we get the following statement about fairness of the Cl_β-based refinement.

COROLLARY 4.7. *Let $\mathcal{L}'$ be a characterizable fragment. Then the Cl_β-based refinement of both $\Delta^{d_D,\Sigma}$ and $\Delta^{d_D,\mathrm{GMax}}$ for $\mathcal{L}'$ is fair.*

Fairness turns out to be a sufficient property to preserve the postulate (IC4) as stated in the following proposition.

PROPOSITION 4.8. *Let Δ be a merging operator satisfying postulate* (IC4)*, and $\mathcal{L}' \subseteq \mathcal{L}$ a characterizable fragment. Then every fair Δ-refinement for $\mathcal{L}'$ satisfies* (IC4) *as well.*

PROOF. Consider Δ a merging operator satisfying postulate (IC4). Let $\Delta^\star$ be a fair Δ-refinement for $\mathcal{L}'$. If $\Delta^\star$ does not satisfy (IC4), then there exist $E = \{K_1, K_2\}$ with $K_1, K_2 \in \mathcal{L}'$ and $\mu \in \mathcal{L}'$, with $K_1 \models \mu$ and $K_2 \models \mu$ such that $\mathrm{Mod}(\Delta^\star_\mu(E)) \cap \mathrm{Mod}(K_1) \neq \emptyset$ and $\mathrm{Mod}(\Delta^\star_\mu(E)) \cap \mathrm{Mod}(K_2) = \emptyset$, i.e., such that $\#(\Delta^\star_\mu(E), E) = 1$. Since Δ satisfies postulate (IC4) we have $\#(\Delta_\mu(E), E) \neq 1$, thus contradicting the fairness property in Definition **??**. □

With Proposition **??** at hand, we can conclude that the Cl_β-based refinement of both $\Delta^{d_D,\Sigma}$ and $\Delta^{d_D,\mathrm{GMax}}$ for $\mathcal{L}'$ as well as the Min/Cl_β-based refinement of any merging operator satisfies (IC4).

Remark 4.9. Observe that the distance which is used in distance-based operators matters for the preservation of (IC4), as well as for fairness. Indeed, while the Cl_β-refinement of $\Delta^{d_D,\mathrm{GMax}}$

is fair, and therefore satisfies (IC4), the Cl_β-refinement of $\Delta^{d,\mathrm{GMax}}$ where d is an arbitrary non-drastic counting distance violates postulate (IC4) in $\mathcal{L}_{Horn}$ and $\mathcal{L}_{Krom}$, and therefore is not fair.

For all refinements considered so far we know whether (IC4) is preserved or not, with one single exception: the Cl_β-refinement of $\Delta^{d,\Sigma}$ where d is an arbitrary non-drastic counting distance. In this case we get a partial positive result.

PROPOSITION 4.10. *Let Δ be a merging operator with $\Delta = \Delta^{d,\Sigma}$, where d is an arbitrary counting distance that satisfies the triangular inequality. Then the closure-based refined operator Δ^{Cl_β} satisfies postulate* (IC4) *in any characterizable fragment.*

Remark 4.11. Proposition **??** together with Proposition **??** shows that the aggregation function that is used in distance-based operators matters for the preservation of the postulate (IC4).

Interestingly Proposition **??** (recall that the Hamming distance satisfies the triangular inequality) together with the following proposition show that fairness, which is a sufficient condition for preserving (IC4) is not a necessary one.

PROPOSITION 4.12. *The Cl_β-refinement of $\Delta^{d_H,\Sigma}$ is neither fair in $\mathcal{L}_{Horn}$ nor in $\mathcal{L}_{Krom}$.*

4.3. Postulates IC 5 – IC 8

It turns out that our refined operators have a similar behavior regarding the satisfaction of postulates (IC5) & (IC7) as well as (IC6) & (IC8). Therefore we will deal with the remaining postulates in pairs.

In the case of belief revision [**?**] the only refinement which is proved to preserve the postulate (R5) is the Min-based one. We obtain a similar result in the context of belief merging for the postulate (IC7). We prove that the Min-based refinement satisfies (IC5) and (IC7), whereas the refined operators Δ^{Cl_β} and $\Delta^{\mathrm{Min}/Cl_\beta}$ violate these two postulates.

PROPOSITION 4.13. *Let Δ be a merging operator satisfying postulates* (IC5) *and* (IC6) *(resp.* (IC7) *and* (IC8)*), and $\mathcal{L}' \subseteq \mathcal{L}$ a characterizable fragment. Then the refined operator Δ^{Min} for $\mathcal{L}'$ satisfies* (IC5) *(resp.* (IC7)*) in $\mathcal{L}'$ as well.*

PROPOSITION 4.14. *Let Δ be a merging operator with $\Delta \in \{\Delta^{d,\Sigma}, \Delta^{d,\mathrm{GMax}}\}$, where d is an arbitrary counting distance. Then the refined operators Δ^{Cl_β} and $\Delta^{\mathrm{Min}/Cl_\beta}$ violate postulates* (IC5) *and* (IC7) *in $\mathcal{L}_{Horn}$ and in $\mathcal{L}_{Krom}$.*

In the Horn fragment the negative results of Proposition **??** can be extended to any fair refinement.

PROPOSITION 4.15. *Let Δ be a merging operator with $\Delta \in \{\Delta^{d,\Sigma}, \Delta^{d,\mathrm{GMax}}\}$, where d is an arbitrary counting distance. Then any fair refined operator Δ^* violates* (IC5) *and* (IC7) *in $\mathcal{L}_{Horn}$.*

We leave it as an open question whether this proposition can be extended to Krom.

In the case of belief revision [**?**] all the studied refinements are proved to violate the postulate (R6). We obtain similar results in the context of belief merging for the postulate (IC8). We prove that any refinement of the two kinds of operators we considered violates both (IC6) and (IC8) in $\mathcal{L}_{Horn}$ and in $\mathcal{L}_{Krom}$.

PROPOSITION 4.16. *Let Δ be a merging operator with $\Delta \in \{\Delta^{d,\Sigma}, \Delta^{d,\mathrm{GMax}}\}$, where d is an arbitrary counting distance. Then any refined operator $\Delta^\star$ violates postulates* (IC6) *and* (IC8) *in $\mathcal{L}_{Horn}$ and in $\mathcal{L}_{Krom}$.*

This last result shows us that there is a clear limit with respect to which postulates we can satisfy with the technique of refinements if the original merging operator is based on a counting distance. On the other hand, there exists a distance-based Horn merging operator [**?**], that satisfies all postulates but uses a distance that does not classify as a counting distance. This leads to the following open question: Does there exist a refined operator satisfying postulates (IC6) and (IC8) if the original operator is based on a counting distance but uses a different aggregation function or can Proposition **??** be generalized to arbitrary counting distance-based merging operators?

4.4. Majority

As we said in the introduction there are two main families of merging operators, the majority and the arbitration families. As previously observed the arbitration postulate does not make sense in our framework. Indeed this postulate involves disjunction of formulae, while our fragments are not closed under disjunction, i.e., given two formulae μ_1 and μ_2 in a fragment $\mathcal{L}'$, there is no reason that still $\mu_1 \vee \mu_2 \in \mathcal{L}'$. So we will focus on the majority postulate. A natural question is whether a refinement of a majority merging operator is still a majority operator. As we show below, while the closure-based refinement of a majority merging operator is still a majority operator, it seems not to be the case for a large variety of other refinements.

First we show a positive result for the refinement by closure.

PROPOSITION 4.17. *Let Δ be a merging operator with $\Delta = \Delta^{d,\Sigma}$, where d is an arbitrary counting distance. Then the closure-based refined operator Δ^{Cl_β} satisfies postulate (Maj) in any characterizable fragment.*

PROOF. Since $\Delta = \Delta^{d,\Sigma}$ satisfies (Maj), there exists an integer n such that $\mathrm{Mod}(\Delta_\mu(E_1 \sqcup E_2^n)) \subseteq \mathrm{Mod}(\Delta_\mu(E_2))$. Thus, $Cl_\beta(\mathrm{Mod}(\Delta_\mu(E_1 \sqcup E_2^n))) \subseteq Cl_\beta(\mathrm{Mod}(\Delta_\mu(E_2)))$ and it follows $\Delta_\mu^{Cl_\beta}(E_1 \sqcup E_2^n) \models \Delta_\mu^{Cl_\beta}(E_2)$. □

Second we investigate refinements that do not make use of the closure, and for them obtain a general negative result.

PROPOSITION 4.18. *Let Δ be a merging operator with $\Delta = \Delta^{d,\Sigma}$, where d is a counting distance. Then any refinement Δ^{f_β} such that $f_\beta(\mathcal{M}, \mathcal{X}) \neq Cl_\beta(\mathcal{M})$ for every non-β-closed set of models $\mathcal{M}$ violates postulate (Maj) in $\mathcal{L}_{Horn}$ and $\mathcal{L}_{Krom}$.*

The Min-based refinement is a special case of Proposition **??**. Therefore, Δ^{Min} violates postulate (Maj) in $\mathcal{L}_{Horn}$ and $\mathcal{L}_{Krom}$. The next proposition shows that it is also the case for $\Delta^{\mathrm{Min}/Cl_\beta}$.

PROPOSITION 4.19. *Let Δ be a merging operator with $\Delta = \Delta^{d,\Sigma}$, where d is a non-drastic counting distance. Then $\Delta^{\mathrm{Min}/Cl_\beta}$ violates postulate (Maj) in $\mathcal{L}_{Horn}$ and $\mathcal{L}_{Krom}$.*

PROOF. If $\#(\mathcal{M}, E) = 0$ then $\Delta_\mu^{\mathrm{Min}/Cl_\beta}(E) = \Delta_\mu^{\mathrm{Min}}(E)$. Therefore the examples developed in the first part of proof of Proposition **??** show that $\Delta^{\mathrm{Min}/Cl_\beta}$ violates postulate (Maj) as well. □

Concerning the drastic distance and the Min/Cl_β-based refinement, recall that $\Delta^{\mathrm{Min}/Cl_\beta}$ and Δ^{Cl_β} coincide in this situation (cf. Proposition **??**). Thus by Proposition **??**, postulate (Maj) remains satisfied.

COROLLARY 4.20. *Let Δ be a merging operator with $\Delta = \Delta^{d_D,\Sigma}$. Then $\Delta^{\mathrm{Min}/Cl_\beta}$ satisfies postulate (Maj) in any characterizable fragment.*

5. GENERALIZATIONS

A natural extension of this work is to study merging when only the belief bases of the profile are in the fragment, or when only the formula representing the integrity constraint is in the fragment. Given a merging operator Δ, we call $\Delta^*\colon \mathcal{E}_{\mathcal{L}'} \times \mathcal{L} \to \mathcal{K}_{\mathcal{L}'}$ a Δ*-left-refinement (for $\mathcal{L}'$)* if it satisfies all properties given in Definition **??** with profiles in $\mathcal{E}_{\mathcal{L}'}$ and integrity constraints in $\mathcal{L}$. Similarly we call $\Delta^*\colon \mathcal{E}_{\mathcal{L}} \times \mathcal{L}' \to \mathcal{K}_{\mathcal{L}'}$ a Δ*-right-refinement (for $\mathcal{L}'$)* if it satisfies all properties given in Definition **??** with profiles in $\mathcal{E}_{\mathcal{L}}$ and integrity constraints in $\mathcal{L}'$. It is then easy to check that the characterization given in Proposition **??** still holds, that is that any Δ-left-refinement (resp., Δ-right-refinement) can be defined as Δ^{f_β} for some β-mapping f_β.

Table III. Overview of some results for (IC0)–(IC8) for refinements in the Horn and Krom fragment ($x \in \{\Sigma, \mathrm{GMax}\}$, $d \in \{d_H, d_D\}$).

	$(\Delta^{d_H,\Sigma})^{Cl_\beta}$	$(\Delta^{d_H,\mathrm{GMax}})^{Cl_\beta}$	$(\Delta^{d_D,x})^{Cl_\beta}$	$(\Delta^{d,x})^{\mathrm{Min}}$	$(\Delta^{d,x})^{\mathrm{Min}/Cl_\beta}$
IC0 - IC3	+	+	+	+	+
IC4	+	−	+	−	+
IC5, IC7	−	−	−	+	−
IC6, IC8	−	−	−	−	−

Let us now study whether such more general refined operators still preserve the basic postulates. It is immediate to prove that they violate the postulate (IC2).

PROPOSITION 5.1. *Let Δ be a merging operator, and $\mathcal{L}' \subseteq \mathcal{L}$ a characterizable fragment. Then each Δ-left-refinement for $\mathcal{L}'$, $\Delta^*\colon \mathcal{E}_{\mathcal{L}'} \times \mathcal{L} \to \mathcal{K}_{\mathcal{L}'}$ (resp., each Δ-right-refinement for $\mathcal{L}'$, $\Delta^*\colon \mathcal{E}_{\mathcal{L}} \times \mathcal{L}' \to \mathcal{K}_{\mathcal{L}'}$) violates* (IC2).

PROOF. Let $\phi \in \mathcal{L} \setminus \mathcal{L}'$ such that $Cl_\beta(\mathrm{Mod}(\phi)) \neq \mathrm{Mod}(\phi)$. Consider $E = \{\{\top\}\}$ and $\mu = \phi$ (resp., $E = \{\{\phi\}\}$ and $\mu = \top$). If Δ^* satisfied (IC2), then we would have $\mathrm{Mod}(\Delta^*_\mu(E)) = \mathrm{Mod}(\phi)$, which provides a contradiction since by assumption $\mathrm{Mod}(\phi)$ is not closed under β. □

For right-refinements, the case where only the integrity constraint has to belong to the fragment, we prove that all other basic postulates are preserved.

PROPOSITION 5.2. *Let Δ be a merging operator satisfying postulates* (IC0)–(IC3)*, and $\mathcal{L}' \subseteq \mathcal{L}$ a characterizable fragment. Then each Δ-right-refinement for $\mathcal{L}'$, $\Delta^*\colon \mathcal{E}_{\mathcal{L}} \times \mathcal{L}' \to \mathcal{K}_{\mathcal{L}'}$ satisfies* (IC0)*,* (IC1) *and* (IC3)

PROOF. Let $E \in \mathcal{E}_{\mathcal{L}}$ and $\mu \in \mathcal{L}'$. We have $\mathrm{Mod}(\Delta^*_\mu(E)) \subseteq Cl_\beta(\mathrm{Mod}(\Delta_\mu(E)))$, thus $\mathrm{Mod}(\Delta^*_\mu(E)) \subseteq Cl_\beta(\mathrm{Mod}(\mu))$ since Δ satisfies (IC0) and Cl_β is a monotone function. Since $\mu \in \mathcal{L}'$, we get $\mathrm{Mod}(\Delta^*_\mu(E)) \subseteq \mathrm{Mod}(\mu)$, thus proving that Δ^* satisfies (IC0).
The proof of the preservation of (IC1) and (IC3) is straightforward and similar to the one in Proposition **??**. □

The preservation of postulate of (IC0) is less clear for left-refinements, that is in the case where only the belief bases that are in the profile have to be in the fragment.

PROPOSITION 5.3. *Let Δ be a merging operator satisfying postulates* (IC0)–(IC3)*, and $\mathcal{L}' \subseteq \mathcal{L}$ a characterizable fragment. Then each Δ-left-refinement for $\mathcal{L}'$, $\Delta^*\colon \mathcal{E}_{\mathcal{L}'} \times \mathcal{L} \to \mathcal{K}_{\mathcal{L}'}$ satisfies* (IC1) *and* (IC3)*. The preservation of* (IC0) *depends on the associated β-mapping, in particular Δ^{Min} satisfies* (IC0)*, while Δ^{Cl_β} and $\Delta^{\mathrm{Min}/Cl_\beta}$ violate it.*

PROOF. Let us deal with (IC0). Suppose $\Delta^* = \Delta^{f_\beta}$ where f_β is a contracting β-mapping, that is for any $\mathcal{M}$ and $\mathcal{X}$, $f_\beta(\mathcal{M}, \mathcal{X}) \subseteq \mathcal{M}$. Then $\mathrm{Mod}(\Delta^{f_\beta}_\mu(E)) = f_\beta(\mathrm{Mod}(\Delta_\mu(E), \mathrm{Mod}(E)) \subseteq \mathrm{Mod}(\Delta_\mu(E))$. Thus, $\mathrm{Mod}(\Delta^{f_\beta}_\mu(E)) \subseteq \mathrm{Mod}(\mu)$ since Δ satisfies (IC0). Therefore Δ^{f_β} satisfies (IC0) for any contracting β-mapping, in particular when $f_\beta = \mathrm{Min}$. Now, consider $E = \{\{\top\}\}$ and $\mu \in \mathcal{L} \setminus \mathcal{L}'$ such that $Cl_\beta(\mathrm{Mod}(\mu)) \neq \mathrm{Mod}(\mu)$. Since Δ satisfies (IC2), we have $\mathrm{Mod}(\Delta_\mu(E)) = \mathrm{Mod}(\mu)$. Since $\mathrm{Mod}(\mu)$ is not closed under β, we have $\mathrm{Mod}(\Delta^{Cl_\beta}_\mu(E)) = \mathrm{Mod}(\Delta^{\mathrm{Min}/Cl_\beta}_\mu(E)) = Cl_\beta(\mathrm{Mod}(\mu))$. Hence, we get $\Delta^{Cl_\beta}_\mu(E) \not\models \mu$ and $\Delta^{\mathrm{Min}/Cl_\beta}_\mu(E) \not\models \mu$, thus showing that (IC0) is not preserved for these refinements. □

Observe that all negative results that have been obtained in the previous sections still hold in this broader context. Moreover it is easy to check that the positive results reported in Proposition **??**, Proposition **??**, Proposition **??** and Proposition **??** are still valid when either only the belief bases of the profile or only the integrity constraint are in the fragment.

6. CONCLUSION

We have investigated to which extent known merging operators can be refined to work within fragments of propositional logic. Compared to revision, this task is more involved since merging operators have many parameters that have to be taken into account.

We have first defined desired properties any refined merging operator should satisfy and provided a characterization of all refined merging operators. We have shown that the refined merging operators preserve the basic postulates, namely $(\mathrm{IC0})$–$(\mathrm{IC3})$. The situation is more complex for the other postulates. For the postulate $(\mathrm{IC4})$ we have provided a sufficient condition for its preservation by a refinement (fairness). For the other postulates, we have focused on two representative families of distance-based merging operators that satisfy the postulates $(\mathrm{IC0})$–$(\mathrm{IC8})$. For these two families the preservation of $(\mathrm{IC5})$ and $(\mathrm{IC7})$ depends on the used refinement and it would be interesting to obtain a necessary and sufficient condition for this. In contrast, there is no hope for such a condition for $(\mathrm{IC6})$ and $(\mathrm{IC8})$, since we have shown that any refinement of merging operators belonging to these families violates these postulates in $\mathcal{L}_{Horn}$ and $\mathcal{L}_{Krom}$. We also studied the majority postulate and showed that the refinement by the closure of majority operators provides refined majority operators. Finally, we had a brief look on relaxations of merging scenarios in Section **??**, where not all ingredients need to be from the fragment.

Table **??** summarizes some results for refinements in $\mathcal{L}_{Horn}$ and $\mathcal{L}_{Krom}$. This table shows the apparent trade-off one has to expect when choosing a refinement. The closure-based refinements tend to satisfy $(\mathrm{IC4})$ but violate $(\mathrm{IC5})$ and $(\mathrm{IC7})$. On the other hand, while the Min-based refinement satisfies $(\mathrm{IC5})$ and $(\mathrm{IC7})$, it violates $(\mathrm{IC4})$. An interesting issue is whether the postulate $(\mathrm{IC4})$ is compatible with $(\mathrm{IC5})$ and $(\mathrm{IC7})$ for some refinements and whether this can depend on the fragment under consideration.

More generally, we plan to study further refinements and see whether they may yield better results than the natural refinements investigated in the current paper. As mentioned in Section **??**, the arbitration postulate (Arb) is stated in a way that seems incompatible with our notion of fragments. Hence, an interesting open problem is to find a modified or weaker version of this postulate, which is applicable in the fragment setting. Finally, another interesting issue is to apply our findings to other domains of merging, for instance merging in (fragments) of Answer-Set programs [**?**].

APPENDIX

PROOF OF PROPOSITION **??**. First consider d is a drastic distance. We show that Δ^{Min} violates postulate $(\mathrm{IC4})$ in every characterizable fragment $\mathcal{L}' \subset \mathcal{L}$. Since $\mathcal{L}'$ is a characterizable fragment there exists $\beta \in \mathcal{B}$ such that $\mathcal{L}'$ is a β-fragment. Consider a set of models $\mathcal{M}$ that is not closed under β and that is cardinality-minimum with this property. Such a set exists since $\mathcal{L}'$ is a proper subset of $\mathcal{L}$. Observe that necessarily $|\mathcal{M}| > 1$. Let $m \in \mathcal{M}$, consider the knowledge bases K_1 and K_2 such that $\mathrm{Mod}(K_1) = \{m\}$ and $\mathrm{Mod}(K_2) = \mathcal{M} \setminus \{m\}$. By the choice of $\mathcal{M}$ both K_1 and K_2 are in $\mathcal{K}_{\mathcal{L}'}$, whereas $K_1 \cup K_2$ is not. Let $\mu = \top$. Since the merging operator uses a drastic distance it is easy to see that $\Delta_\mu(\{K_1, K_2\}) = \mathrm{Mod}(K_1) \cup \mathrm{Mod}(K_2)$. Therefore, $\mathrm{Mod}(\Delta_\mu^{\mathrm{Min}}(\{K_1, K_2\})) = \mathrm{Min}(\mathrm{Mod}(K_1) \cup \mathrm{Mod}(K_2))$, and this single element is either a model of K_1 or a model of K_2 (but not of both since they do not share any model). This shows that Δ^{Min} violates $(\mathrm{IC4})$.

Otherwise, consider d is a non-drastic counting distance. Let g be the function which we used to define counting distances in Definition **??**. Since d is non-drastic, there exists an $x > 0$, such that $g(x) < g(x + 1)$. We first show that then Δ^{Min} violates postulate $(\mathrm{IC4})$ in $\mathcal{L}_{Horn}$. Let A be a set of atoms such that $|A| = x - 1$ and $A \cap \{a, b\} = \emptyset$. Moreover, consider $E = \{K_1, K_2\}$ with $\mathrm{Mod}(K_1) = \{\emptyset, \{a\}, \{b\}\}$, $\mathrm{Mod}(K_2) = \{A \cup \{a, b\}\}$, and let μ such that $\mathrm{Mod}(\mu) = \{\emptyset, \{a\}, \{b\}, A \cup \{a, b\}\}$. Such profile and constraint exist in $\mathcal{L}_{Horn}$. We get:

	K_1	K_2	Σ	GMax
$\emptyset$	0	$g(x+1)$	$g(x+1)$	$(g(x+1),0)$
$\{a\}$	0	$g(x)$	$g(x)$	$(g(x),0)$
$\{b\}$	0	$g(x)$	$g(x)$	$(g(x),0)$
$A\cup\{a,b\}$	$g(x)$	0	$g(x)$	$(g(x),0)$

Since $g(x) < g(x+1)$, we have $\mathcal{M} = \mathrm{Mod}(\Delta_\mu(E)) = \{\{a\},\{b\},A\cup\{a,b\}\}$, which is not closed under intersection. Hence, $\mathrm{Mod}(\Delta_\mu^{\mathrm{Min}}(E))$ contains exactly one of the three models depending on the ordering. Therefore, $\#(\mathrm{Mod}(\Delta_\mu^{\mathrm{Min}}(E)),E) = 1$, thus violating (IC4).

For $\mathcal{L}_{Krom}$, let $x > 0$ be the smallest index such that $g(x) < g(x+1)$ in the definition of distance d. For any y with $0 < y < x$, thus $g(y) = g(x)$ holds. Let A, A' be two disjoint set of atoms with cardinality $x-1$ and $A\cap\{a,b,c,d\} = A'\cap\{a,b,c,d\} = \emptyset$. Let us consider $E = \{K_1,K_2\}$ with $\mathrm{Mod}(K_1) = \{\emptyset,\{a\},\{b\},\{c\},\{d\},\{a,b\},\{c,d\}\}$ (in case $x > 1$) resp. $\mathrm{Mod}(K_1) = \{\emptyset,\{a\},\{b\},\{c\},\{d\}\}$ (in case $x = 1$), $\mathrm{Mod}(K_2) = \{A\cup\{a,b\},A'\cup\{c,d\}\}$, and μ such that $\mathrm{Mod}(\mu) = \{\emptyset,\{a\},\{b\},\{c\},\{d\},\{a,b\},\{c,d\},A\cup\{a,b\},A'\cup\{c,d\}\}$. Such profile and constraint exist in $\mathcal{L}_{Krom}$. The following table represents the case $x > 1$.

	K_1	K_2	E
$\emptyset$	0	$g(x+1)$	$(g(x+1),0)$
$\{a\}$	0	$g(x)$	$(g(x),0)$
$\{b\}$	0	$g(x)$	$(g(x),0)$
$\{c\}$	0	$g(x)$	$(g(x),0)$
$\{d\}$	0	$g(x)$	$(g(x),0)$
$\{a,b\}$	0	$g(x-1)$	$(g(x-1),0)$
$\{c,d\}$	0	$g(x-1)$	$(g(x-1),0)$
$A\cup\{a,b\}$	$g(x-1)$	0	$(g(x-1),0)$
$A'\cup\{c,d\}$	$g(x-1)$	0	$(g(x-1),0)$

For the case $x > 1$, observe $g(x-1) = g(x) < g(x+1)$, and we have $\mathcal{M} = \mathrm{Mod}(\Delta_\mu(E)) = \{\{a\},\{b\},\{c\},\{d\},\{a,b\},\{c,d\},A\cup\{a,b\},A'\cup\{c,d\}\}$. For the case $x = 1$, note that A and A' are empty, thus the two last rows of the table coincide with the two rows before. Recall that K_1 is defined differently for this case. Hence, the distances of $\{a,b\}$ and $\{c,d\}$ to K_1 are $g(x) = g(1)$. Thus, we have $\mathcal{M} = \mathrm{Mod}(\Delta_\mu(E)) = \{\{a\},\{b\},\{c\},\{d\},\{a,b\},\{c,d\}\}$. Neither of the $\mathcal{M}$ is closed under ternary majority. Hence, $\mathrm{Mod}(\Delta_\mu^{\mathrm{Min}}(E))$ contains exactly one of the six resp. eight models depending on the ordering. Therefore, $\#(\mathrm{Mod}(\Delta_\mu^{\mathrm{Min}}(E)),E) = 1$, thus violating (IC4). □

PROOF OF PROPOSITION **??**. Since d is not drastic, there exists an $x > 0$ such that $g(x) < g(x+1)$. In what follows, we select the smallest such x.

We start with the case $\mathcal{L}_{Horn}$. Let A be a set of atoms of cardinality $x-1$ not containing a,b. Let us consider $E = \{K_1,K_2\}$ with $\mathrm{Mod}(K_1) = \{\emptyset\}$ and $\mathrm{Mod}(K_2) = \{A\cup\{a,b\}\}$, and μ such that $\mathrm{Mod}(\mu) = \{\emptyset,\{a\},\{b\},A\cup\{a,b\}\}$. Such profile and constraint exist in $\mathcal{L}_{Horn}$.

	K_1	K_2	E
$\emptyset$	0	$g(x+1)$	$(g(x+1),0)$
$\{a\}$	$g(1)$	$g(x)$	$(g(x),g(1))$
$\{b\}$	$g(1)$	$g(x)$	$(g(x),g(1))$
$A\cup\{a,b\}$	$g(x+1)$	0	$(g(x+1),0)$

Since $g(x) < g(x+1)$, we have $\mathcal{M} = \mathrm{Mod}(\Delta_\mu(E)) = \{\{a\},\{b\}\}$, which is not closed under intersection. Hence, $\mathrm{Mod}(\Delta_\mu^{Cl_\wedge}(E)) = \{\{a\},\{b\},\emptyset\}$. Therefore, $\#(\mathrm{Mod}(\Delta_\mu^{Cl_\wedge}(E)),E) = 1$, thus violating (IC4).

For the case $\mathcal{L}_{Krom}$, let us consider two disjoint sets A, A' of atoms not containing a,b,c,d of cardinality $x-1$, the profile $E = \{K_1,K_2\}$ with $\mathrm{Mod}(K_1) = \{\emptyset\}$ and $\mathrm{Mod}(K_2) = \{A\cup$

$\{a,b\}, A' \cup \{c,d\}\}$, and constraint μ such that $\mathrm{Mod}(\mu) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{c,d\}, A \cup \{a,b\}, A' \cup \{c,d\}\}$. Such profile and constraint exist in $\mathcal{L}_{Krom}$.

	K_1	K_2	E
$\emptyset$	0	$g(x+1)$	$(g(x+1), g(0))$
$\{a\}$	$g(1)$	$g(x)$	$(g(x), g(1))$
$\{b\}$	$g(1)$	$g(x)$	$(g(x), g(1))$
$\{c\}$	$g(1)$	$g(x)$	$(g(x), g(1))$
$\{d\}$	$g(1)$	$g(x)$	$(g(x), g(1))$
$\{a,b\}$	$g(2)$	$g(x-1)$	$(g(x-1), g(2))$
$\{c,d\}$	$g(2)$	$g(x-1)$	$(g(x-1), g(2))$
$A \cup \{a,b\}$	$g(x+1)$	$g(0)$	$(g(x+1), g(0))$
$A' \cup \{c,d\}$	$g(x+1)$	$g(0)$	$(g(x+1), g(0))$

If $x = 1$ note that A and A' are empty and $g(2) > g(x) > g(x-1) = g(0)$ (thus the last four lines collapse into two lines). We have $\mathcal{M} = \mathrm{Mod}(\Delta_\mu(E)) = \{\{a\}, \{b\}, \{c\}, \{d\}\}$, which is not closed under ternary majority. Hence, $\mathrm{Mod}(\Delta_\mu^{Cl_{\mathrm{maj}_3}}(E)) = \{\{a\}, \{b\}, \{c\}, \{d\}, \emptyset\}$. If $x > 1$, we have $g(x+1) > g(x) = g(x-1) = g(2) = g(1)$. Thus, $\mathcal{M} = \mathrm{Mod}(\Delta_\mu(E)) = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{c,d\}\}$, which is not closed under ternary majority either and one has to add $\emptyset$. Therefore, in both cases $\#(\mathrm{Mod}(\Delta_\mu^{Cl_{\mathrm{maj}_3}}(E)), E) = 1$, thus violating (IC4). □

PROOF OF PROPOSITION **??**. Let $\mathcal{L}'$ be a β-fragment. Let profile $E \in \mathcal{E}_{\mathcal{L}'}$ such that $E = \{K_1, \dots K_n\}$, $\mu \in \mathcal{L}'$ and let Δ be an arbitrary merging operator.

If $\#(\Delta_\mu(E), E) = 0$ then $\mathrm{Mod}(\Delta_\mu(E)) \cap \bigcup_i \mathrm{Mod}(K_i) = \emptyset$. By Definition **??** $\Delta_\mu^{\mathrm{Min}/Cl_\beta}(E) = \Delta_\mu^{\mathrm{Min}}(E)$, therefore $\#(\Delta_\mu^{\mathrm{Min}/Cl_\beta}(E), E) = 0$ as well. If $\#(\Delta_\mu(E), E) > 1$ then by Definition **??**, $\mathrm{Mod}(\Delta_\mu^{\mathrm{Min}/Cl_\beta}(E)) = \mathrm{Mod}(\Delta_\mu^{Cl_\beta}(E))$, thus $\#(\Delta_\mu^{\mathrm{Min}/Cl_\beta}(E), E) \geq \#(\Delta_\mu(E), E) > 1$. □

PROOF OF PROPOSITION **??**. Let $\mathcal{L}'$ be a β-fragment. Let profile $E \in \mathcal{E}_{\mathcal{L}'}$ such that $E = \{K_1, \dots K_n\}$, $\mu \in \mathcal{L}'$ and let Δ be a merging operator with $\Delta = \Delta^{d_D, f}$ and $f \in \{\Sigma, \mathrm{GMax}\}$. Let $\mathcal{M} = \mathrm{Mod}(\Delta_\mu(E))$. We will show that $\mathrm{Mod}(\Delta_\mu^{Cl_\beta}(E)) = \mathrm{Mod}(\Delta_\mu^{\mathrm{Min}/Cl_\beta}(E))$.

If $\mathcal{M} = Cl_\beta(\mathcal{M})$ we immediately get from Definition **??** that $\mathrm{Mod}(\Delta_\mu^{Cl_\beta}(E)) = \mathcal{M} = \mathrm{Mod}(\Delta_\mu^{\mathrm{Min}/Cl_\beta}(E))$.

Hence, assume that $\mathcal{M} \neq Cl_\beta(\mathcal{M})$. We proceed by case distinction on $\#(\mathcal{M}, E)$. First, consider the case $\#(\mathcal{M}, E) \geq 1$. Again, it follows immediately from Definition **??** that $\mathrm{Mod}(\Delta_\mu^{Cl_\beta}(E)) = \mathrm{Mod}(\Delta_\mu^{\mathrm{Min}/Cl_\beta}(E))$.

Hence, the only interesting case is $\#(\mathcal{M}, E) = 0$. In this case we know that $\mathcal{M} \cap K_i = \emptyset$ for $1 \leq i \leq n$. Therefore, for all $\omega \in \mathcal{M}$ and for all $K_i \in E$ we have $d_D(\omega, K_i) = g(1)$ and hence $d_D(\omega, E) = f(g(1), \dots, g(1))$. Since we are using drastic distance, we have for all $\omega' \in \mathrm{Mod}(\mu)$ and for all $K_i \in E$ either $d_D(\omega, K_i) = 0$ or $d_D(\omega, K_i) = g(1)$. Assume there exists $\omega' \in \mathrm{Mod}(\mu)$ and $K_i \in E$ such that $d_D(\omega, K_i) = 0$. Since $f(g(1), \dots, 0, \dots, g(1)) < f(g(1), \dots, g(1))$ we have $\omega' <_E \omega$ for all $\omega \in \mathcal{M}$. But this contradicts that $\mathcal{M}$ contains the minimal models according to $\leq_E$. Hence it follows that $\mathcal{M} = \mathrm{Mod}(\mu)$. But then $\mathcal{M} = Cl_\beta(\mathcal{M})$ which contradicts our assumption. Therefore, the Cl_β-based refinement and the Min/Cl_β-based refinement coincide. □

PROOF OF PROPOSITION **??**. Let $\mathcal{L}'$ be a β-fragment. Let $E = \{K_1, K_2\}$ with $K_1, K_2 \in \mathcal{L}'$ and $\mu \in \mathcal{L}'$, with $K_1 \models \mu$ and $K_2 \models \mu$. The merging operator Δ satisfies (IC4) therefore $\Delta_\mu(E) \wedge K_1$ is consistent if and only if $\Delta_\mu(E) \wedge K_2$.

If both $\Delta_\mu(E) \wedge K_1$ and $\Delta_\mu(E) \wedge K_2$ are consistent, then so are *a fortiori* $\Delta_\mu^{Cl_\beta}(E) \wedge K_1$ and $\Delta_\mu^{Cl_\beta}(E) \wedge K_2$. Therefore a violation of (IC4) can only occur when both $\Delta_\mu(E) \wedge K_1$ and $\Delta_\mu(E) \wedge K_2$ are inconsistent. We prove that this never occurs. Suppose that $\Delta_\mu(E) \wedge K_1$ is inconsistent, this means that there exists $m \notin K_1$ such that $min(\mathrm{Mod}(\mu), \leq_E) = d(m, E)$ and that for all $m_1 \in K_1$, $d(m, E) < d(m_1, E)$, i.e., $d(m, K_1) + d(m, K_2) < d(m_1, K_1) + d(m_1, K_2)$ since Σ is the aggregation function. Choose now $m_1 \in K_1$ such that $d(m, K_1) = d(m, m_1)$ and $m_2 \in K_2$ such that $d(m, K_2) = d(m, m_2)$. We have $d(m, K_1) + d(m, K_2) = d(m, m_1) + d(m, m_2) < d(m_1, K_1) + d(m_1, K_2) = d(m_1, K_2)$ since $m_1 \in K_1$ and hence $d(m_1, K_1) = 0$. Since d satisfies the triangular inequality we have $d(m_1, m_2) \leq d(m_1, m) + d(m, m_2)$. But this contradicts $d(m, m_1) + d(m, m_2) < d(m_1, K_2) \leq d(m_1, m_2)$, thus $\Delta_\mu(E) \wedge K_1$ cannot be inconsistent. □

PROOF OF PROPOSITION **??**. We give the proof for $\mathcal{L}_{Horn}$. One can verify that the same example works for $\mathcal{L}_{Krom}$ as well.

Let us consider $E = \{K_1, K_2\}$ and μ in $\mathcal{L}_{Horn}$ with

$$\begin{aligned} \mathrm{Mod}(K_1) &= \{\{a\}, \{a, b\}, \{a, d\}, \{a, f\}\} \\ \mathrm{Mod}(K_2) &= \{\{a, b, c, d, e, f, g\}\} \text{ and} \\ \mathrm{Mod}(\mu) &= \{\{a\}, \{a, b, c\}, \{a, d, e\}, \{a, f, g\}\}. \end{aligned}$$

We have the following situation:

	K_1	K_2	E
$\{a\}$	0	6	6
$\{a, b, c\}$	1	4	5
$\{a, d, e\}$	1	4	5
$\{a, f, g\}$	1	4	5

Therefore, we have $\mathrm{Mod}(\Delta_\mu^{d_H,\Sigma}(E)) = \{\{a, b, c\}, \{a, d, e\}, \{a, f, g\}\}$, and $\mathrm{Mod}(\Delta_\mu^{Cl_\wedge}(E)) = \{\{a\}, \{a, b, c\}, \{a, d, e\}, \{a, f, g\}\}$. Hence, $\#(\mathrm{Mod}(\Delta_\mu^{d_H,\Sigma}(E)), E) = 0$. On the other hand $\#(\mathrm{Mod}(\Delta_\mu^{Cl_\wedge}(E)), E) = 1$, thus proving that fairness is not satisfied. □

PROOF OF PROPOSITION **??**. Since $\mathcal{L}'$ is characterizable there exists a $\beta \in \mathcal{B}$, such that $\mathcal{L}'$ is a β-fragment.

(IC5): If $\Delta_\mu^{\mathrm{Min}}(E_1) \wedge \Delta_\mu^{\mathrm{Min}}(E_2)$ is inconsistent, then (IC5) is satisfied. Assume that $\Delta_\mu^{\mathrm{Min}}(E_1) \wedge \Delta_\mu^{\mathrm{Min}}(E_2)$ is consistent. Then, by definition of Δ^{Min} we know that $\Delta_\mu(E_1) \wedge \Delta_\mu(E_2)$ is consistent as well. From (IC5) and (IC6) it follows that $\mathrm{Mod}(\Delta_\mu(E_1)) \cap \mathrm{Mod}(\Delta_\mu(E_2)) = \mathrm{Mod}(\Delta_\mu(E_1 \sqcup E_2))$. We distinguish two cases. First assume that both $\mathrm{Mod}(\Delta_\mu(E_1))$ and $\mathrm{Mod}(\Delta_\mu(E_2))$ are closed under β. By Definition **??** we know that $\mathrm{Mod}(\Delta_\mu(E_1)) \cap \mathrm{Mod}(\Delta_\mu(E_2)) = \mathrm{Mod}(\Delta_\mu(E_1 \sqcup E_2))$ is closed under β as well. Hence, (IC5) is satisfied. For the second case assume that not both $\mathrm{Mod}(\Delta_\mu(E_1))$ and $\mathrm{Mod}(\Delta_\mu(E_2))$ are closed under β. From the definition of Δ^{Min} it follows that $\mathrm{Mod}(\Delta_\mu^{\mathrm{Min}}(E_1)) \cap \mathrm{Mod}(\Delta_\mu^{\mathrm{Min}}(E_2))$ consists of a single interpretation, say ω with $\omega \in \mathrm{Mod}(\Delta_\mu(E_1)) \cap \mathrm{Mod}(\Delta_\mu(E_2))$. If $\mathrm{Mod}(\Delta_\mu(E_1 \sqcup E_2))$ is closed under β we have $\omega \in \mathrm{Mod}(\Delta_\mu^{\mathrm{Min}}(E_1 \sqcup E_2))$ and (IC5) is satisfied. If $\mathrm{Mod}(\Delta_\mu(E_1 \sqcup E_2))$ is not closed under β, then $\mathrm{Mod}(\Delta_\mu^{\mathrm{Min}}(E_1 \sqcup E_2))$ consists of a single interpretation, say $\omega' \in \mathrm{Mod}(\Delta_\mu(E_1)) \cap \mathrm{Mod}(\Delta_\mu(E_2))$. From $\mathrm{Mod}(\Delta_\mu^{\mathrm{Min}}(E_1)) \cap \mathrm{Mod}^{\mathrm{Min}}(\Delta_\mu(E_2)) = \{\omega\}$ it follows that $\mathrm{Min}(\{\omega, \omega'\}) = \omega$ and from $\mathrm{Mod}(\Delta_\mu^{\mathrm{Min}}(E_1 \sqcup E_2)) = \{w'\}$ it follows that $\mathrm{Min}(\{\omega, \omega'\}) = \omega'$. Hence, $\omega = \omega'$ and (IC5) is satisfied.

(IC7): If $\Delta_{\mu_1}^{\mathrm{Min}}(E) \wedge \mu_2$ is inconsistent, then (IC7) is satisfied. Assume that $\Delta_{\mu_1}^{\mathrm{Min}}(E) \wedge \mu_2$ is consistent. Then, by definition of Δ^{Min} we know that $\Delta_{\mu_1}(E) \wedge \mu_2$ is consistent as well. From (IC7) and (IC8) it follows that $\mathrm{Mod}(\Delta_{\mu_1}(E)) \cap \mathrm{Mod}(\mu_2) = \mathrm{Mod}(\Delta_{\mu_1 \wedge \mu_2}(E))$. We distinguish two cases. First assume that $\mathrm{Mod}(\Delta_{\mu_1}(E))$ is closed under β. By Definition **??** we know that $\mathrm{Mod}(\Delta_{\mu_1}(E)) \cap \mathrm{Mod}(\mu_2) = \mathrm{Mod}(\Delta_{\mu_1 \wedge \mu_2}(E))$ is closed under β as well. Hence, (IC7) is satisfied. For the second case assume that $\mathrm{Mod}(\Delta_{\mu_1}(E))$ is not closed under β. From the definition of Δ^{Min} it follows that $\mathrm{Mod}(\Delta_{\mu_1}^{\mathrm{Min}}(E)) \cap \mathrm{Mod}(\mu_2)$ consists of a single interpretation, say ω with $\omega \in \mathrm{Mod}(\Delta_{\mu_1}(E)) \cap \mathrm{Mod}(\mu_2)$. If $\mathrm{Mod}(\Delta_{\mu_1 \wedge \mu_2}(E))$ is closed under β we have $\omega \in \mathrm{Mod}(\Delta_{\mu_1 \wedge \mu_2}^{\mathrm{Min}}(E))$

and (IC7) is satisfied. If $\mathrm{Mod}(\Delta_{\mu_1\wedge\mu_2}(E))$ is not closed under β, then $\mathrm{Mod}(\Delta^{\mathrm{Min}}_{\mu_1\wedge\mu_2}(E))$ consists of a single interpretation, say $\omega' \in \mathrm{Mod}(\Delta_{\mu_1}(E))\cap\mathrm{Mod}(\mu_2)$. From $\mathrm{Mod}(\Delta^{\mathrm{Min}}_{\mu_1}(E))\cap\mathrm{Mod}(\mu_2) = \{\omega\}$ it follows that $\mathrm{Min}(\{\omega,\omega'\}) = \omega$ and from $\mathrm{Mod}(\Delta^{\mathrm{Min}}_{\mu_1\wedge\mu_2}(E)) = \{\omega'\}$ it follows that $\mathrm{Min}(\{\omega,\omega'\}) = \omega'$. Hence, $\omega = \omega'$ and (IC7) is satisfied. □

PROOF OF PROPOSITION **??**. We give the proof for Δ^{Cl_β} with $\Delta = \Delta^{d,\Sigma}$ where d is associated with a function g (see Definition **??**). The given examples also apply to GMax and for the refinement $\Delta^{\mathrm{Min}/Cl_\beta}$.

(IC5): Let $\beta \in \{\wedge, \mathrm{maj}_3\}$. Consider profiles $E_1 = \{K_1,K_2,K_3\}$, $E_2 = \{K_4\}$ and integrity constraint μ with $\mathrm{Mod}(K_1) = \{\{a\},\{a,b\},\{a,c\}\}$, $\mathrm{Mod}(K_2) = \{\{b\},\{a,b\},\{b,c\}\}$, $\mathrm{Mod}(K_3) = \{\{c\},\{a,c\},\{b,c\}\}$, $\mathrm{Mod}(K_4) = \{\emptyset,\{b\}\}$, and $\mathrm{Mod}(\mu) = \{\emptyset,\{a\},\{b\},\{c\}\}$. Such profiles and constraint exist in $\mathcal{L}_{Horn}$ and $\mathcal{L}_{Krom}$.

	K_1	K_2	K_3	K_4	E_1	$E_1 \sqcup E_2$
$\emptyset$	$g(1)$	$g(1)$	$g(1)$	0	$3g(1)$	$3g(1)$
$\{a\}$	0	$g(1)$	$g(1)$	$g(1)$	$2g(1)$	$3g(1)$
$\{b\}$	$g(1)$	0	$g(1)$	0	$2g(1)$	$2g(1)$
$\{c\}$	$g(1)$	$g(1)$	0	$g(1)$	$2g(1)$	$3g(1)$

By definition of counting distances we know that $g(1) > 0$. Hence, we have $\mathrm{Mod}(\Delta^{Cl_\beta}_{\mu}(E_1)) = \{\emptyset,\{a\},\{b\},\{c\}\}$, $\mathrm{Mod}(\Delta^{Cl_\beta}_{\mu}(E_2)) = \{\emptyset,\{b\}\}$, and $\mathrm{Mod}(\Delta^{Cl_\beta}_{\mu}(E_1 \sqcup E_2)) = \{\{b\}\}$, thus violating (IC5).

(IC7): For $\mathcal{L}_{Horn}$, consider $E = \{K_1,K_2,K_3\}$ with $\mathrm{Mod}(K_1) = \{\{a\}\}$, $\mathrm{Mod}(K_2) = \{\{b\}\}$, $\mathrm{Mod}(K_3) = \{\{a,b\}\}$, and assume $\mathrm{Mod}(\mu_1) = \{\emptyset,\{a\},\{b\}\}$ and $\mathrm{Mod}(\mu_2) = \{\emptyset,\{a\}\}$.

	K_1	K_2	K_3	E
$\emptyset$	$g(1)$	$g(1)$	$g(2)$	$2g(1)+g(2)$
$\{a\}$	0	$g(2)$	$g(1)$	$g(1)+g(2)$
$\{b\}$	$g(2)$	0	$g(1)$	$g(1)+g(2)$

We have $\mathrm{Mod}(\Delta_{\mu_1}(E)) = \{\{a\},\{b\}\}$, thus $\mathrm{Mod}(\Delta^{Cl_\wedge}_{\mu_1}(E)) = \{\emptyset,\{a\},\{b\}\}$. Therefore, $\mathrm{Mod}(\Delta^{Cl_\wedge}_{\mu_1}(E)\wedge\mu_2) = \{\emptyset,\{a\}\}$, whereas $\mathrm{Mod}(\Delta^{Cl_\wedge}_{\mu_1\wedge\mu_2}(E)) = \{\{a\}\}$, violating (IC7).

For $\mathcal{L}_{Krom}$ let $E = \{K_1,K_2,K_3,K_4,K_5\}$, μ_1 and μ_2 with $\mathrm{Mod}(K_1) = \{\{a\}\}$, $\mathrm{Mod}(K_2) = \{\{b\}\}$, $\mathrm{Mod}(K_3) = \{\{c\}\}$, $\mathrm{Mod}(K_4) = \{\{a,b\},\{a,c\}\}$, $\mathrm{Mod}(K_5) = \{\{a,b\},\{b,c\}\}$, $\mathrm{Mod}(\mu_1) = \{\emptyset,\{a\},\{b\},\{c\}\}$, and $\mathrm{Mod}(\mu_2) = \{\emptyset,\{a\}\}$.

	K_1	K_2	K_3	K_4	K_5	E
$\emptyset$	$g(1)$	$g(1)$	$g(1)$	$g(2)$	$g(2)$	$2g(2)+3g(1)$
$\{a\}$	0	$g(2)$	$g(2)$	$g(1)$	$g(1)$	$2g(2)+2g(1)$
$\{b\}$	$g(2)$	0	$g(2)$	$g(1)$	$g(1)$	$2g(2)+2g(1)$
$\{c\}$	$g(2)$	$g(2)$	0	$g(1)$	$g(1)$	$2g(2)+2g(1)$

We have $\mathrm{Mod}(\Delta^{Cl_{\mathrm{maj}_3}}_{\mu_1}(E)) = \{\emptyset,\{a\},\{b\},\{c\}\}$, thus $\mathrm{Mod}(\Delta^{Cl_{\mathrm{maj}_3}}_{\mu_1}(E)\wedge\mu_2) = \{\emptyset,\{a\}\}$, and $\mathrm{Mod}(\Delta^{Cl_{\mathrm{maj}_3}}_{\mu_1\wedge\mu_2}(E)) = \{\{a\}\}$. This violates postulate (IC7). □

PROOF OF PROPOSITION **??**. The same, or simpler examples as in the proof of the previous proposition will work here. We give the proof in the case of $\Delta^{d,\Sigma}$ where d is a counting distance associated with the function g. The given counter-examples work as well when using the aggregation function GMax. Any involved set of models is closed under intersection and hence it can be represented by a Horn formula.

(IC5): Let us consider $E_1 = \{K_1, K_2\}$, $E_2 = \{K_3\}$ and μ with

$$\begin{aligned}\mathrm{Mod}(K_1) &= \{\{a\},\{a,b\}\} \text{ and}\\ \mathrm{Mod}(K_2) &= \{\{b\},\{a,b\}\} \text{ and}\\ \mathrm{Mod}(K_3) &= \{\emptyset,\{b\}\} \text{ and}\\ \mathrm{Mod}(\mu) &= \{\emptyset,\{a\},\{b\}\}.\end{aligned}$$

	K_1	K_2	K_3	E_1	$E_1 \cup E_2$
$\emptyset$	$g(1)$	$g(1)$	0	$2g(1)$	$2g(1)$
$\{a\}$	0	$g(1)$	$g(1)$	$g(1)$	$2g(1)$
$\{b\}$	$g(1)$	0	0	$g(1)$	$g(1)$

Since $g(1) > 0$ by definition of a counting distance, we have $\mathrm{Mod}(\Delta_\mu(E_1)) = \{\{a\},\{b\}\}$, and thus $\mathrm{Mod}(\Delta^*_\mu(E_1)) \subseteq \{\emptyset,\{a\},\{b\}\}$. We can exclude $\mathrm{Mod}(\Delta^*_\mu(E_1)) = \{\{a\},\{b\}\}$ since it is not closed under $\wedge$. By Definition **??** (Δ^* is fair) we can exclude $\mathrm{Mod}(\Delta^*_\mu(E_1)) = \{\{a\}\}$ and $\mathrm{Mod}(\Delta^*_\mu(E_1)) = \{\{b\}\}$. Therefore either $\mathrm{Mod}(\Delta^*_\mu(E_1)) = \{\emptyset\}$ or $\mathrm{Mod}(\Delta^*_\mu(E_1)) = \{\emptyset,\{a\},\{b\}\}$. On the one hand, since $\mathrm{Mod}(\Delta^*_\mu(E_2)) = \{\emptyset,\{b\}\}$, in any case $\mathrm{Mod}(\Delta^*_\mu(E_1) \wedge \Delta^*_\mu(E_2))$ contains $\emptyset$. On the other hand $\mathrm{Mod}(\Delta^*_\mu(E_1 \sqcup E_2)) = \{\{b\}\}$. This violates postulate (IC5).

(IC7): Let us consider $E = E_1$ and $\mu_1 = \mu$ as above, and μ_2 such that $\mathrm{Mod}(\mu_2) = \{\emptyset,\{a\}\}$. There we have $\mathrm{Mod}(\Delta_{\mu_1\wedge\mu_2}(E)) = \{\{a\}\}$. By Properties 3 and 4 of Definition **??** it holds that $\mathrm{Mod}(\Delta^\star_{\mu_1\wedge\mu_2}(E)) = \{\{a\}\}$. Since $\mathrm{Mod}(\Delta_{\mu_1}(E)) = \{\{a\},\{b\}\}$, it follows that $\mathrm{Mod}(\Delta^\star_{\mu_1}(E)) \subseteq \{\emptyset,\{a\},\{b\}\}$. We can exclude $\mathrm{Mod}(\Delta^\star_{\mu_1}(E)) = \{\{a\},\{b\}\}$ since it is not closed under $\wedge$. By Definition **??** we can exclude $\mathrm{Mod}(\Delta^\star_{\mu_1}(E)) = \{\{a\}\}$ and $\mathrm{Mod}(\Delta^\star_{\mu_1}(E)) = \{\{b\}\}$. Hence, $\emptyset \in \mathrm{Mod}(\Delta^\star_{\mu_1}(E))$. Therefore $\emptyset \in \mathrm{Mod}(\Delta^\star_{\mu_1}(E)) \cap \mathrm{Mod}(\mu_2)$ but $\emptyset \notin \mathrm{Mod}(\Delta^\star_{\mu_1\wedge\mu_2}(E))$, which violates (IC7). □

PROOF OF PROPOSITION **??**. (IC6): We start with the $\mathcal{L}_{Horn}$ case. Since $\mathcal{L}_{Horn}$ is an $\wedge$-fragment, there exists an $\wedge$-mapping f such that $\Delta^\star = \Delta^f$ and we have $f(\mathcal{M},\mathcal{X}) \subseteq Cl_\wedge(\mathcal{M})$ with $Cl_\wedge(f(\mathcal{M},\mathcal{X})) = f(\mathcal{M},\mathcal{X})$. Let us consider $E_1 = \{K_1, K_2, K_3\}$ and μ with $\mathrm{Mod}(K_1) = \{\{a\},\{a,b\}\}$, $\mathrm{Mod}(K_2) = \{\{b\},\{a,b\}\}$, $\mathrm{Mod}(K_3) = \{\emptyset,\{a\},\{b\}\}$ and $\mathrm{Mod}(\mu) = \{\emptyset,\{a\},\{b\},\{a,b\}\}$.

	K_1	K_2	K_3	E_1
$\emptyset$	$g(1)$	$g(1)$	0	$(g(1), g(1), 0)$
$\{a\}$	0	$g(1)$	0	$(g(1), 0, 0)$
$\{b\}$	$g(1)$	0	0	$(g(1), 0, 0)$
$\{a,b\}$	0	0	$g(1)$	$(g(1), 0, 0)$

We have $\mathcal{M} = \mathrm{Mod}(\Delta_\mu(E_1)) = \{\{a\},\{b\},\{a,b\}\}$. Let us consider the possibilities for $\mathrm{Mod}(\Delta^\star_\mu(E_1)) = f(\mathcal{M},\mathcal{M}\mathrm{od}(E_1))$. If $\emptyset \in f(\mathcal{M},\mathcal{M}\mathrm{od}(E_1))$, then let $E_2 = \{K_4\}$ with K_4 in $\mathcal{L}_{Horn}$ be such that $\mathrm{Mod}(K_4) = \{\emptyset\}$. Thus, $\mathrm{Mod}(\Delta^\star_\mu(E_2)) = \{\emptyset\}$ and $\mathrm{Mod}(\Delta^\star_\mu(E_1) \wedge \Delta^\star_\mu(E_2)) = \{\emptyset\}$. Moreover, $\mathrm{Mod}(\Delta_\mu(E_1 \sqcup E_2)) = \{\emptyset,\{a\},\{b\}\}$ or $\{\emptyset,\{a\},\{b\},\{a,b\}\}$ depending on whether $g(1) < g(2)$ or $g(1) = g(2)$. Since both sets are closed under intersection, we have $\mathrm{Mod}(\Delta^\star_\mu(E_1 \sqcup E_2)) = \mathrm{Mod}(\Delta_\mu(E_1 \sqcup E_2))$. Thus $\mathrm{Mod}(\Delta^\star_\mu(E_1 \sqcup E_2)) \not\subseteq \{\emptyset\}$ and (IC6) does not hold.

On the other hand, let $f(\mathcal{M},\mathcal{M}\mathrm{od}(E_1)) \subseteq \{\{a\},\{b\},\{a,b\}\}$. By symmetry assume w.l.o.g. that $f(\mathcal{M},\mathcal{M}\mathrm{od}(E_1)) \subseteq \{\{a,b\},\{a\}\}$ (note that $\{\{a\},\{b\}\} \subseteq f(\mathcal{M},\mathcal{M}\mathrm{od}(E_1))$ would imply $\emptyset \in f(\mathcal{M},\mathcal{M}\mathrm{od}(E_1))$). If $f(\mathcal{M},\mathcal{M}\mathrm{od}(E_1)) = \{\{a\}\}$ or $\{\{a,b\}\}$, then let $E_2 = \{K_1\}$. Then, $\mathrm{Mod}(\Delta_\mu(E_2)) = \{\{a\},\{a,b\}\} = \mathrm{Mod}(\Delta^\star_\mu(E_2))$, and $\mathrm{Mod}(\Delta^\star_\mu(E_1) \wedge \Delta^\star_\mu(E_2)) = \{\{a\}\}$ or $\{\{a,b\}\}$. Furthermore, $\mathrm{Mod}(\Delta_\mu(E_1 \sqcup E_2)) = \{\{a\},\{a,b\}\} = \mathrm{Mod}(\Delta^\star_\mu(E_1 \sqcup E_2))$, thus violating (IC6). If $f(\mathcal{M},\mathcal{M}\mathrm{od}(E_1)) = \{\{a,b\},\{a\}\}$, then let $E_2 = \{K_2\}$. Then, $\mathrm{Mod}(\Delta_\mu(E_2)) =$

$\{\{b\},\{a,b\}\} = \mathrm{Mod}(\Delta^\star_\mu(E_2))$, and $\mathrm{Mod}(\Delta^\star_\mu(E_1) \wedge \Delta^\star_\mu(E_2)) = \{\{a,b\}\}$. Furthermore, $\mathrm{Mod}(\Delta_\mu(E_1 \sqcup E_2)) = \{\{b\},\{a,b\}\} = \mathrm{Mod}(\Delta^\star_\mu(E_1 \sqcup E_2))$, and thus (IC6) does not hold.

Let us now turn to the Krom case. Let us consider $E_1 = \{K_1, K_2, K_3\}$ and μ with

$$\begin{aligned}\mathrm{Mod}(K_1) &= \{\{a\},\{b\},\{a,c\}\},\\ \mathrm{Mod}(K_2) &= \{\{a\},\{c\},\{b,c\}\},\\ \mathrm{Mod}(K_3) &= \{\{b\},\{c\},\{a,b\}\},\\ \mathrm{Mod}(\mu) &= \{\emptyset,\{a\},\{b\},\{c\}\}.\end{aligned}$$

	K_1	K_2	K_3	E_1
$\emptyset$	$g(1)$	$g(1)$	$g(1)$	$(g(1),g(1),g(1))$
$\{a\}$	0	0	$g(1)$	$(g(1),0,0)$
$\{b\}$	0	$g(1)$	0	$(g(1),0,0)$
$\{c\}$	$g(1)$	0	0	$(g(1),0,0)$

We have $\mathcal{M} = \mathrm{Mod}(\Delta_\mu(E_1)) = \{\{a\},\{b\},\{c\}\}$. Let us consider the possibilities for $\mathrm{Mod}(\Delta^\star_\mu(E_1)) = f(\mathcal{M}, \mathcal{M}\mathrm{od}(E_1))$. First, assume $\emptyset \in f(\mathcal{M}, \mathcal{M}\mathrm{od}(E_1))$: Let $E_2 = \{K_4, K_4\}$ (recall that a profile is a multiset) with K_4 in $\mathcal{L}_{Krom}$ be such that $\mathrm{Mod}(K_4) = \{\emptyset\}$. Then, $\mathrm{Mod}(\Delta^\star_\mu(E_2)) = \{\emptyset\}$ and $\mathrm{Mod}(\Delta^\star_\mu(E_1) \wedge \Delta^\star_\mu(E_2)) = \{\emptyset\}$. Furthermore, $\mathrm{Mod}(\Delta_\mu(E_1 \sqcup E_2)) = \{\emptyset,\{a\},\{b\},\{c\}\} = \mathrm{Mod}(\Delta^\star_\mu(E_1 \sqcup E_2))$ and thus (IC6) does not hold.

Otherwise, $f(\mathcal{M}, \mathcal{M}\mathrm{od}(E_1))$ is one of the following six cases: $\{\{a\}\}$, $\{\{b\}\}$, $\{\{c\}\}$, $\{\{a\},\{b\}\}$, $\{\{a\},\{c\}\}$, or $\{\{b\},\{c\}\}$. The set $\{\{a\},\{b\},\{c\}\}$ is excluded, because $\{\{a\},\{b\},\{c\}\} \subseteq f(\mathcal{M}, \mathcal{M}\mathrm{od}(E_1))$ would imply $\emptyset \in f(\mathcal{M}, \mathcal{M}\mathrm{od}(E_1))$. Let us suppose $f(\mathcal{M}, \mathcal{M}\mathrm{od}(E_1)) = \{\{a\}\}$ or $f(\mathcal{M}, \mathcal{M}\mathrm{od}(E_1)) = \{\{a\},\{b\}\}$. The other cases are symmetric. Let $E_2 = \{K_2\}$ Then, $\mathrm{Mod}(\Delta^\star_\mu(E_2)) = \{\{a\},\{c\}\}$ and $\mathrm{Mod}(\Delta^\star_\mu(E_1) \wedge \Delta^\star_\mu(E_2)) = \{\{a\}\}$. On the other hand $\mathrm{Mod}(\Delta^\star_\mu(E_1 \sqcup E_2)) = \{\{a\},\{c\}\} \not\subseteq \{\{a\}\}$ and thus (IC6) does not hold.

(IC8): We start with the $\mathcal{L}_{Horn}$ case. Since $\mathcal{L}_{Horn}$ is an $\wedge$-fragment, there is an $\wedge$-mapping f such that $\Delta^\star = \Delta^f$ and we have $f(\mathcal{M}, \mathcal{X}) \subseteq Cl_\wedge(\mathcal{M})$ with $Cl_\wedge(f(\mathcal{M}, \mathcal{X})) = f(\mathcal{M}, \mathcal{X})$. Let us consider $E = \{K_1, K_2\}$ and μ_1 with

$$\begin{aligned}\mathrm{Mod}(K_1) &= \{\{a\},\{a,b\},\{a,c\},\{a,b,d\}\},\\ \mathrm{Mod}(K_2) &= \{\{b\},\{a,b\},\{b,c\}\} \text{ and}\\ \mathrm{Mod}(\mu_1) &= \{\emptyset,\{a\},\{b\},\{c\},\{a,b,d\}\}.\end{aligned}$$

	K_1	K_2	Σ	GMax
$\emptyset$	$g(1)$	$g(1)$	$2g(1)$	$(g(1),g(1))$
$\{a\}$	0	$g(1)$	$g(1)$	$(g(1),0)$
$\{b\}$	$g(1)$	0	$g(1)$	$(g(1),0)$
$\{c\}$	$g(1)$	$g(1)$	$2g(1)$	$(g(1),g(1))$
$\{a,b,d\}$	0	$g(1)$	$g(1)$	$(g(1),0)$

We have $\mathcal{M} = \mathrm{Mod}(\Delta_{\mu_1}(E)) = \{\{a\},\{b\},\{a,b,d\}\}$. Let us consider the possibilities for $\mathrm{Mod}(\Delta^\star_{\mu_1}(E)) = f(\mathcal{M}, \mathcal{M}\mathrm{od}(E))$. By the definition of refined operators, we know that $\{c\} \notin f(\mathcal{M}, \mathcal{M}\mathrm{od}(E))$ since $\{c\} \notin Cl_\wedge(\mathcal{M})$. First, assume $\emptyset \in f(\mathcal{M}, \mathcal{M}\mathrm{od}(E))$: let μ_2 in $\mathcal{L}_{Horn}$ be such that $\mathrm{Mod}(\mu_2) = \{\emptyset,\{c\}\} = \mathcal{N}$. Then, $\Delta^\star_{\mu_1}(E) \wedge \mu_2$ is consistent, and $\Delta_{\mu_1 \wedge \mu_2}(E) = \Delta_{\mu_2}(E) = \mathcal{N}$ is closed under $\wedge$ and thus $\Delta^\star_{\mu_1 \wedge \mu_2}(E) = \mathcal{N}$. Thus, $\Delta^\star_{\mu_1 \wedge \mu_2}(E) \not\models \Delta^\star_{\mu_1}(E)$, since $\{c\} \notin \mathrm{Mod}(\Delta^\star_{\mu_1}(E))$. Otherwise, $f(\mathcal{M}, \mathcal{M}\mathrm{od}(E)) \subseteq \{\{a,b,d\},\{a\}\}$ or $f(\mathcal{M}, \mathcal{M}\mathrm{od}(E)) \subseteq \{\{a,b,d\},\{b\}\}$, because $\{\{a\},\{b\}\} \subseteq f(\mathcal{M}, \mathcal{M}\mathrm{od}(E))$ would imply $\emptyset \in f(\mathcal{M}, \mathcal{M}\mathrm{od}(E))$. For the cases with $|f(\mathcal{M}, \mathcal{M}\mathrm{od}(E))| = 1$, we select $\mu_2 \in \mathcal{L}_{Horn}$ where $\mathrm{Mod}(\mu_2)$ is $\{\{a,b,d\},\{a\}\}$ or $\{\{a,b,d\},\{b\}\}$ such that $f(\mathcal{M}, \mathcal{M}\mathrm{od}(E)) \subseteq \mathrm{Mod}(\mu_2)$ holds. $\mathrm{Mod}(\Delta^\star_{\mu_1 \wedge \mu_2}(E)) = \mathrm{Mod}(\Delta^\star_{\mu_2}(E)) = f(\mathrm{Mod}(\Delta_{\mu_2}(E))) = \mathrm{Mod}(\mu_2)$ (again since $\mathrm{Mod}(\mu_2)$

is closed under $\wedge$) but $f(\mathcal{M}, \mathcal{M}\mathrm{od}(E)) \subset \mathrm{Mod}(\mu_2)$, since $|f(\mathcal{M}, \mathcal{M}\mathrm{od}(E))| = 1$ by assumption. Thus $\Delta^{\star}_{\mu_1 \wedge \mu_2}(E) \not\models \Delta^{\star}_{\mu_1}(E)$. Two cases remain. Let us suppose $f(\mathcal{M}, \mathcal{M}\mathrm{od}(E)) = \{\{a,b,d\},\{a\}\}$; the final case is then symmetric. We now use $\mu_2 \in \mathcal{L}_{Horn}$ with $\mathrm{Mod}(\mu_2) = \{\{a,b,d\},\{b\}\}$. Again, $\Delta^{\star}_{\mu_1}(E) \wedge \mu_2$ is consistent, $\mathrm{Mod}(\Delta^{\star}_{\mu_1 \wedge \mu_2}(E)) = \mathrm{Mod}(\Delta^{\star}_{\mu_2}(E)) = f(\mathrm{Mod}(\Delta_{\mu_2}(E))) = \mathrm{Mod}(\mu_2)$. Now, since $\{b\} \notin f(\mathcal{M}, \mathcal{M}\mathrm{od}(E))$, $\Delta^{\star}_{\mu_1 \wedge \mu_2}(E) \not\models \Delta^{\star}_{\mu_1}(E)$.

Let us now turn to the Krom Case. Let us consider $E = \{K_1, K_2, K_3\}$ and μ_1 with:

$$\begin{aligned}
\mathrm{Mod}(K_1) &= \{\{a\},\{a,b\},\{a,c\},\{a,d\}\},\\
\mathrm{Mod}(K_2) &= \{\{b\},\{a,b\},\{b,c\},\{b,d\}\},\\
\mathrm{Mod}(K_3) &= \{\{c\},\{a,c\},\{b,c\},\{c,d\}\} \text{ and}\\
\mathrm{Mod}(\mu_1) &= \{\emptyset,\{a\},\{b\},\{c\},\{d\}\}.
\end{aligned}$$

	K_1	K_2	K_3	Σ	GMax
$\emptyset$	$g(1)$	$g(1)$	$g(1)$	$3g(1)$	$(g(1),g(1),g(1))$
$\{a\}$	0	$g(1)$	$g(1)$	$2g(1)$	$(g(1),g(1),0)$
$\{b\}$	$g(1)$	0	$g(1)$	$2g(1)$	$(g(1),g(1),0)$
$\{c\}$	$g(1)$	$g(1)$	0	$2g(1)$	$(g(1),g(1),0)$
$\{d\}$	$g(1)$	$g(1)$	$g(1)$	$3g(1)$	$(g(1),g(1),g(1))$

We have $\mathcal{M} = \mathrm{Mod}(\Delta_{\mu_1}(E)) = \{\{a\},\{b\},\{c\}\}$. Let us consider the possibilities for $\mathrm{Mod}(\Delta^{\star}_{\mu_1}) = f(\mathcal{M}, \mathcal{M}\mathrm{od}(E))$. By the definition of refined operators, we know that $\{d\} \notin f(\mathcal{M}, \mathcal{M}\mathrm{od}(E))$. For $\emptyset \in f(\mathcal{M}, \mathcal{M}\mathrm{od}(E))$, let $\mathrm{Mod}(\mu_2) = \{\emptyset,\{d\}\}$. Otherwise there exist $\alpha, \beta \in \{a,b,c\}$ such that $\alpha \in f(\mathcal{M}, \mathcal{M}\mathrm{od}(E))$ and $\beta \notin f(\mathcal{M}, \mathcal{M}\mathrm{od}(E))$. Then, take $\mathrm{Mod}(\mu_2) = \{\{\alpha\},\{\beta\}\}$. □

PROOF OF PROPOSITION **??**. We have to make a case distinction depending on whether d is drastic or not.

We first consider the case where d is non-drastic. Let g be the function associated with the counting distance d and $k \geq 1$ be the minimum number such that $g(k) < g(k+1)$. Moreover, let X be a set of atoms such that $X \cap \{a,b,c\} = \emptyset$ and $|X| = k-1$.

We start with the case $\mathcal{L}_{Horn}$ and define μ with $\mathrm{Mod}(\mu) = \{X, X \cup \{a\}, X \cup \{b\}\}$ and $E_2 = \{K_2\}$ with $\mathrm{Mod}(K_2) = \{\{a,b\}\}$. Such profile and constraint exist in $\mathcal{L}_{Horn}$. Thus $\mathcal{M} = \mathrm{Mod}(\Delta_\mu(E_2)) = \{X \cup \{a\}, X \cup \{b\}\}$. Since $\mathcal{M}$ is non-β-closed, we have $\mathrm{Mod}(\Delta^{f_\beta}_\mu(E_2)) = f_\beta(\mathcal{M}, \mathcal{M}\mathrm{od}(E_2))$. Since by assumption $f_\beta(\mathcal{M}, \mathcal{M}\mathrm{od}(E_2)) \neq Cl_\beta(\mathcal{M})$ we can suppose w.l.o.g. that $f_\beta(\mathcal{M}, \mathcal{M}\mathrm{od}(E_2))$ is given by $\{X \cup \{a\}\}$, $\{X\}$ or $\{X, X \cup \{a\}\}$. Then, let us consider $E_1 = \{K_1\}$ with $\mathrm{Mod}(K_1) = \{\{b,c\}\}$. Such a profile exists in $\mathcal{L}_{Horn}$. We have the following situation:

	E_1	E_2	$E_1 \sqcup E_2$	$E_1 \sqcup E_2^n$
X	$g(k+1)$	$g(k+1)$	$g(k+1)+g(k+1)$	$(n+1)g(k+1)$
$X \cup \{a\}$	$g(k+2)$	$g(k)$	$g(k+2)+g(k)$	$ng(k)+g(k+2)$
$X \cup \{b\}$	$g(k)$	$g(k)$	$2g(k)$	$(n+1)g(k)$

By assumption $g(k) < g(k+1) \leq g(k+2)$. Moreover, for all $n \geq 1$, $(n+1)g(k) < ng(k) + g(k+2)$ and $(n+1)g(k) < (n+1)g(k+1)$. Consequently, for all $n \geq 1$, $\mathrm{Mod}(\Delta^{f_\beta}_\mu(E_1 \sqcup E_2^n)) = \mathrm{Mod}(\Delta_\mu(E_1 \sqcup E_2^n)) = \{X \cup \{b\}\}$. Therefore, for all $n \geq 1$, $\mathrm{Mod}(\Delta^{f_\beta}_\mu(E_1 \sqcup E_2^n)) \not\subseteq \mathrm{Mod}(\Delta^{f_\beta}_\mu(E_2))$, thus proving that Δ^{f_β} violates the postulate (Maj).

Let us now turn to $\mathcal{L}_{Krom}$. Here we use $\mathrm{Mod}(\mu) = \{X, X \cup \{a\}, X \cup \{b\}, X \cup \{c\}\}$ and $E_2 = \{K_2\}$ with $\mathrm{Mod}(K_2) = \{\{a,b\},\{b,c\}\}$. Again, such profile and formula exist in $\mathcal{L}_{Krom}$. We have $\mathcal{M} = \mathrm{Mod}(\Delta_\mu(E_2)) = \{X \cup \{a\}, X \cup \{b\}, X \cup \{c\}\}$. Since $\mathcal{M}$ is non-maj_3-closed, we have $\mathrm{Mod}(\Delta^{f_\beta}_\mu(E_2)) = f_\beta(\mathcal{M}, \mathcal{M}\mathrm{od}(E_2))$. Since by assumption $f_\beta(\mathcal{M}, \mathcal{M}\mathrm{od}(E_2)) \neq Cl_\beta(\mathcal{M})$ and since $X \cup \{a\}$, $X \cup \{b\}$ and $X \cup \{c\}$ play a symmetric role in $\mathcal{M}$ and in E_2, we can suppose w.l.o.g. that $f_\beta(\mathcal{M}, \mathcal{M}\mathrm{od}(E_2))$ is a non-empty subset of $\{X, X \cup \{a\}, X \cup \{b\}\}$. Then, let us consider $E_1 = \{K_1\}$ with $\mathrm{Mod}(K_1) = \{\{b, c\}\}$. Such a profile exists in $\mathcal{L}_{Krom}$. We obtain:

	E_1	E_2	$E_1 \sqcup E_2$	$E_1 \sqcup E_2^n$
X	$g(k+1)$	$g(k+1)$	$2g(k+1)$	$(n+1)g(k+1)$
$X \cup \{a\}$	$g(k+2)$	$g(k)$	$g(k+2) + g(k)$	$ng(k) + g(k+2)$
$X \cup \{b\}$	$g(k)$	$g(k)$	$2g(k)$	$(n+1)g(k)$
$X \cup \{c\}$	$g(k)$	$g(k)$	$2g(k)$	$(n+1)g(k)$

By the same calculations as before, $\mathrm{Mod}(\Delta_\mu^{f_\beta}(E_1 \sqcup E_2^n)) = \mathrm{Mod}(\Delta_\mu(E_1 \sqcup E_2^n)) = \{X \cup \{b\}, X \cup \{c\}\}$ for all $n \geq 1$. Therefore, in all cases $\mathrm{Mod}(\Delta_\mu^{f_\beta}(E_1 \sqcup E_2^n)) \not\subseteq \mathrm{Mod}(\Delta_\mu^{f_\beta}(E_2))$, thus proving that Δ^{f_β} violates the postulate (Maj).

Let us now consider the drastic distance. We start with the case $\mathcal{L}_{Horn}$ and define μ with $\mathrm{Mod}(\mu) = \{\emptyset, \{a\}, \{b\}\}$ and $E_2 = \{K_1, K_2\}$ with $\mathrm{Mod}(K_1) = \{\{a\}\}$ and $\mathrm{Mod}(K_2) = \{\{b\}\}$. Such profile and constraint exist in $\mathcal{L}_{Horn}$. Thus $\mathcal{M} = \mathrm{Mod}(\Delta_\mu(E_2)) = \{\{a\}, \{b\}\}$. Since $\mathcal{M}$ is non-β-closed, we have $\mathrm{Mod}(\Delta_\mu^{f_\beta}(E_2)) = f_\beta(\mathcal{M}, \{E_2\})$. Since by assumption $f_\beta(\mathcal{M}, \{E_2\}) \neq Cl_\beta(\mathcal{M})$ we can suppose w.l.o.g. that $f_\beta(\mathcal{M}, \{E_2\})$ is given by $\{\emptyset\}$, $\{\{a\}\}$ or $\{\emptyset, \{a\}\}$. Then, let us consider $E_1 = \{K_0\}$ with $\mathrm{Mod}(K_0) = \{\{b\}\}$. Such a profile exists in $\mathcal{L}_{Horn}$. We have the following situation:

	E_1	E_2	$E_1 \sqcup E_2$	$E_1 \sqcup E_2^n$
$\emptyset$	1	2	3	$2n+1$
$\{a\}$	1	1	2	$n+1$
$\{b\}$	0	1	1	n

Consequently, for all $n \geq 1$, $\mathrm{Mod}(\Delta_\mu^{f_\beta}(E_1 \sqcup E_2^n)) = \mathrm{Mod}(\Delta_\mu(E_1 \sqcup E_2^n)) = \{\{b\}\}$. Therefore, for all $n \geq 1$, $\mathrm{Mod}(\Delta_\mu^{f_\beta}(E_1 \sqcup E_2^n)) \not\subseteq \mathrm{Mod}(\Delta_\mu^{f_\beta}(E_2))$, thus proving that Δ^{f_β} violates the postulate (Maj).

Let us now turn to $\mathcal{L}_{Krom}$. Here we use $\mathrm{Mod}(\mu) = \{\emptyset, \{a\}, \{b\}, \{c\}\}$ and $E_2 = \{K_1, K_2, K_3\}$ with $\mathrm{Mod}(K_1) = \{\{a\}\}$, $\mathrm{Mod}(K_2) = \{\{b\}\}$ and $\mathrm{Mod}(K_1) = \{\{c\}\}$. Again, such profile and formula exist in $\mathcal{L}_{Krom}$. We have $\mathcal{M} = \mathrm{Mod}(\Delta_\mu(E_2)) = \{\{a\}, \{b\}, \{c\}\}$. Since $\mathcal{M}$ is non-maj_3-closed, we have $\mathrm{Mod}(\Delta_\mu^{f_\beta}(E_2)) = f_\beta(\mathcal{M}, \{E_2\})$. Since by assumption $f_\beta(\mathcal{M}, \{E_2\}) \neq Cl_\beta(\mathcal{M})$ and since $X \cup \{a\}$, $X \cup \{b\}$ and $X \cup \{c\}$ play a symmetric role in $\mathcal{M}$ and in E_2, we can suppose w.l.o.g. that $f_\beta(\mathcal{M}, \{E_2\})$ is a non-empty subset of $\{\emptyset, \{a\}, \{b\}\}$. Then, let us consider $E_1 = \{K_1\}$ with $\mathrm{Mod}(K_1) = \{\{c\}\}$. Such a profile exists in $\mathcal{L}_{Krom}$. We obtain:

	E_1	E_2	$E_1 \sqcup E_2$	$E_1 \sqcup E_2^n$
$\emptyset$	1	3	4	$3n+1$
$\{a\}$	1	2	3	$2n+1$
$\{b\}$	1	2	3	$2n+1$
$\{c\}$	0	2	2	$2n$

By the same calculations as before, $\mathrm{Mod}(\Delta_\mu^{f_\beta}(E_1 \sqcup E_2^n)) = \mathrm{Mod}(\Delta_\mu(E_1 \sqcup E_2^n)) = \{\{c\}\}$. Therefore, in all cases $\mathrm{Mod}(\Delta_\mu^{f_\beta}(E_1 \sqcup E_2^n)) \not\subseteq \mathrm{Mod}(\Delta_\mu^{f_\beta}(E_2))$, thus proving that Δ^{f_β} violates the postulate (Maj). □