1 Beweisen Sie Satz 1.2.2 der Vorlesung

$\boldsymbol{\gamma} = \mathbf{x} + i\mathbf{y}$
$\lambda = a + ib$
$\boldsymbol{\gamma} \neq \mathbf{0}$
$\mathbf{A}\boldsymbol{\gamma} = \lambda\boldsymbol{\gamma}$

$\mathbf{A}(\mathbf{x} + i\mathbf{y}) = (a + ib)(\mathbf{x} + i\mathbf{y})$
$\mathbf{A}\mathbf{x} + \mathbf{A}i\mathbf{y} = a\mathbf{x} + ib\mathbf{x} + ai\mathbf{y} + i^2 b\mathbf{y}$
$\mathbf{A}\mathbf{x} + \mathbf{A}i\mathbf{y} = a\mathbf{x} + ib\mathbf{x} + ai\mathbf{y} - b\mathbf{y}$

spliten in Real- und Imaginärteil

$\mathbf{A}\mathbf{x} = a\mathbf{x} - b\mathbf{y}$
$\mathbf{A}\mathbf{x} = a\mathbf{x} - b\mathbf{y}$

$\mathbf{y}^T\mathbf{A}\mathbf{x} = \mathbf{y}^T a\mathbf{x} - \mathbf{y}^T b\mathbf{y}$

$\mathbf{A}i\mathbf{y} = ib\mathbf{x} + ai\mathbf{y}$
$\mathbf{A}\mathbf{y} = b\mathbf{x} + a\mathbf{y}$

$\mathbf{x}^T\mathbf{A}\mathbf{y} = \mathbf{x}^T b\mathbf{x} + \mathbf{x}^T a\mathbf{y}$
$\mathbf{y}^T\mathbf{A}\mathbf{x} = \mathbf{x}^T b\mathbf{x} + \mathbf{y}^T a\mathbf{x}$

$0 = -b(\mathbf{x}^T\mathbf{x} + \mathbf{y}^T\mathbf{y})$
$b = 0$ da $\boldsymbol{\gamma} \neq 0$ sein muß

2 Zeigen Sie

$$\boldsymbol{\Gamma}\boldsymbol{\Lambda}\boldsymbol{\Gamma}^T = \sum_{i=1}^{p} \lambda_i \boldsymbol{\gamma}_{.i}\boldsymbol{\gamma}_{.i}^T$$

$$\boldsymbol{\Gamma} = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1p} \\ \vdots & \ddots & \vdots \\ \gamma_{p1} & \cdots & \gamma_{pp} \end{pmatrix} \quad \boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_p \end{pmatrix}$$

$$\boldsymbol{\Gamma}\boldsymbol{\Lambda} = \begin{pmatrix} \gamma_{11}\lambda_1 & \dots & \gamma_{1p}\lambda_p \\ \vdots & \ddots & \vdots \\ \gamma_{p1}\lambda_1 & \dots & \gamma_{pp}\lambda_p \end{pmatrix} \quad \boldsymbol{\Gamma}\boldsymbol{\Lambda}\boldsymbol{\Gamma}^T = \begin{pmatrix} \sum_{i=1}^p \gamma_{1i}\lambda_i\gamma_{1i} & \dots & \sum_{i=1}^p \gamma_{1i}\lambda_i\gamma_{pi} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^p \gamma_{pi}\lambda_i\gamma_{1i} & \dots & \sum_{i=1}^p \gamma_{pi}\lambda_i\gamma_{pi} \end{pmatrix}$$

$$\sum_{i=1}^{p} \lambda_i \boldsymbol{\gamma}_{.i}\boldsymbol{\gamma}_{.i}^T = \lambda_i \begin{pmatrix} \gamma_{1i} \\ \vdots \\ \gamma_{pi} \end{pmatrix} \begin{pmatrix} \gamma_{1i} & \dots & \gamma_{pi} \end{pmatrix}$$
$$= \begin{pmatrix} \sum_{i=1}^p \lambda_i\gamma_{1i}\gamma_{1i} & \dots & \sum_{i=1}^p \lambda_i\gamma_{1i}\gamma_{pi} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^p \lambda_i\gamma_{pi}\gamma_{1i} & \dots & \sum_{i=1}^p \lambda_i\gamma_{pi}\gamma_{pi} \end{pmatrix}$$

3 Zeigen Sie

$Cov(\mathbf{Ax}, \mathbf{By}) = \mathbf{A}Cov(\mathbf{x}, \mathbf{y})\mathbf{B}^T$

$E[(\mathbf{Ax} - \mathbf{A}\boldsymbol{\mu}_x)(\mathbf{By} - \mathbf{B}\boldsymbol{\mu}_y)^T] =$
$E[\mathbf{A}(\mathbf{x} - \boldsymbol{\mu}_x)(\mathbf{B}(\mathbf{y} - \boldsymbol{\mu}_y))^T] =$
$E[\mathbf{A}(\mathbf{x} - \boldsymbol{\mu}_x)(\mathbf{y} - \boldsymbol{\mu}_y)^T\mathbf{B}^T] =$
$\mathbf{A}\underbrace{E[(\mathbf{x} - \boldsymbol{\mu}_x)(\mathbf{y} - \boldsymbol{\mu}_y)^T]}_{Cov(\mathbf{x},\mathbf{y})}\mathbf{B}^T = \mathbf{A}Cov(\mathbf{x}, \mathbf{y})\mathbf{B}^T$

4 Zeigen Sie

$E(\mathbf{x}^T\mathbf{Ax}) = \boldsymbol{\mu}^T\mathbf{A}\boldsymbol{\mu} + \sigma^2 tr(\mathbf{A})$

$$\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} =$$

$$\begin{pmatrix} \sum_{i=1}^n x_i a_{i1} & \dots & \sum_{i=1}^n x_i a_{in} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} =$$

$$\sum_{j=1}^n \sum_{i=1}^n x_i a_{ij} x_i =$$

$$\underbrace{\sum_{i=1}^n x_i a_{ii} x_i}_{Haupdiagonale} + \underbrace{\sum_{j=1}^n \sum_{i=1}^n}_{i \neq j} x_i a_{ij} x_i$$

$$E(\mathbf{x}^T\mathbf{Ax}) = \sum_{i=1}^n a_{ii} E(x_i^2) + \underbrace{\sum_{j=1}^n \sum_{i=1}^n}_{i \neq j} a_{ij} E(x_i x_j)$$

$$E(x_i x_j) = E(x_i)E(x_j) \qquad \text{(da unabhängig !!!)}$$

$$E(x_i x_j) = \mu_i \mu_j$$

$$\sigma^2 = Var(x_i) = E(x_i^2) - [E(x_i)]^2 \rightarrow E(x_i^2) = \sigma^2 + [E(x_i)]^2 = \sigma^2 + \mu_i^2$$

$$\underbrace{\underbrace{\sum_{i=1}^n}_{tr(\boldsymbol{A}} a_{ii}(\sigma^2 + \mu_i^2)}_{tr(\boldsymbol{A})+\sigma^2+} + \underbrace{\underbrace{\sum_{j=1}^n \sum_{i=1}^n}_{i \neq j} a_{ij} \mu_i \mu_j =}_{\sum_{j=1}^n \sum_{i=1}^n a_{ij}\mu_i\mu_j}$$

$$tr(\boldsymbol{A})\sigma^2 + \mu^T\boldsymbol{A}\mu$$