Delete and Insert Symbols on a Tape of a Turing Machine

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1 Problem 2.8.8

Suppose that Turing machines can *delete* and *insert symbols* in their string, instead of only overwriting.

(a) Define carefully the transition function and the computation of such machines.

(b) Show that such a machine can be simulated by an ordinary Turing Machine with a quadratic loss of efficiency.

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2 Definitons

2.1 Symbols and Terms

$M = (K, \Sigma, \delta, s)$	Turing Machine
K	possible States
Σ	possible Symbols
D	possible Cursor Movements
$\delta(q, \sigma) = (p, \rho, D)$	transition function
q	current state
σ	current symbol
p	next state
ρ	symbol to be written
s	starting state
h	halting state
$\triangleright$	starting symbol
$\sqcup$	space
$\leftarrow$	left
$-$	stay
$\rightarrow$	right

2.2 Assumption for all Examples

We want to generate Turing Machines that are able to *insert* a *space* after each *1* and *remove* every *0* from the input string.

This will be the base for all the following examples in this paper.

2.3 Examples

Input	Output
$\triangleright\sqcup$	$\triangleright\sqcup$
$\triangleright 0\sqcup$	$\triangleright \sqcup \sqcup$
$\triangleright 1\sqcup$	$\triangleright 1\sqcup\sqcup$
$\triangleright 01\sqcup$	$\triangleright 1\sqcup\sqcup$
$\triangleright 010\sqcup$	$\triangleright 1\sqcup \sqcup \sqcup$
$\triangleright 101\sqcup$	$\triangleright 1\sqcup 1\sqcup\sqcup$
...	...

3 Transition Function Definition

3.1 Transition Functions

According to the three columns of the transition funcion table there are three possible ways to solve point (a) of the exercise. We can define special

- states,
- symbols or
- cursor movements

to realize such a Turing Machine.

The convention for Turing Machines says that it is not possible to fall of the left end of the tape, and that's the reason why there is only one possible way left to insert a symbol into a string. We avoid the possibility of trying to insert a symbol before the $\triangleright$, if we allow only to insert after the current symbol. On the other side we have an undefined transition function for the combination of the $\triangleright$ and trying to delete this symbol.

With an execption of adapting the cursor movements we can use the movement symbols to decide where the cursor should stay after. I have written all possible combinations down in the following a table because it is easier to understand than in form of a text-block.

Action	Movement	Before	After
delete	$\leftarrow$	$\triangleright 1\underline{0}1\sqcup$	$\triangleright \underline{1}1\sqcup$
delete	$-$	$\triangleright 1\underline{0}1\sqcup$	$\triangleright 1\underline{1}\sqcup$
delete	$\rightarrow$	$\triangleright 1\underline{0}1\sqcup$	$\triangleright 11\underline{\sqcup}$
insert	$\leftarrow$	$\triangleright 0\underline{1}0\sqcup$	$\triangleright 0\underline{1} \sqcup 0\sqcup$
insert	$-$	$\triangleright 0\underline{1}0\sqcup$	$\triangleright 01\underline{\sqcup}0\sqcup$
insert	$\rightarrow$	$\triangleright 0\underline{1}0\sqcup$	$\triangleright 01 \sqcup \underline{0}\sqcup$

3.2 States

3.2.1 Changes

We add two new states *del* and *ins* to K. There is one problem if we want to use this for a general Turing Machine with more than the two standard states s and h because we want to be able to *delete* or *insert* without having to forget the previous state of the Turing Machine. For the examples I have made the following assumption. These two new states will return immediately to the state s, after they had performed their task. That is the reason why there is not more than one - the initial - state defined in the transition function table.

The Turing Machine M with
$K = \{s, del, ins\}$,
$D = \{\leftarrow, -, \rightarrow\}$ and
$\Sigma = \{\triangleright, 0, 1, \sqcup\}$.

3.2.2 Transition Function Table

#	$q \in K$	$\sigma \in \Sigma$	$\delta(q, \sigma)$
1.	s	$\triangleright$	$(s, \triangleright, \rightarrow)$
2.	s	0	$(del, 0, -)$
3.	s	1	$(ins, 1, \rightarrow)$
4.	s	$\sqcup$	$(h, \sqcup, -)$

3.2.3 Examples

Step	Current Tape	q	#
0.	$\underline{\triangleright}010\sqcup$	s	1.
1.	$\triangleright\underline{0}10\sqcup$	del	2.
2.	$\triangleright\underline{1}0\sqcup\sqcup$	ins	3.
3.	$\triangleright 1\sqcup\underline{0}\sqcup$	del	2.
4.	$\triangleright 1\sqcup\underline{\sqcup}\sqcup$	s	4.
5.	$\triangleright 1\sqcup\underline{\sqcup}\sqcup$	h	

Step	Current Tape	q	#
0.	$\underline{\triangleright}101\sqcup$	s	1.
1.	$\triangleright\underline{1}01\sqcup$	ins	3.
2.	$\triangleright 1\sqcup\underline{0}1\sqcup$	del	2.
3.	$\triangleright 1\sqcup\underline{1}\sqcup\sqcup$	ins	3.
4.	$\triangleright 1\sqcup 1\sqcup\underline{\sqcup}$	s	4.
5.	$\triangleright 1\sqcup 1\sqcup\underline{\sqcup}$	h	

3.3 Symbols

3.3.1 Changes

There are two new symbols d and i. Whenever the Turing Machine finds one of these symbols on its input an endless loop will be the result. If we wrote an d on the tape, the Turing Machine will remove the actual symbol from the tape, and an i will insert a $\sqcup$ after the current symbol.

The Turing Machine M with
$K = \{s\}$,
$D = \{\leftarrow, -, \rightarrow\}$ and
$\Sigma = \{\triangleright, 0, 1, \sqcup, d, i\}$.

3.3.2 Transition Function Table

#	$q \in K$	$\sigma \in \Sigma$	$\delta(q, \sigma)$
1.	s	$\triangleright$	$(s, \triangleright, \rightarrow)$
2.	s	0	$(s, d, -)$
3.	s	1	$(s, i, \rightarrow)$
4.	s	$\sqcup$	$(h, \sqcup, -)$

3.3.3 Examples

Step	Current Tape	q	#
0.	$\underline{\triangleright}010\sqcup$	s	1.
1.	$\triangleright\underline{0}10\sqcup$	s	2.
2.	$\triangleright\underline{1}0\sqcup\sqcup$	s	3.
3.	$\triangleright1\sqcup\underline{0}\sqcup$	s	2.
4.	$\triangleright1\sqcup\underline{\sqcup}\sqcup$	s	4.
5.	$\triangleright1\sqcup\underline{\sqcup}\sqcup$	h	

Step	Current Tape	q	#
0.	$\underline{\triangleright}101\sqcup$	s	1.
1.	$\triangleright\underline{1}01\sqcup$	s	3.
2.	$\triangleright1\sqcup\underline{0}1\sqcup$	s	2.
3.	$\triangleright1\sqcup\underline{1}\sqcup\sqcup$	s	3.
4.	$\triangleright1\sqcup1\sqcup\underline{\sqcup}$	s	4.
5.	$\triangleright1\sqcup1\sqcup\underline{\sqcup}$	h	

3.4 Movements

3.4.1 Changes

We expand the set of D by del and ins. Now we have no possibility to define a direction where the cursor should be after the state. So we assume that there is no movement to the left or to the right. After removing the actual symbol from the tape the cursor will be under the next symbol. And after inserting a symbol into the string the cursor will point to the inserted symbol, but we want to stay after this symbol to be able to process with the next symbol on the tape. So I have expanded the Turing Machine by an additional state j which will jump over the inserted symbol.

The Turing Machine M with
$K = \{s, j\}$,
$D = \{\leftarrow, -, \rightarrow, ins, del\}$ and
$\Sigma = \{\triangleright, 0, 1, \sqcup\}$.

3.4.2 Transition Function Table

#	$q \in K$	$\sigma \in \Sigma$	$\delta(q, \sigma)$
1.	s	$\triangleright$	$(s, \triangleright, \rightarrow)$
2.	s	0	$(s, 0, del)$
3.	s	1	$(j, 1, ins)$
4.	s	$\sqcup$	$(h, \sqcup, -)$
5.	j	$\triangleright$	$(s, \triangleright, \rightarrow)$[1]
8.	j	$\sqcup$	$(s, \sqcup, \rightarrow)$

3.4.3 Examples

Step	Current Tape	q	#
0.	$\underline{\triangleright}010\sqcup$	s	1.
1.	$\triangleright\underline{0}10\sqcup$	s	2.
2.	$\triangleright\underline{1}0\sqcup\sqcup$	s	3.
3.	$\triangleright 1\underline{\sqcup}0\sqcup$	j	8.
4.	$\triangleright 1\sqcup\underline{0}\sqcup$	s	2.
5.	$\triangleright 1\sqcup\underline{\sqcup}\sqcup$	s	4.
6.	$\triangleright 1\sqcup\underline{\sqcup}\sqcup$	h	

Step	Current Tape	q	#
0.	$\underline{\triangleright}101\sqcup$	s	1.
1.	$\triangleright\underline{1}01\sqcup$	s	3.
2.	$\triangleright 1\underline{\sqcup}01\sqcup$	j	8.
3.	$\triangleright 1\sqcup\underline{0}1\sqcup$	s	2.
4.	$\triangleright 1\sqcup\underline{1}\sqcup\sqcup$	s	3.
5.	$\triangleright 1\sqcup 1\underline{\sqcup}\sqcup$	j	8.
6.	$\triangleright 1\sqcup 1\sqcup\underline{\sqcup}$	s	4.
7.	$\triangleright 1\sqcup 1\sqcup\underline{\sqcup}$	h	

[1] Some Transition Functions are not necessary to be defined. Do not wonder when there are missing functions in the Transition Function Tables.

4 Simulation with a standard Turing Machine

4.1 Proof

Now we should show that a normal Turing Machine will work with a quadratic loss of efficiency. In O-notation $O(N^2)$, which means that the computing time will grow in a quadratic order when doubbing the length of the input string.

4.1.1 Proof for deleting symbols

First we will deal with a string which contains n elements, and we want to remove all elements form the tape, because it is a little bit easier to find a general formula for the number of steps the Turing Machine will need to process before halting.

For an empty string the Turing Machine needs two steps to halt. If there are n symols on the string we have to go twice over the tape until one symbol is removed from the tape. These are $(2n+1)$ steps, $+1$ because we have to change the direction at the empty (last) symbol of the string. So now there are only $n-1$ symols left on the tape. And that is it.

Z_n ...total number of steps until the halting state is reached

$$\sum_{i=m}^{n} i \qquad (m \leq n)$$
$$\frac{n+m}{2}(n-(m-1))$$

General form of dissolving sums (Gauß).

$$Z_n = (2*n+1)+(2*(n-1)+1)+\ldots+(2*1+1)+\underbrace{(2*0+1)+1}_{empty\ \ string} \tag{1}$$
$$Z_n = \left(\sum_{i=0}^{n}(2i+1)\right)+1 \tag{2}$$
$$Z_n = \left(2\left(\frac{n}{2}\right)(n+1)\right)+(n+1)+1 \tag{3}$$
$$Z_n = \left(n^2+n\right)+n+1+1 \tag{4}$$

$$
\begin{aligned}
Z_n &= \left(n^2 + 2n + 1\right) + 1 && (5)\\
Z_n &= (n+1)^2 + 1 && (6)
\end{aligned}
$$

(1) Summing up the steps for removing n, n-1, ..., 1, 0 symbols on the tape.

(2) Same as in the previous line, but using the sum formula.

(3) Breaking up the sum formula.

(4) Removing the unnecessary multiplication and division and transforming $n\,(n+1)$ to $n^2 + n$.

(5) Combining for simplfing the term.

(6) $\Rightarrow O(N^2)$

4.1.2 Proof for inserting symbols

Now we suppose an empty string, and we want to insert n elements on the beginning of the tape. This will be the worst case for our Turing Machine because we must return after each insertion process to the $\triangleright$ on the tape. over the inserted symbol.

$$
\begin{aligned}
Z_n &= (2 * 1 + 2) + \ldots + (2(n-1) + 2) + (2n + 2) && (7)\\
Z_n &= \sum_{i=1}^{n} (2i + 2) && (8)\\
Z_n &= 2\left(\frac{n+1}{2}\right) n + 2n && (9)\\
Z_n &= n^2 + 3n && (10)
\end{aligned}
$$

(7) Summing up the steps for inserting 1, 2, ..., n-1, n symbols in the string.

(8) Same as in the previous line, but using the sum formula.

(9) Breaking up the sum formula.

(10) $\Rightarrow O(N^2)$

So we can say, that the costs for simulating a Turing Machine - that can *delete* and *insert* symbols in their string - with a normal Turing Machine are of quadratic order.

4.2 Transition Function Table

The symbol X is used to mark the position where the deleting/inserting process has been started.

The Turing Machine M with
$K = \{s, d, d_0, d_1, d_\sqcup, i, i_0, i_1, i_\sqcup, j\}$,
$D = \{\leftarrow, -, \rightarrow\}$ and
$\Sigma = \{\triangleright, 0, 1, X, \sqcup\}$.

#	$q \in K$	$\sigma \in \Sigma$	$\delta(q, \sigma)$
1.	s	$\triangleright$	$(s, \triangleright, \rightarrow)$
2.	s	0	$(d, X, \rightarrow)$
3.	s	1	$(i_\sqcup, X, \rightarrow)$
5.	s	$\sqcup$	$(h, \sqcup, -)$
7.	d	0	$(d, 0, \rightarrow)$
8.	d	1	$(d, 1, \rightarrow)$
10.	d	$\sqcup$	$(d_\sqcup, \sqcup, \leftarrow)$
12.	d_0	0	$(d_0, 0, \leftarrow)$
13.	d_0	1	$(d_1, 0, \leftarrow)$
14.	d_0	X	$(s, 0, -)$
17.	d_1	0	$(d_0, 1, \leftarrow)$
18.	d_1	1	$(d_1, 1, \leftarrow)$
19.	d_1	X	$(s, 1, -)$
22.	$d_\sqcup$	0	$(d_0, \sqcup, \leftarrow)$
23.	$d_\sqcup$	1	$(d_1, \sqcup, \leftarrow)$
24.	$d_\sqcup$	X	$(s, \sqcup, -)$
27.	i	0	$(i, 0, \leftarrow)$
28.	i	1	$(i, 1, \leftarrow)$
29.	i	X	$(j, 1, \rightarrow)$
30.	i	$\sqcup$	$(i, \sqcup, \leftarrow)$
35.	j	$\sqcup$	$(s, \sqcup, \rightarrow)$
37.	i_0	0	$(i_0, 0, \rightarrow)$
38.	i_0	1	$(i_1, 0, \rightarrow)$
40.	i_0	$\sqcup$	$(i, 0, \leftarrow)$
42.	i_1	0	$(i_0, 1, \rightarrow)$
43.	i_1	1	$(i_1, 1, \rightarrow)$
45.	i_1	$\sqcup$	$(i, 1, \leftarrow)$
47.	$i_\sqcup$	0	$(i_0, \sqcup, \rightarrow)$
48.	$i_\sqcup$	1	$(i_1, \sqcup, \rightarrow)$
50.	$i_\sqcup$	$\sqcup$	$(i, \sqcup, \leftarrow)$

4.2.1 Examples

Step	Current Tape	q	#
0.	$\underline{\triangleright}010\sqcup$	s	1.
1.	$\triangleright\underline{0}10\sqcup$	s	2.
2.	$\triangleright X\underline{1}0\sqcup$	d	8.
3.	$\triangleright X1\underline{0}\sqcup$	d	7.
4.	$\triangleright X10\underline{\sqcup}$	d	10.
5.	$\triangleright X1\underline{0}\sqcup$	$d_\sqcup$	22.
6.	$\triangleright X\underline{1}\sqcup\sqcup$	d_0	13.
7.	$\triangleright\underline{X}0\sqcup\sqcup$	d_1	19.
8.	$\triangleright\underline{1}0\sqcup\sqcup$	s	3.
9.	$\triangleright X\underline{0}\sqcup\sqcup$	$i_\sqcup$	47.
10.	$\triangleright X\sqcup\underline{\sqcup}\sqcup$	i_0	40.
11.	$\triangleright X\underline{\sqcup}0\sqcup$	i	30.
12.	$\triangleright\underline{X}\sqcup 0\sqcup$	i	29.
13.	$\triangleright 1\underline{\sqcup}0\sqcup$	j	35.
14.	$\triangleright 1\sqcup\underline{0}\sqcup$	s	2.
15.	$\triangleright 1\sqcup X\underline{\sqcup}$	d	10.
16.	$\triangleright 1\sqcup\underline{X}\sqcup$	$d_\sqcup$	24.
17.	$\triangleright 1\sqcup\underline{\sqcup}\sqcup$	s	5.
18.	$\triangleright 1\sqcup\underline{\sqcup}\sqcup$	h	

Step	Current Tape	q	#
0.	$\underline{\triangleright}101\sqcup$	s	1.
1.	$\triangleright\underline{1}01\sqcup$	s	3.
2.	$\triangleright X\underline{0}1\sqcup$	$i_\sqcup s$	47.
3.	$\triangleright X\sqcup\underline{1}\sqcup$	i_0	38.
4.	$\triangleright X\sqcup 0\underline{\sqcup}$	i_1	45.
5.	$\triangleright X\sqcup\underline{0}1$	i	27.
6.	$\triangleright X\underline{\sqcup}01$	i	30.
7.	$\triangleright\underline{X}\sqcup 01$	i	29.
8.	$\triangleright 1\underline{\sqcup}01$	j	35.
9.	$\triangleright 1\sqcup\underline{0}1$	s	2.
10.	$\triangleright 1\sqcup X\underline{1}$	d	8.
11.	$\triangleright 1\sqcup X1\underline{\sqcup}$	d	10.
12.	$\triangleright 1\sqcup X\underline{1}\sqcup$	$d_\sqcup$	23.
13.	$\triangleright 1\sqcup\underline{X}\sqcup\sqcup$	d_1	19.
14.	$\triangleright 1\sqcup\underline{1}\sqcup\sqcup$	s	3.
15.	$\triangleright 1\sqcup X\underline{\sqcup}\sqcup$	$i_\sqcup$	50.
16.	$\triangleright 1\sqcup\underline{X}\sqcup\sqcup$	i	29.
17.	$\triangleright 1\sqcup 1\underline{\sqcup}\sqcup$	j	35.
18.	$\triangleright 1\sqcup 1\sqcup\underline{\sqcup}$	s	5.
19.	$\triangleright 1\sqcup 1\sqcup\underline{\sqcup}$	h	

Step	Current Tape	q	#
0.	$\underline{\triangleright}000\sqcup$	s	1.
1.	$\triangleright\underline{0}00\sqcup$	s	2.
2.	$\triangleright X\underline{0}0\sqcup$	d	7.
3.	$\triangleright X0\underline{0}\sqcup$	d	7.
4.	$\triangleright X00\underline{\sqcup}$	d	10.
5.	$\triangleright X0\underline{0}\sqcup$	$d_\sqcup$	22.
6.	$\triangleright X\underline{0}\sqcup\sqcup$	d_0	12.
7.	$\triangleright \underline{X}0\sqcup\sqcup$	d_0	14.
8.	$\triangleright\underline{0}0\sqcup\sqcup$	s	2.
9.	$\triangleright X\underline{0}\sqcup\sqcup$	d	7.
10.	$\triangleright X0\underline{\sqcup}\sqcup$	d	10.
11.	$\triangleright X\underline{0}\sqcup\sqcup$	$d_\sqcup$	22.
12.	$\triangleright \underline{X}\sqcup\ \sqcup\ \sqcup$	d_0	14.
13.	$\triangleright\underline{0}\sqcup\ \sqcup\ \sqcup$	s	2.
14.	$\triangleright X\underline{\sqcup}\ \sqcup\ \sqcup$	d	10.
15.	$\triangleright \underline{X}\sqcup\ \sqcup\ \sqcup$	$d_\sqcup$	24.
16.	$\triangleright\underline{\sqcup}\ \sqcup\ \sqcup\sqcup$	s	5.
17.	$\triangleright\underline{\sqcup}\ \sqcup\ \sqcup\sqcup$	h	

Step	Current Tape	q	#
0.	$\underline{\triangleright}111\sqcup$	s	1.
1.	$\triangleright\underline{1}11\sqcup$	s	3.
2.	$\triangleright X\underline{1}1\sqcup$	$i_\sqcup$	48.
3.	$\triangleright X\sqcup\underline{1}\sqcup$	i_1	43.
4.	$\triangleright X\sqcup 1\underline{\sqcup}$	i_1	45.
5.	$\triangleright X\sqcup\underline{1}1$	i	28.
6.	$\triangleright X\underline{\sqcup}11$	i	30.
7.	$\triangleright\underline{X}\sqcup 11$	i	29.
8.	$\triangleright 1\underline{\sqcup}11$	j	35.
9.	$\triangleright 1\sqcup\underline{1}1$	s	3.
10.	$\triangleright 1\sqcup X\underline{1}$	$i_\sqcup$	48.
11.	$\triangleright 1\sqcup X\underline{\sqcup}$	i_1	45.
12.	$\triangleright 1\sqcup X\underline{\sqcup}1$	i	30.
13.	$\triangleright 1\sqcup\underline{X}\sqcup 1$	i	29.
14.	$\triangleright 1\sqcup 1\underline{\sqcup}1$	j	35.
15.	$\triangleright 1\sqcup 1\sqcup\underline{1}$	s	3.
16.	$\triangleright 1\sqcup 1\sqcup X\underline{\sqcup}$	$i_\sqcup$	50.
17.	$\triangleright 1\sqcup 1\sqcup\underline{X}\sqcup$	i	29.
18.	$\triangleright 1\sqcup 1\sqcup 1\underline{\sqcup}$	j	35.
19.	$\triangleright 1\sqcup 1\sqcup 1\sqcup\underline{\sqcup}$	s	5.
20.	$\triangleright 1\sqcup 1\sqcup 1\sqcup\underline{\sqcup}$	h	

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