Quantitative Disjunctive Logic Programming: Semantics and Computation *

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A new knowledge representation language, called QDLP, which extends DLP to deal with uncertain values is introduced. Each (quantitative) rule is assigned a certainty degree interval (a subinterval of $[0,1]$). The propagation of uncertainty information from the premises to the conclusion of a quantitative rule is achieved by means of triangular norms (T-norms). Different T-norms induce different semantics for one given quantitative program. In this sense, QDLP is parameterized and each choice of a T-norm induces a different QDLP language. Each T-norm is eligible for events with determinate relationships (e.g., independence, exclusiveness) between them. Since there are infinitely many T-norms, it turns out that there is a family of infinitely many QDLP languages. This family is carefully studied and the set of QDLP languages which generalize traditional DLP is precisely singled out. Algorithms for computing the minimal models of quantitative programs are proposed.

1. Introduction

Disjunctive logic programs are logic programs where disjunction is allowed in the heads of the rules and negation may occur in the bodies of the rules. The extension of logic programming by disjunction has been pointed out as a necessary requirement for knowledge representation and commonsense reasoning [2,18,24]. This view has been confirmed by results which prove that without disjunction, the expressive capability of logic programming is limited such that relevant problems can not be expressed [15,7]. Disjunctive logic programming (DLP) has a very high expressive power. In [16] it is proved that, under stable model semantics, disjunctive programs capture the complexity class Σ_2^P, that is, they allow us to express every property which is decidable in non-deterministic polynomial time with an oracle in NP. Thus, DLP can express real world situations that cannot be represented by disjunction-free programs. Moreover, disjunction has been recognized as an important feature for a declarative knowledge representation language, which allows to express knowledge in a simple and natural way.

However, real-life applications often need to deal with uncertain information and quantitative data which cannot be represented in DLP. The usual logical reasoning in terms of the truth values **true** and **false** are insufficient for the purposes of several real-life applications. Image databases, sensor data, temporal indeterminacy, information retrieval are only a few of the domains where uncertainty occurs [22]. Consider for instance a robot which moves and changes direction according to a prefixed route and to the coordinates received from a sensor. Since sensor data may be subject to error and sensors may have different reliability, a formalism able to deal with uncertain information is needed to encode the control mechanism of the robot. (See section 3.2 for an example on this subject.)

Many frameworks for multivalued logic programming have been proposed to handle uncertain information. There is a split in the AI community between (i) those who attempt to deal with uncertainty using non-numerical techniques [8,9,14], (ii) those who use numerical representations of uncertainty but, believing that probability calculus is inadequate for the task, invent entirely new calculi, such as Dempster-Shafer calculus [11,19,35], fuzzy logic [5,6,17,20,21,36,37], and (iii) those who remain within the traditional framework of probability theory, while attempting to equip the theory with computational facilities needed to perform AI tasks [1,10,13,27,28,29,30,32,33].

*Work partially supported by the Austrian Science Fund (FWF) under grants N Z29-INF, P12344-INF and P11580-MAT.

ISSN 0921-7126, IOS Press.

We propose an approach to define the representation, inference, and control of uncertain information in the framework of DLP which is closely related to the second of the above categories.

The main contributions of the paper are the following.

- We define a new knowledge representation language, called *Quantitative Disjunctive Logic Programming* (QDLP), extending DLP to deal with uncertain values.
- We define a mechanism of reasoning with uncertainty through rule chaining by using the well-studied and mathematically clean notion of T-norm. In particular, we consider a p-parameterized family of T-norms. Each T-norm is eligible for events with determinate relationships (e.g., independence, exclusiveness) between them. Different T-norms induce different semantics for one given quantitative program. Thus, QDLP is parameterized and each choice of a T-norm induces a different QDLP language. There are infinitely many T-norms, hence there are infinitely many QDLP languages. Importantly, the T-norm may be chosen according to the level of knowledge of the relationships between the atoms (events) of the program.
- We propose algorithms which compute the minimal models of quantitative programs for every T-norm in the non-disjunctive case, and for the T-norm T_3 in the disjunctive case.
- We show that the Quantitative Logic Programming Language proposed by van Emden in [37] coincides with the disjunction-free fragment of QDLP induced by the T-norm T_3.
- We single out precisely the fragments from the QDLP family which are generalizations of DLP. Basically, a fragment QF of QDLP induced by a T-norm $T^{(p)}$, $p \in [-\infty, +\infty]$, is a generalization of DLP iff to each program P from DLP corresponds a program QP in QF such that the set of the minimal models of P is exactly the set of the minimal models of QP under the semantics induced by $T^{(p)}$.

The proofs of all theorems, lemmas, and propositions which appear throughout this paper are given in Appendix B.

2. Preliminaries: Triangular Norms and Conorms

The triangular norms (T-norms) and conorms (T-conorms) form the basis for the various uncertainty calculi discussed in this paper. They are the most general families of binary functions that satisfy the requirements of the conjunction and disjunction operators, respectively. We will denote a T-norm by T and a T-conorm by S. One of the advantages of these operators is their low computational complexity.

The T-norms and T-conorms are functions $T, S : [0,1] \times [0,1] \to [0,1]$ which satisfy the following properties:

$$\begin{array}{ll}
T(a,0) = T(0,a) = 0 & [boundary] \\
T(a,1) = T(1,a) = a & [boundary] \\
T(a,b) \leq T(c,d) \text{ if } a \leq c,\ b \leq d & [monotonicity] \\
T(a,b) = T(b,a) & [commutativity] \\
T(a,T(b,c)) = T(T(a,b),c) & [associativity]
\end{array}$$

$$\begin{array}{ll}
S(1,a) = S(a,1) = 1 & [boundary] \\
S(0,a) = S(a,0) = a & [boundary] \\
S(a,b) \leq S(c,d) \text{ if } a \leq c,\ b \leq d & [monotonicity] \\
S(a,b) = S(b,a) & [commutativity] \\
S(a,S(b,c)) = S(S(a,b),c) & [associativity]
\end{array}$$

Intuitively, $T(a,b)$ (resp., $S(a,b)$) assigns a certainty value to the composition of two events e_1 and e_2 whose certainty values are a and b. Usually, the composition of e_1 and e_2 is the conjunction (resp., disjunction) under certain conditions (e.g., independence, mutual exclusiveness).

Although defined as two-place functions, the T-norms and T-conorms can be used to represent the composition of a larger number of events. Because of the associativity property, it is possible to define recursively $T(x_1,\ldots,x_n,x_{n+1})$ and $S(x_1,\ldots,x_n,x_{n+1})$ for $x_1,\ldots,x_{n+1} \in [0,1]$ as:

$$T(x_1,\ldots,x_n,x_{n+1}) = T(T(x_1,\ldots,x_n),x_{n+1})$$
$$S(x_1,\ldots,x_n,x_{n+1}) = S(S(x_1,\ldots,x_n),x_{n+1})$$

Some typical T-norms and T-conorms are the following:

$$T_0(a,b) = \begin{cases} min(a,b) & \text{if } max(a,b) = 1 \\ 0 & \text{otherwise} \end{cases}$$
$$T_1(a,b) = max(0, a+b-1)$$
$$T_{1.5}(a,b) = max(0, \sqrt{a} + \sqrt{b} - 1)^2$$
$$T_2(a,b) = ab$$
$$T_{2.5}(a,b) = \frac{ab}{a+b-ab}$$
$$T_3(a,b) = min(a,b)$$

$$S_0(a,b) = \begin{cases} max(a,b) & \text{if } min(a,b) = 0 \\ 1 & \text{otherwise} \end{cases}$$
$$S_1(a,b) = min(1, a+b)$$
$$S_{1.5}(a,b) = 1 - max(0, \sqrt{1-a} + \sqrt{1-b} - 1)^2$$
$$S_2(a,b) = a+b-ab$$
$$S_{2.5}(a,b) = \frac{a+b-2ab}{1-ab}$$
$$S_3(a,b) = max(a,b)$$

p	$-\infty$	-1	-0.5	0	1	$+\infty$
$T(a,b,p)$	T_0	T_1	$T_{1.5}$	T_2	$T_{2.5}$	T_3

Fig. 1. Spanning of the T-norms over the real numbers

It is important to note that

$$T_0 \leq T_1 \leq T_{1.5} \leq T_2 \leq T_{2.5} \leq T_3$$
$$S_3 \leq S_{2.5} \leq S_2 \leq S_{1.5} \leq S_1 \leq S_0$$

T_1 is appropriate to perform the intersection of lower probability bounds (uncertainty values) and captures the notion of the worst case, where the arguments are considered as mutually exclusive as possible. T_3 is appropriate to represent the intersection of upper probability bounds and captures the notion of the best case, where one argument attempts to subsume the others. T_2 is the classical probabilistic operator that assumes independence of arguments and its dual T-conorm S_2 is the usual additive measure for the union.

Schweizer and Sklar [34] proposed a parameterized family, denoted by $T(a,b,p)$, where a and b are the T-norm's arguments and p is the parameter that spans the space of T-norms from T_0 to T_3:

$$T(a,b,p) = \begin{cases} (a^{-p} + b^{-p} - 1)^{-\frac{1}{p}} & \text{if } a^{-p} + b^{-p} \geq 1 \\ & \text{when } p < 0 \\ 0 & \text{if } a^{-p} + b^{-p} \leq 1 \\ & \text{when } p < 0 \\ lim_{p \to 0} T(a,b,p) = ab & \text{when } p \to 0 \\ (a^{-p} + b^{-p} - 1)^{-\frac{1}{p}} & \text{when } p > 0 \end{cases}$$

Let $\overline{\mathbf{R}} = [-\infty, +\infty]$ and $\overline{\mathbf{R}^+} = [0, +\infty]$. Given a real number $p \in \overline{\mathbf{R}}$, we denote by $T^{(p)}$ the member of the family of T-norms induced by p. Note that we allow p to be assigned the infinite values $-\infty$ and $+\infty$. Figure 1 illustrates how $T^{(p)}$ spans over the real numbers, so for example $T^{(-\infty)} = T_0$, $T^{(-1)} = T_1$, $T^{(0)} = T_2$, and $T^{(+\infty)} = T_3$.

For suitable negation operators $N(a)$, such as $N(a) = 1 - a$, T-norms and T-conorms are duals in the sense of the following generalization of DeMorgan's law:

$$S(a,b) = N(T(N(a), N(b)))$$
$$T(a,b) = N(S(N(a), N(b)))$$

This duality implies that given the negation operator $N(a) = 1 - a$, the selection of a T-norm uniquely constrains the selection of the T-conorm.

The dual parameterized family of T-conorm, denoted by $S(a,b,p)$ is defined as $S(a,b,p) = 1 - T(1-a, 1-b, p)$. Given a real number $p \in \overline{\mathbf{R}}$ we denote by $S^{(p)}$ the member of the family of T-conorms induced by p. So for example $S^{(-\infty)} = S_0$, $S^{(-1)} = S_1$, $S^{(0)} = S_2$, and $S^{(+\infty)} = S_3$.

Theorem 1 *The evaluation of the T-norms and T-conorms at the extremes of the unity interval $[0,1]$ satisfies the truth tables of the logical operators AND and OR, respectively.* □

The previous theorem proves that the T-norms (resp., T-conorms) are generalizations of the logical operator AND (resp., OR) in the sense that the T-norms (resp., T-conorms) deal with all real numbers between 0 and 1 and their behavior coincides with the behavior of the logical operator AND (resp., OR) when applied to logical values (e.g., 0 and 1).

3. The QDLP Language

3.1. Syntax

A $term$ is either a constant or a variable[1]. An $atom$ is $a(t_1, ..., t_n)$, where a is a $predicate$ of arity n and $t_1, ..., t_n$ are terms. A $literal$ is either a $positive\ literal$ p or a $negative\ literal$ $\neg p$, where p is an atom.

A *(disjunctive) quantitative rule* r is a clause of the form:

$$h_1 \vee \cdots \vee h_n \xleftarrow{[x,y]} b_1, \cdots, b_k, \quad n \geq 1,\ \ k \geq 0$$

where $h_1, \cdots, h_n, b_1, \cdots, b_k$ are atoms and $0 < x \leq y \leq 1$. The interval $[x, y]$ is the certainty degree interval of the rule (i.e., the strength of the rule implication) and it is a measure of the reliability of the rule. $h_1 \vee \cdots \vee h_n$ is the head of the quantitative rule and it is a non-empty disjunction of atoms. $b_1, \cdots, b_k$ is the body of the quantitative rule and it is a (possibly empty) conjunction of atoms. If the body is empty (i.e., $k = 0$) and the head contains exactly one atom (i.e., $n = 1$), the rule is a fact whose certainty degree interval coincides with the strength of the implication.

A *(disjunctive) quantitative program* is a finite set of quantitative disjunctive rules.

Note that we use intervals to assign certainty degrees to rules instead of adopting single coefficients. The reason is that in practice it may be difficult to assign a pre-

[1] Note that function symbols are not considered in this work.

cise value to the certainty degree of a rule (fact). One might be able to only say that the certainty degree of the strength of a rule implication ranges in some interval $[x, y] \subseteq [0, 1]$ and might not be able to say that it is a precise value p in $[0, 1]$. This allows a greater flexibility and subsumes the case of single coefficients. If the strength of the implication is known to be a precise number p, then $x = y = p$ and the interval $[x, y]$ collapses into a single point p.

3.2. Semantics

Let $\mathcal{P}$ be a disjunctive quantitative program. The *Herbrand universe* $U_{\mathcal{P}}$, the *Herbrand base* $B_{\mathcal{P}}$, and $ground(\mathcal{P})$ of $\mathcal{P}$ are defined like in DLP.

Once we defined the syntax of quantitative rules, we need to evaluate the satisfiability of premises, to propagate uncertainty through rule chaining and to consolidate the same conclusion derived from different rules.

A *quantitative interpretation* I of $\mathcal{P}$ is a mapping which assigns to each atom $A \in B_{\mathcal{P}}$ a certainty degree interval $[x_A, y_A] \subseteq [0, 1]$. We write $I(A) = [x_A, y_A]$, $\forall\ A \in B_{\mathcal{P}}$, $[x_A, y_A] \subseteq [0, 1]$.

It is worth noting that a quantitative program $\mathcal{P}$ has infinitely many quantitative interpretations because each atom $A \in B_{\mathcal{P}}$ can be assigned infinitely many intervals $[x_A, y_A] \subseteq [0, 1]$. This is an important difference w.r.t. (function-free) DLP, where each program has always a finite number of Herbrand interpretations.

Let p be any real number inducing $T^{(p)}$ from the family of T-norms. We denote by $\mathcal{T}^{(p)}$ (resp., $\mathcal{S}^{(p)}$) the generalization of the T-norm $T^{(p)}$ (resp., T-conorm $S^{(p)}$) whose arguments are intervals instead of single values, e.g., $\mathcal{T}^{(p)}([a, b], [c, d]) = [T^{(p)}(a, c), T^{(p)}(b, d)]$.

Now that we know what a quantitative interpretation I is, the first thing to straighten out is when a rule r is true w.r.t. I and what is the role of p. To this end, we first define the way the certainty degree intervals of the atoms of a conjunction or disjunction are combined. In particular, we define

1. The certainty degree interval of a (possibly empty) conjunction C of atoms from $B_{\mathcal{P}}$, $C = b_1 \wedge \ldots \wedge b_m$, w.r.t. I and p:

$$I^{(p)}(C) = \begin{cases} [1, 1], & \text{if } m = 0 \ \ (i.e., C = \emptyset) \\ \mathcal{T}^{(p)}(I(b_1), \ldots, I(b_m)), & \text{if } m > 0 \end{cases}$$

2. The certainty degree interval of a non-empty disjunction D of atoms from $B_{\mathcal{P}}$, $D = h_1 \vee \ldots \vee h_n$, w.r.t. I and p:

$$I^{(p)}(D) = \mathcal{S}^{(p)}(I(h_1), \ldots, I(h_n)).$$

Given two certainty degree intervals $[a, b]$ and $[p, q]$, then $[a, b] \leq [p, q]$ iff $a \leq p$ and $b \leq q$. Moreover, $[a, b] < [p, q]$ iff (i) $[a, b] \leq [p, q]$, and (ii) $a < p$ or $b < q$.

We say that a rule $r \in ground(\mathcal{P})$, $H(r) \xleftarrow{[x,y]} B(r)$, is p-satisfied w.r.t. I iff the following inequality is satisfied

$$I^{(p)}(H(r)) \geq \mathcal{T}_2(I^{(p)}(B(r)), [x, y]) \qquad (1)$$

The member on the right-hand side of the inequality (1) represents the certainty degree interval propagated through the rule w.r.t. I and p.

The head event $H(r)$ depends on two events: (i) the rule reliability event, expressed through $[x, y]$, and (ii) the reliability event of the body of r w.r.t. I and p, given by $I^{(p)}(B(r))$. Intuitively, we can assume that the rule reliability is independent of the certainty degree intervals of the body literals, so that the two events are to be considered independent and for this reason we use $\mathcal{T}_2$ in (1).

A quantitative p-model of $\mathcal{P}$ is a quantitative interpretation M of $\mathcal{P}$ such that each rule $r \in ground(\mathcal{P})$ is p-satisfied w.r.t. M. Since the definition of quantitative p-model relies completely on the instantiation $ground(\mathcal{P})$ of $\mathcal{P}$, for simplicity, throughout the rest of this paper, we assume that $\mathcal{P}$ is a ground program (that can be either ground originally, or it is the instantiation $ground(\mathcal{P}')$ of a program $\mathcal{P}'$). The set of all p-models of $\mathcal{P}$ is denoted by $\mathcal{M}^{(p)}(\mathcal{P})$.

As previously noted, a quantitative program $\mathcal{P}$ has infinitely many quantitative interpretations. Thus, $\mathcal{P}$ may have (infinitely) many p-models. Therefore, it is useful to define an order relation between the p-models of $\mathcal{P}$ which makes possible to prefer some p-models to others. Since a p-model assigns certainty degree intervals to all atoms in $B_{\mathcal{P}}$, an order relation between p-models should be defined in terms of an order relation between intervals.

Given $M_1, M_2 \in \mathcal{M}^{(p)}(\mathcal{P})$, $M_1 \leq M_2$ iff $M_1(A) \leq M_2(A)$ for each $A \in B_{\mathcal{P}}$. Moreover, $M_1 < M_2$ iff (i) $M_1 \leq M_2$, and (ii) $\exists A \in B_{\mathcal{P}}$ s.t. $M_1(A) < M_2(A)$.

We are now in a position to define what a *minimal p-model* is.

Definition 1 *A p-model $M \in \mathcal{M}^{(p)}(\mathcal{P})$ is minimal iff there is no $N \in \mathcal{M}^{(p)}(\mathcal{P})$ such that $N < M$.* □

The minimal p-model semantics of $\mathcal{P}$ is the set of all minimal p-models of $\mathcal{P}$ and is denoted by $\mathcal{MM}^{(p)}(\mathcal{P})$.

Once we fix p, we uniquely select a T-norm and its dual T-conorm which completely describe an uncertainty calculus. That is, according to the previous definitions, once we fix p, we define a semantics for $\mathcal{P}$, called the p-semantics. In this sense, we say that the semantics of the quantitative programs is parameterized and the choice of a T-norm induces the semantics of a quantitative program. Moreover, different T-norms induce different semantics in general. Since we can fix p in infinitely many ways, we can define infinitely many semantics for $\mathcal{P}$. The T-norm may be chosen according to the level of knowledge of the relationships between the atoms of $\mathcal{P}$.

Example 1 *Consider the ground program $\mathcal{P}$ consisting of the following rules*

$$\begin{array}{lll} a \vee c \overset{[0.9,1]}{\longleftarrow} . & u \overset{[0.8,0.8]}{\longleftarrow} a,\ b\ . & w \overset{[0.5,0.6]}{\longleftarrow} u\ . \\ b \overset{[0.5,0.5]}{\longleftarrow} . & v \overset{[0.4,0.8]}{\longleftarrow} b\ . & w \overset{[1,1]}{\longleftarrow} v\ . \end{array}$$

and the interpretations I_1, I_2 and I_3,

$$I_1 = \{a : [0.9, 1], b : [0.5, 0.5], c : [0, 0], u : [0.4, 0.4], \\ v : [0.2, 0.4], w : [0.2, 0.4]\}$$

$$I_2 = \{a : [0.9, 1], b : [0.5, 0.5], c : [0, 0], u : [0.4, 0.6], \\ v : [0.2, 0.4], w : [0.2, 0.4]\}$$

$$I_3 = \{a : [0.9, 1], b : [0.5, 0.5], c : [0, 0], u : [0.2, 0.5], \\ v : [0.2, 0.4], w : [0.2, 0.4]\}$$

If $p = +\infty$ (i.e., $T^{(p)} = T_3$) then $I_1, I_2 \in \mathcal{M}^{(p)}(\mathcal{P})$. $I_3 \notin \mathcal{M}^{(p)}(\mathcal{P})$ because the rule $u \overset{[0.8,0.8]}{\longleftarrow} a, b$ is not p-satisfied w.r.t. I_3. Moreover, $I_1 < I_2$ and I_1 is minimal. □

Example 2 *Consider a robot which moves and changes direction according to a prefixed route and to the coordinates received from a sensor. Sensor data is subject to error and different sensors may have different reliabilities. The control mechanism of the robot can be encoded in QDLP as follows. Consider the atoms moveRight, moveLeft, moveUp, moveDown, xCoord(X), yCoord(Y), sensorX(X), and sensorY(Y). At regular intervals of time, the sensors return instances of the atoms $sensorX(X)$ and $sensorY(Y)$ which are used to derive the actual coordinates according to the following quantitative rules*

$$xCoord(X) \overset{[0.9,1]}{\longleftarrow} sensorX(Z), |X - Z| \leq 0.5$$
$$yCoord(Y) \overset{[0.8,1]}{\longleftarrow} sensorY(Z), |Y - Z| \leq 0.5$$

where the strength of the implication of each rule represents the reliability of the corresponding sensor in normal environment conditions (e.g., good visibility, low level of usage, etc). The built-in predicates have always the maximal reliability (i.e., $[1, 1]$). The atoms $sensorX(X)$ and $sensorY(Y)$ are assigned reliabilities according to the current environment conditions. For each turning point (x, y) of the assigned route, we define a rule like

$$atom \overset{[1,1]}{\longleftarrow} xCoord(x), yCoord(y)$$

where $atom \in \{moveRight, moveLeft, moveUp, moveDown\}$. The robot turns to the right when the certainty degree interval of $moveRight$ is at least $[0.75, 1]$, and so on. □

The question which arises now is, given a quantitative program $\mathcal{P}$ and some $p \in \overline{\mathbf{R}}$ how do we find $\mathcal{MM}^{(p)}(\mathcal{P})$? This is subject of the next section.

4. Evaluation Algorithms

In this section the computational semantics for the classes of both non-disjunctive and disjunctive quantitative programs is analyzed. We delimit the fragments of quantitative programs and the values of p for which the number of the minimal p-models is finite, and then find methods (e.g., algorithms) for computing all minimal p-models of these fragments.

4.1. Non-Disjunctive Quantitative Programs

Let $\mathcal{P}$ be a non-disjunctive quantitative program, and p a real number which induces a semantics for $\mathcal{P}$. The operator $\Phi_{\mathcal{P}}^{(p)}$ is a mapping from interpretations to interpretations such that to each atom $A \in B_{\mathcal{P}}$ it assigns the value given by

$$\Phi_{\mathcal{P}}^{(p)}(I)(A) = \mathcal{S}_3(\{\mathcal{T}_2(I^{(p)}(B(r_i)), [x_i, y_i]) \mid r_i \in \mathcal{P} \\ \text{and } r_i = A \overset{[x_i,y_i]}{\longleftarrow} B_1 \wedge \ldots \wedge B_m\})$$

(Recall that $I^{(p)}(B(r_i)) = \mathcal{T}^{(p)}(I(B_1), \ldots, I(B_m))$.)

In words, given an interpretation I for $\mathcal{P}$ and a ground atom $A \in B_{\mathcal{P}}$, the operator $\Phi_{\mathcal{P}}^{(p)}$ proceeds as follows:

(i) It finds all ground instances of rules in $\mathcal{P}$ such that the head of the rule instance is A.
(ii) For each rule r_i found at the first step, it obtains the certainty degree interval propagated through the rule r_i w.r.t. I and p, obtaining a set of intervals.

(iii) It assigns to $\Phi_{\mathcal{P}}^{(p)}(I)(A)$ the maximum certainty degree interval of the set found at the second step.

We say that $\Phi_{\mathcal{P}}^{(p)}$ increases the value of the atom A through the rule r, if the value assigned to A by $\Phi_{\mathcal{P}}^{(p)}$ at the step (iii) was inferred from the rule r at the step (ii). Note that the set of the ground instances of rules in $\mathcal{P}$ such that the head of the rule instance is A, could be the empty set. In this case, according to the world closed assumption, $\Phi_{\mathcal{P}}^{(p)}(I)(A)$ is assigned the value $[0, 0]$.

We will see that like for classical logic programs [38], (i) $\Phi_{\mathcal{P}}^{(p)}$ is monotonic, (ii) the intersection of two p-models of $\mathcal{P}$ is a p-model of $\mathcal{P}$, and (iii) $\mathcal{P}$ has a unique minimal p-model, called the least p-model of $\mathcal{P}$.

We say that a quantitative interpretation I entails an atom $A : [x, y]$ iff $I(A) \geq [x, y]$.

Given two non-empty interpretations I_1, I_2 of $\mathcal{P}$, it is reasonable to consider that the intersection $I_1 \cap I_2$ must be an interpretation I of $\mathcal{P}$ satisfying the following property:

$$\forall A \in B_{\mathcal{P}} \text{ and } \forall\, 0 \leq x \leq y \leq 1, I \text{ entails } A : [x, y] \\ \text{iff} \\ I_1 \text{ entails } A : [x, y] \text{ and } I_2 \text{ entails } A : [x, y].$$

The only T-norm satisfying this property is $\mathcal{T}_3$, hence we define $I = I_1 \cap I_2$ such that $I(A) = \mathcal{T}_3(I_1(A), I_2(A))$ for each $A \in B_{\mathcal{P}}$. Similarly, we find that the union $I_1 \cup I_2$ is the interpretation I of $\mathcal{P}$ such that $I(A) = \mathcal{S}_3(I_1(A), I_2(A))$ for each $A \in B_{\mathcal{P}}$.

The minimal p-model of $\mathcal{P}$ exists for each $p \in \overline{\mathbf{R}}$ and is unique. The unique minimal p-model of $\mathcal{P}$ is called also the least p-model of $\mathcal{P}$.

From the monotonicity of the T-norms and T-conorms it follows that $\Phi_{\mathcal{P}}^{(p)}$ is a monotonic function. That is, if I_1 and I_2 are two interpretations of $\mathcal{P}$ such that $I_1 \leq I_2$ then $\Phi_{\mathcal{P}}^{(p)}(I_1) \leq \Phi_{\mathcal{P}}^{(p)}(I_2)$.

Let $I^0 = \emptyset$, $I^0(A) = [0, 0]$ for each $A \in B_{\mathcal{P}}$, and apply $\Phi_{\mathcal{P}}^{(p)}$ iteratively. Then at the i-th iteration, $i > 0$, we have

$$I^i(A) = \Phi_{\mathcal{P}}^{(p)}(I^{i-1})(A) \qquad \forall\, A \in B_{\mathcal{P}}.$$

The monotonicity of $\Phi_{\mathcal{P}}^{(p)}$ implies that the least fixpoint $lfp(\Phi_{\mathcal{P}}^{(p)})$, namely

$$\bigcap\{I \mid \Phi_{\mathcal{P}}^{(p)}(I) = I\}$$

exists and is equal to

$$lfp(\Phi_{\mathcal{P}}^{(p)}) = \bigcup\{{\Phi_{\mathcal{P}}^{(p)}}^n(I^0) \mid n \in \mathbf{N}\}$$

where $\mathbf{N}$ is the set of natural numbers.

Input: A non-disj. quant. program $\mathcal{P}$ and $p \in \overline{\mathbf{R}}$.
Output: The least p-model of $\mathcal{P}$.

Notation: Given a quantitative program $\mathcal{P}$, $p \in \overline{\mathbf{R}}$, and a quantitative interpretation I, $unsat^{(p)}(\mathcal{P}, I)$ denotes the set of rules in $\mathcal{P}$ which are not p-satisfied w.r.t. I (i.e., $r \in unsat^{(p)}(\mathcal{P})$ iff $I^{(p)}(H(r)) < \mathcal{T}_2(I^{(p)}(B(r)), [x, y])$, where $H(r)$, $B(r)$ and $[x, y]$ are the head, the body and the strength of the implication of r, respectively).

function $PMinMod(\mathcal{P} : program,\ p : real) : model$
var $I : interpretation$;
  $\mathcal{P}', \mathcal{P}'' : program$;
begin
  $I := \emptyset$;
  $\mathcal{P}' := unsat^{(p)}(\mathcal{P}, I)$;
  $\mathcal{P}'' := \mathcal{P}'$;
  while $\mathcal{P}' \neq \emptyset$ **do begin**
    for each $r \in \mathcal{P}'$, $H(r) \xleftarrow{[x,y]} B(r)$ **do**
      if $r \in \mathcal{P}''$ **then begin**
        $I(H(r)) := \mathcal{T}_2(I^{(p)}(B(r), [x, y])$;
        $\mathcal{P}'' := unsat^{(p)}(\mathcal{P}, I)$;
        if $\mathcal{P}'' = \emptyset$ **then**
          return (I)
      end;
    $\mathcal{P}' := \mathcal{P}''$;
  end;
  return (I)
end $PMinMod$;

Fig. 2. Algorithm for Computing the Least p-Model of Non-Disjunctive Quantitative Programs

Theorem 2 *Let $\mathcal{P}$ be an arbitrary non-disjunctive quantitative program. The operator $\Phi_{\mathcal{P}}^{(p)}$ converges and computing $lfp(\Phi_{\mathcal{P}}^{(p)})$ is polynomial in the number of rules of $\mathcal{P}$, whatever is $p \in \overline{\mathbf{R}}$.* □

Theorem 3 *Let $\mathcal{P}$ be an arbitrary non-disjunctive quantitative program. The least fixpoint $lfp(\Phi_{\mathcal{P}}^{(p)})$ is a minimal p-model of $\mathcal{P}$, whatever is $p \in \overline{\mathbf{R}}$.* □

Any minimal p-model of $\mathcal{P}$ must be constructible by a computational sequence in which at each step the value of an atom is incremented only if the new value is derivable from a rule which is not p-satisfied. That is, any minimal p-model must be a least fixpoint of $\Phi_{\mathcal{P}}^{(p)}$.

Theorem 4 *Every non-disjunctive quantitative program has exactly one minimal p-model, whatever is $p \in \overline{\mathbf{R}}$.* □

An algorithm which computes the fixpoint of $\Phi_{\mathcal{P}}^{(p)}$ of a non-disjunctive quantitative program $\mathcal{P}$, hence according to Theorem 3 the least p-model of $\mathcal{P}$, is given in Figure 2.

4.2. Disjunctive Quantitative Programs

A positive quantitative disjunctive rule r is of the form

$$a_1 \vee \ldots \vee a_n \xleftarrow{[x,y]} b_1, \ldots, b_m \qquad n \geq 1, \ \ m \geq 0$$

where $a_1, \ldots, a_n, b_1, \ldots, b_m$ are atoms and $0 < x \leq y \leq 1$.

As in the qualitative case, a disjunctive quantitative program $\mathcal{P}$ can have more than one minimal p-model. The non-unicity of the minimal p-model is a consequence of the presence of disjunction in the rules' heads and of the choice of $\mathcal{S}^{(p)}$ (i.e., of the choice of p). While in the qualitative case the number of minimal models of a program is finite, a quantitative program may have an infinite number of minimal p-models because it has infinitely many quantitative interpretations.

Given $p \in \overline{\mathbf{R}}$, a program $\mathcal{P}$, a quantitative interpretation I of $\mathcal{P}$, and a disjunctive rule $r \in \mathcal{P}$ which is not p-satisfied w.r.t. I, a question which arises is which are the minimal intervals that $\Phi_{\mathcal{P}}^{(p)}(I)$ should assign to the head atoms of r in order to p-satisfy the rule r and how many solutions there are. This question reduces to the following problem: given $p \in \overline{\mathbf{R}}$ and $m \in (0,1]$, which are the minimal solutions $\{x, y\}$ of $S^{(p)}(x,y) = m$ and how many they are. A solution $\{x, y\}$ is minimal iff there is no other solution $\{x', y'\}$ such that either $x' < x$ and $y' \leq y$ or $x' \leq x$ and $y' < y$. We split the latter problem in two cases: (i) $m = 1$, and (ii) $m \in (0,1)$.

Lemma 1 *The equation $S^{(p)}(x,y) = 1$ has a finite number of minimal solutions iff $p \in \{-\infty\} \cup \overline{\mathbf{R}^{+}}$.* $\square$

Consider the very simple *boolean*[2] quantitative disjunctive program $\mathcal{P} = \{a \vee b \xleftarrow{[1,1]}\}$. From Lemma 1 it follows that if $p \in \{-\infty\} \cup \overline{\mathbf{R}^{+}}$, then $\mathcal{P}$ has only two minimal p-models, $M_1 = \{a : [0,0], b : [1,1]\}$ and $M_2 = \{a : [1,1], b : [0,0]\}$, whereas for each $p \in (-\infty, 0)$ $\mathcal{P}$ has infinitely many minimal p-models.

We wonder now how many minimal p-models has a *pure* quantitative disjunctive program like $\mathcal{P} = \{a \vee b \xleftarrow{[0.8,0.8]}\}$. In other terms, how many minimal solutions has the equation $S^{(p)}(x,y) = 0.8$? If $S^{(p)} \neq S_3$ and $S^{(p)} \neq S_0$ (i.e., p is finite) we may even have an infinite number of minimal p-models, as shown in the following example.

Example 3 *Consider $\mathcal{P} = \{a \vee b \xleftarrow{[0.6,0.8]}\}$.*

If $S^{(p)} \in \{S_0, S_3\}$, it is easy to see that $\mathcal{P}$ has only two minimal p-models, namely $M_1 = \{a : [0.6, 0.8]\}$ and $M_2 = \{b : [0.6, 0.8]\}$.

If $S^{(p)} \notin \{S_0, S_3\}$, it is easy to see that $\mathcal{P}$ has as many minimal p-models as the number of solutions of the following equations system.

$$\begin{cases} S^{(p)}(x_1, x_2) = 0.6 \\ S^{(p)}(y_1, y_2) = 0.8 \\ x_1 \leq y_1, \quad x_2 \leq y_2 \end{cases}$$

For each solution $s_i = (x_1^i, x_2^i, y_1^i, y_2^i)$ of the equations system, the p-model $M^{(s_i)} = \{a : [x_1^i, y_1^i], b : [x_2^i, y_2^i]\}$ is minimal for $\mathcal{P}$ because the T-conorms are so that for each $p \in \mathbf{R}$[3]*, if $S^{(p)}(x,y) = const$ and x decreases then y increases. In our example, assuming that y_1 and y_2 do not vary, if x_1 decreases (i.e., the interval assigned to a decreases) then x_2 increases (i.e., the interval assigned to b increases) and we get minimal p-models which are not comparable. Since the equations system has an infinite number of solutions, it follows that the program $\mathcal{P}$ has an infinite number of minimal p-models. For instance, if $S^{(p)} = S_2$ we get the following minimal p-models of $\mathcal{P}$:*

$\{a : [0.6, 0.8], b : [0, 0]\}$, $\{a : [0.5, 0.5], b : [0.2, 0.6]\}$, $\{a : [0.5, 0.6], b : [0.2, 0.5]\}$, $\{a : [0.2, 0.5], b : [0.5, 0.6]\}$, $\{a : [0.2, 0.6], b : [0.5, 0.5]\}, \ldots$

$\square$

The following lemma confirms the observations of the previous example.

Lemma 2 *The equation $S^{(p)}(x,y) = m$, where m is an arbitrary value such that $0 < m < 1$, has a finite number of minimal solutions iff $p \in \{-\infty, +\infty\}$.* $\square$

To see the difference between $p = -\infty$ and $p = +\infty$, consider the program $\mathcal{P} = \{a \vee b \xleftarrow{[0.5,0.5]}, a \xleftarrow{[0.2,0.2]}\}$ and the quantitative interpretations $M_1 = \{a : [0.2, 0.2], \ b : [0.5, 0.5]\}$ and $M_2 = \{a : [0.5, 0.5], \ b : [0, 0]\}$. If $p = +\infty$, both M_1 and M_2 are minimal p-models for $\mathcal{P}$, whereas if $p = -\infty$, only M_2 is a minimal p-model for $\mathcal{P}$.

[2] A quantitative program $\mathcal{P}$ is boolean iff the strengths of all rules' implications are assigned $[1, 1]$.

[3] p is finite, hence $S^{(p)} \notin \{S_0, S_3\}$.

A consequence of Lemma 1 is that if $p \in (-\infty, 0)$, we cannot find general algorithms which compute all minimal p-models for all disjunctive quantitative programs whose implications' strengths of the rules are assigned one interval of $[1, 1]$ and $[0, 1]$. Similarly, from Lemma 2 it follows that if $p \in \mathbf{R}$, we cannot find general algorithms which compute all minimal p-models for all disjunctive quantitative programs whose implications' strengths of the rules are assigned only pure quantitative intervals (intervals different from $[0, 0]$, $[1, 1]$ and $[0, 1]$). It turns out that if $p \notin \{-\infty, +\infty\}$, in general, a disjunctive quantitative program may have infinitely many minimal p-models. However, if $p \in \{-\infty, +\infty\}$, from Lemma 1 and 2 it does not turn out that every disjunctive quantitative program has finitely many minimal p-models. We will see that if $p = +\infty$, the disjunctive quantitative programs have always finitely many minimal p-models. The case $p = -\infty$ is a particular case of $p = +\infty$ (see the definition of the T-norms and T-conorms for $p = -\infty$ and $p = +\infty$) and like for $p = +\infty$, the disjunctive programs should have finitely many minimal p-models.

In this work we will consider only $p = +\infty$[4] and the corresponding T-norm T_3 (resp., T-conorm S_3) will be employed for reasoning. The T-norm T_3 generalizes the semantics of van Emden (see Theorem 11) and we will show that it has nice computational properties.

In the sequel we will still use the parameter p (i.e., notations like p-model, p-satisfied, $M^{(p)}(H(r))$, are still used), but remember that the value of p is $+\infty$ and the results could not be valid for other values.

Given a quantitative program $\mathcal{P}$ and a quantitative interpretation I of $\mathcal{P}$, $unsat^{(p)}(\mathcal{P}, I)$ denotes the set of rules in $\mathcal{P}$ which are not p-satisfied w.r.t. I.

A rule $r \in \mathcal{P}$ is in $unsat^{(p)}(\mathcal{P}, I)$ iff $I^{(p)}(H(r)) < \mathcal{T}_2(I^{(p)}(B(r)), [x, y])$, where $H(r)$, $B(r)$ and $[x, y]$ are the head, the body and the strength of the implication of r, respectively.

A rule $r \in \mathcal{P}$, $H(r) \xleftarrow{[x,y]} B(r)$, is p-supported w.r.t. a quantitative interpretation I of $\mathcal{P}$ iff $I^{(p)}(H(r)) = \mathcal{T}_2(I^{(p)}(B(r)), [x, y])$.

If $A \in H(r)$ and $I(A) = I^{(p)}(H(r))$, we say that r p-supports the atom A and the atom A is p-supported by r.

If r p-supports exactly n distinct atoms, we say that r is n-fold p-supported. In the sequel, if r is 1-fold p-supported, we will say shortly that r is p-supported.

If $I^{(p)}(H(r)) = [0, 0]$, we say that r is trivially p-supported.

An atom A is p-superfluous for a rule r w.r.t. a quantitative interpretation I, iff (i) $A \in H(r)$, (ii) r is p-satisfied w.r.t. I, and (iii) the rule obtained from r by deleting A from $H(r)$ is still p-satisfied w.r.t. I.

Note that a n-fold p-supported rule has at least $n - 1$ p-superfluous atoms.

Definition 2 *Let $\mathcal{P}$ be an arbitrary disjunctive quantitative program. A* computation *$\mathcal{C}$ of $\mathcal{P}$ is a sequence $\langle I_0, I_1, \ldots I_n \rangle$ of quantitative interpretations for $\mathcal{P}$ such that the following hold:*

- *(i) $I_0 = \emptyset$ (i.e., I_0 assigns $[0, 0]$ to all atoms from $B_{\mathcal{P}}$);*
- *(ii) $unsat^{(p)}(\mathcal{P}, I_n) = \emptyset$ (i.e., I_n is a p-model of $\mathcal{P}$);*
- *(iii) for each i, $1 \le i \le n$, there is an atom $A' \in B_{\mathcal{P}}$ and a rule $r \in unsat^{(p)}(\mathcal{P}, I_{i-1})$, $H(r) \xleftarrow{[x,y]} B(r)$, with $A' \in H(r)$, such that*
  - *$I_i(A) = I_{i-1}(A)$ for each $A \in B_{\mathcal{P}} \setminus \{A'\}$;*
  - *$I_i(A') = \mathcal{T}_2(I_{i-1}^{(p)}(B(r)), [x, y])$;*

  Note that $r \notin unsat^{(p)}(\mathcal{P}, I_i)$, r is p-supported w.r.t. I_i, r p-supports A', and $I_i(A') > I_j(A')$ for each $j < i$.

The computation $\mathcal{C}$ is said to be partial *if the condition (ii) does not hold. If $\mathcal{C}$ is not partial, we say that $\mathcal{C}$* derives *the p-model I_n.* □

To each computation $\mathcal{C}$ we associate two functions $\mathcal{A}_{\mathcal{C}} : \{1, \ldots, n\} \to B_{\mathcal{P}}$ and $\mathcal{R}_{\mathcal{C}} : \{1, \ldots, n\} \to \mathcal{P}$, where n is the index of the last interpretation in the sequence of $\mathcal{C}$. The former function associates to each index i the atom which is updated in I_i w.r.t. I_{i-1} (i.e., the atom A' such that $I_i(A') > I_{i-1}(A')$). The latter function associates to each index i the rule r which is used at iteration i to update the atom A' in I_i. In the sequel, we will use sometimes for simplicity the expression "the computation $\mathcal{C}$ contains (resp., uses, considers) the rule r" which stays for "there exists an index i such that $r = \mathcal{R}_{\mathcal{C}}(i)$". Similarly, the expression "(the value of) an atom A' is updated in the computation $\mathcal{C}$" stays for "there exists an index i such that $A' = \mathcal{A}_{\mathcal{C}}(i)$".

We denote by $\mathcal{F}_{\mathcal{P}}$ the family of all the computations of $\mathcal{P}$. We will show that each computation of $\mathcal{P}$ is finite

[4] For disjunctive quantitative programs whose implications' strengths of the rules are $[1, 1]$ or $[0, 1]$, it should be possible also for $p \in [0, +\infty)$ to find feasible algorithms which compute all minimal p-models, but since the T-norms and T-conorms would handle only with 0 and 1, the results would be the same as for $p = +\infty$.

and there is a finite number of computations of $\mathcal{P}$ (i.e., the size of $\mathcal{F}_{\mathcal{P}}$ is finite).

Informally, the family $\mathcal{F}_{\mathcal{P}}$ can be represented by a rooted tree, where the root represents the null interpretation and each path P from the root to a leaf represents a computation $\mathcal{C}$ of $\mathcal{P}$, the leaf being a p-model of $\mathcal{P}$. We call such a tree the *choice tree* of $\mathcal{P}$, denoted $\mathcal{CT}$, and we call each node of P, apart from the leaves, representing an interpretation I_k of $\mathcal{C}$ a *choice node*. The choice nodes are the points where a p-unsatisfied rule r and an atom $A' \in H(r)$ have to be chosen. Note that each node represents an interpretation I_k of $\mathcal{C}$ and has a number of children equal to $\sum_{r \in unsat(\mathcal{P}, I_k)} |H(r)|$, where $|H(r)|$ is the number of atoms in the head of r.

We will see that there exists a relationship between minimal p-models of $\mathcal{P}$ and the leaves of the choice tree of $\mathcal{P}$ (i.e., the limits of the computations of $\mathcal{P}$). We will show that the set of minimal p-models of $\mathcal{P}$ is a subset of the leaves of the choice tree. Thus, the choice tree of $\mathcal{P}$ both reduces the number of interpretations that are "candidates" to be minimal p-models, and provides us with a constructive way of computing the minimal p-models.

It is worth noting that the choice tree (i.e., the family $\mathcal{F}_{\mathcal{P}}$) of a disjunctive quantitative program can be calculated by a nondeterministic algorithm like in Figure 3.

Theorem 5 *Let $\mathcal{P}$ be an arbitrary disjunctive quantitative program. Each computation $\mathcal{C}$ of $\mathcal{P}$ is finite.* □

Theorem 6 *Let $\mathcal{P}$ be an arbitrary disjunctive quantitative program. The number of computations of $\mathcal{P}$ (i.e., the size of $\mathcal{F}_{\mathcal{P}}$) is finite.* □

Each computation of a program $\mathcal{P}$ finds a p-model of $\mathcal{P}$. In general, a quantitative program may have infinitely many p-models, because real numbers are involved. Therefore, we cannot expect to find all p-models of $\mathcal{P}$ by deriving all its computations. However, we are not interesting in computing all p-models of a program, but only in computing the minimal p-models. The following theorem shows that for each minimal p-model M there exists a computation which derives M.

Theorem 7 *Given an arbitrary disjunctive quantitative program $\mathcal{P}$ as input, the algorithm of Figure 3 derives a superset of the minimal p-models of $\mathcal{P}$ for $p = +\infty$.* □

The set $S = \{\ last(C) \mid C \in allComp\ \}$ contains in general also p-models which are not minimal. (E.g., on the program $\{a \vee b \xleftarrow{[1,1]}, a \xleftarrow{[1,1]} b\}$, S would contain both models $\{a : [1,1], b : [0,0]\}$ and $\{a : [1,1], b : [1,1]\}$.) However, if a p-model I from S is not minimal, then S must contain another p-model which is strictly smaller than I.

Input: A quantitative program $\mathcal{P}$.
Output: The set of all computations of $\mathcal{P}$ for $p = +\infty$.

Notation:

- $unsat^{(p)}(\mathcal{P}, I)$ denotes the set of rules in $\mathcal{P}$ which are not p-satisfied w.r.t. the quantitative interpretation I.
- $computation$ is a type defined as **list of** $interpretation$.
- $last(\mathcal{C})$ is a function which returns the last interpretation of a (partial) computation $\mathcal{C}$.

algorithm $ComputeComp(\mathcal{P} : program)$
var $allComp,\ F$: **set of** $computation$;
  $\mathcal{C} : computation$;
begin
  $allComp := \emptyset$;
  $F := \{\emptyset\}$;
  while $F \neq \emptyset$ **do**
    $allComp := allComp \cup \{\ \mathcal{C} \in F$ s.t. $unsat^{(p)}(\mathcal{P}, last(\mathcal{C})) = \emptyset;\}$;
    $F := \{\ \mathcal{C} \cup I \mid \mathcal{C} \in F$ and $\exists\ r \in unsat^{(p)}(\mathcal{P}, last(\mathcal{C}))$ and $\exists\ A' \in H(r)$ s.t. $I(A) = last(\mathcal{C})(A)\ \ \forall A \in B_{\mathcal{P}} \setminus \{A'\}$ and $I(A') = \mathcal{T}_2(last(\mathcal{C})^{(p)}(B(r)), [x,y])\ \}$; /* $A' \xleftarrow{[x,y]} B(r)$ */
  endwhile;
  output ($allComp$);
end $ComputeComp$;

Fig. 3. Nondeterministic Algorithm for Computing All Computations of Disjunctive Quantitative Programs

Definition 3 *A computation $\mathcal{C} = \langle I_0, I_1, \ldots, I_n\rangle$ deriving a p-model M (i.e., $I_n = M$) of a program $\mathcal{P}$ is* minimal *iff* $n = |\{A \in M | M(A) > [0,0]\}|$. □

Theorem 8 *Let $\mathcal{P}$ be an arbitrary disjunctive quantitative program. If M is a minimal p-model of $\mathcal{P}$, there exists a minimal computation deriving M.* □

Theorem 8 proves that each minimal p-model M of a disjunctive quantitative program can be derived by a minimal computation $\mathcal{C}$ which ends in a number of iterations equal to the number of atoms whose interval value assigned by M are different from $[0,0]$. In other terms, each $M(A) > [0,0]$ is reached in one step at some iteration k (i.e., $I_k(A) = M(A)$ and there is no iteration i, $i < k$, such that $[0,0] < I_i(A) < M(A)$) and this is possible because the final value $M(A')$ of each atom A' from $B(\mathcal{R}_\mathcal{C}(k))$ has been reached (in one step as well) at some iteration before k.

In general, there are more than one (non-minimal) computations deriving a minimal p-model M. Applying the technique described in Theorem 8, we might obtain different minimal computations deriving the same minimal p-model M. It follows that, in general, given a minimal p-model M, there exist more than one minimal computations deriving M.

Theorem 9 *Every disjunctive quantitative program has finitely many minimal p-models for $p = +\infty$.* □

A consequence of Theorem 7 is that we can find all minimal p-models of a program by finding first the set of all the minimal p-models "candidates" with the algorithm from Figure 3 and by discarding from this set the elements violating the minimality condition. It is clear that for each minimal p-model M, the number of atoms $A \in B_\mathcal{P}$ such that $M(A) > [0,0]$ cannot exceed $min\{|\mathcal{P}|, |\{H(r)|r \in \mathcal{P}\}|\}$[5], where $|\mathcal{P}|$ is the number of rules from $\mathcal{P}$ and $|\{H(r)|r \in \mathcal{P}\}|$ is the number of distinct atoms appearing in the rules' heads, because the value of each A such that $M(A) > [0,0]$ must be derived from a rule whose head contains A. By virtue of Theorem 8, in the algorithm of Figure 3, we can cut the "while" loop at the iteration with number $min\{|\mathcal{P}|, |\{H(r)|r \in \mathcal{P}\}|\}$, without loosing any minimal p-model. The algorithm will compute a *reduced* family $\mathcal{F}'_\mathcal{P}$ of computations which are necessary to derive all minimal p-models of $\mathcal{P}$.

An algorithm which computes all minimal p-models of a disjunctive quantitative program $\mathcal{P}$ is shown in Figure 4. First, the quantitative interpretation I_0 is initialized to the null interpretation. Then the procedure $computeMinimalModels$ is invoked to generate one after another the computations of $\mathcal{P}$ contained by the reduced family $\mathcal{F}'_\mathcal{P}$. This procedure is based on a backtracking technique, and its structure is that of a preorder visit of the choice tree $\mathcal{CT}$ of $\mathcal{P}$. Initially, $computeMinimalModels$ is invoked with the actual parameter $I_0 = \emptyset$, which corresponds to a visit to the root of $\mathcal{CT}$. In general, the procedure is invoked with the actual parameter corresponding to a choice node of $\mathcal{CT}$. Then, if the current depth of $\mathcal{CT}$ does not exceed the optimized limit given by Theorem 8, for each p-unsatisfied rule $r \in unsat^{(p)}(\mathcal{P}, I_k)$ (possibly none), one atom $A' \in H(r)$ at a time is chosen and I_{k+1}, corresponding to the next choice node of $\mathcal{CT}$, is computed. The procedure then recursively invokes itself with actual parameter I_{k+1}. Whenever the condition $unsat^{(p)}(\mathcal{P}, I_k) = \emptyset$ is verified, I_k is a minimal p-model "candidate" and its minimality is checked by invoking the function $isMinimal$ on input I_k. The function $isMinimal(I_k)$ returns *true* iff I_k is a minimal p-model of $\mathcal{P}$ and its complexity, according to the results of [25], is coNP. After $computeMinimalModels$ ends, all minimal p-models have been printed out.

[5] $Head(r)$ is intended as the set of the atoms appearing in the head of r, and not as a disjunction.

Input: A disjunctive quantitative program $\mathcal{P}$.
Output: The set of the minimal p-models of $\mathcal{P}$ for $p = +\infty$.

algorithm $MinimalModels(\mathcal{P} : program)$
var $I_0 : interpretation$;
procedure
$computeMinimalModels(I_k : interpretation)$
var I_{k+1} : $interpretation$;
begin
  if $k \le min\{|\mathcal{P}|, |\{H(r)|r \in \mathcal{P}\}|\}$ **then**
    if $unsat^{(p)}(\mathcal{P}, I_k) \neq \emptyset$ **then**
      for each $r \in unsat^{(p)}(\mathcal{P}, I_k)$, $H(r) \xleftarrow{[x,y]} B(r)$ **do**
        for each $A' \in H(r)$ **do begin**
          $I_{k+1}(A') := \mathcal{T}_2(I_k^{(p)}(B(r)), [x,y])$;
          for each atom $A \in B_\mathcal{P} \setminus \{A'\}$ **do**
            $I_{k+1}(A) := I_k(A)$;
          $computeMinimalModels(I_{k+1})$;
        end
    else /* I_k is a p-model of $\mathcal{P}$ */
      if $isMinimal(I_k)$ **then**
        output(I_k);
end $computeMinimalModels$;
begin
  for each $A \in B_\mathcal{P}$ **do**
    $I_0(A) := [0,0]$;
  $computeMinimalModels(I_0)$;
end $MinimalModels$;

Fig. 4. Deterministic Algorithm for Computing the Minimal p-Models of Disjunctive Quantitative Programs

Theorem 10 *Given an arbitrary disjunctive quantitative program $\mathcal{P}$ as input, the algorithm of Figure 4 terminates in a finite amount of time and returns the minimal p-models of $\mathcal{P}$ for $p = +\infty$.* □

It is easy to see that the algorithm of Figure 4 uses polynomial space and terminates in single exponential time (assuming that the function $isMinimal$ is implemented in a proper way [25]). Since the computation of all minimal models is Σ_2^P even for traditional DLP of which QDLP is a generalization (see the section 5.2), the algorithm respects the complexity bounds of the problem; indeed, unless the polynomial hierarchy collapses to polynomial time, single exponential time is needed to solve this problem.

5. Generalization Results

5.1. Van Emden's Approach

One of the most relevant earlier works in this field was accomplished by van Emden in [37]. There, a quantitative rule r is of the form $A \stackrel{\mathrm{f}}{\longleftarrow} B_1, \ldots, B_n$, where $n \geq 0$, A, $B_1, \ldots, B_n$ are all positive atoms, f is a real number in the interval $(0, 1]$. r is true in a quantitative interpretation I iff $I(A) \geq f \times min\{\ I(B_i) \mid i \in \{1, \ldots, n\}\ \}$.

Theorem 11 *The language proposed by van Emden is a particular case of the p-model semantics, where* $p = +\infty$ *(i.e.,* $T^{(p)} = T_3$*).* $\square$

There are important differences between our approach and that of van Emden. First of all, the programs considered in [37] are without disjunction. Moreover, unlike in our approach, each clause implication receives a scalar and not an interval. Finally, van Emden defines a unique uncertainty calculus, based on the T-norm T_3.

5.2. Traditional Disjunctive Logic Programming

From the syntax point of view, QDLP is an extension of DLP. Each $\mathcal{P}$ in DLP can be transformed in a program $\mathcal{P}'$ in QDLP, called the quantitative version of $\mathcal{P}$, by assigning $[1, 1]$ to the strength of the implication of each rule (fact) of $\mathcal{P}$. Remember that in DLP the implications are strict logical true and the logical value **true** is regarded as $[1, 1]$ in QDLP. Thus, $\mathcal{P}$ is equivalent to $\mathcal{P}'$ from the syntax point of view.

Example 4 *Consider the logic program* $\mathcal{P} = \{a \leftarrow\ ;\ b \leftarrow\ ;\ c \vee d \leftarrow a, b\}$*. The quantitative version of* $\mathcal{P}$ *is* $\mathcal{P}' = \{a \stackrel{[1,1]}{\longleftarrow}\ ;\ b \stackrel{[1,1]}{\longleftarrow}\ ;\ c \vee d \stackrel{[1,1]}{\longleftarrow} a, b\}$. $\square$

We wish to see now whether QDLP is an extension of DLP also from the semantics point of view. We say that a minimal p-semantics of QDLP is a generalization of the minimal model semantics of DLP iff $\mathcal{MM}^{(p)}(\mathcal{P}') = \mathcal{MM}(\mathcal{P})$ for each $\mathcal{P}$ in DLP, where $\mathcal{P}'$ is the quantitative version of $\mathcal{P}$. Given p, a priori, it is not guaranteed that the p-semantics of QDLP generalizes the DLP semantics.

It is highly desirable that QDLP semantics coincides with DLP semantics on boolean quantitative programs. Whether the p-semantics of a given class of boolean quantitative programs coincides with the DLP semantics, depends strongly on the value of p and on the features (e.g., positive, disjunctive,etc.) of the QDLP class. We single out the classes of QDLP and the values of p for which the p-semantics on the boolean quantitative programs of these classes coincides with the DLP semantics.

Proposition 1 *Let* $\mathcal{P}$ *be a (disjunctive) logic program and* $\mathcal{P}'$ *be the quantitative version of* $\mathcal{P}$*. If* $M \in \mathcal{M}^{(p)}(\mathcal{P}')$ *then* $M \in \mathcal{M}(\mathcal{P})$*, whatever is* $p \in \overline{\mathbf{R}}$*.* $\square$

Proposition 2 *Let* $\mathcal{P}$ *be a (disjunctive) logic program and* $\mathcal{P}'$ *be the quantitative version of* $\mathcal{P}$*. If* $M \in \mathcal{M}(\mathcal{P})$ *then* $M \in \mathcal{M}^{(p)}(\mathcal{P}')$*, whatever is* $p \in \overline{\mathbf{R}}$*.* $\square$

5.2.1. Non-Disjunctive Programs

In this section we investigate which classes of non-disjunctive quantitative programs are generalizations of the corresponding logic programs classes.

Proposition 3 *Let* $\mathcal{P}'$ *be an arbitrary non-disjunctive boolean quantitative program. The least p-model of* $\mathcal{P}'$ *is boolean, whatever is* $p \in \overline{\mathbf{R}}$*.* $\square$

Proposition 4 *Let* $\mathcal{P}'$ *be an arbitrary non-disjunctive boolean quantitative program. For each p-model* M *of* $\mathcal{P}'$ *there exists a boolean p-model* M' *of* $\mathcal{P}'$ *such that* $M' \leq M$*, whatever is* $p \in \overline{\mathbf{R}}$*.* $\square$

Theorem 12 *Let* $\mathcal{P}$ *be an arbitrary non-disjunctive logic program and* $\mathcal{P}'$ *be the quantitative version of* $\mathcal{P}$*. The least model of* $\mathcal{P}$ *coincides with the least p-model of* $\mathcal{P}'$*, whatever is* $p \in \overline{\mathbf{R}}$*.* $\square$

The following theorem proves that the minimal p-model semantics of the non-disjunctive quantitative programs is a generalization of the minimal model semantics of the non-disjunctive logic programs, whatever is $p \in \overline{\mathbf{R}}$.

Theorem 13 *Let* $\mathcal{P}$ *be an arbitrary non-disjunctive logic program, and* $\mathcal{P}'$ *its quantitative version. Then* $\mathcal{MM}(\mathcal{P}) = \mathcal{MM}^{(p)}(\mathcal{P}')$ *whatever is* $p \in \overline{\mathbf{R}}$*.* $\square$

5.2.2. Disjunctive Programs

In this section we investigate which classes of disjunctive quantitative programs are generalizations of the corresponding logic programs classes.

Lemma 3 *The assertion "if $S^{(p)}(a,b) = 1$ and $min(a,b) \neq 0$ then $max(a,b) = 1$" is true iff $p \in \overline{\mathbf{R}^+}$.* □

Proposition 5 *The assertion "for each boolean quantitative disjunctive program $\mathcal{P}'$, each minimal p-model M of $\mathcal{P}'$ is boolean" holds iff $p \in \{-\infty\} \cup \overline{\mathbf{R}^+}$.* □

Note that the result of Proposition 5 holds for both HCF and non-HCF cases.

Theorem 14 *Let $\mathcal{P}$ be an arbitrary disjunctive logic program and $\mathcal{P}'$ be the quantitative version of $\mathcal{P}$. Each minimal p-model of $\mathcal{P}'$ is a minimal model of $\mathcal{P}$ iff $p \in \{-\infty\} \cup \overline{\mathbf{R}^+}$.* □

Theorem 14 states that for each disjunctive logic program $\mathcal{P}$ and its quantitative version $\mathcal{P}'$, the relation $\mathcal{MM}(\mathcal{P}) \supseteq \mathcal{MM}^{(p)}(\mathcal{P}')$ holds iff $p \in \{-\infty\} \cup \overline{\mathbf{R}^+}$. Note that the HCF case is not "easier" than the non-HCF case, in the sense that the range $\{-\infty\} \cup \overline{\mathbf{R}^+}$ for p cannot be extended even for HCF disjunctive programs.

The analysis in the opposite direction (i.e., from logic program versus their quantitative versions) considers separately the case of HCF programs from the case of non-HCF programs. We show that we obtain different results for the two cases.

The following theorem shows that for each HCF disjunctive logic program $\mathcal{P}$ and its quantitative version $\mathcal{P}'$, the relation $\mathcal{MM}(\mathcal{P}) \subseteq \mathcal{MM}^{(p)}(\mathcal{P}')$ holds for each $p \in \overline{\mathbf{R}}$.

Theorem 15 *Let $\mathcal{P}$ be an arbitrary disjunctive HCF logic program, and $\mathcal{P}'$ be the quantitative version of $\mathcal{P}$. If $M \in \mathcal{MM}(\mathcal{P})$ then $M \in \mathcal{MM}^{(p)}(\mathcal{P}')$, whatever is $p \in \overline{\mathbf{R}}$.* □

The previous theorem shows how disjunctive HCF programs behave, but what can we say about non-HCF programs? Some non-HCF logic programs have minimal models while their quantitative versions have no minimal p-model for a given $p \in \overline{\mathbf{R}}$. Consider for example the non-HCF logic program $\mathcal{P} = \{a \vee b \leftarrow\ ,\ a \leftarrow b,\ b \leftarrow a\}$ which has a minimal model $M = \{a, b\}$ and note that its quantitative version $\mathcal{P}'$ has no minimal p_0-model, where $p_0 = -\infty$. It can be shown that whatever a p_0-model M of $\mathcal{P}'$ is, there exists always a p_0-model N of $\mathcal{P}'$ such that $N < M$, hence $\mathcal{P}'$ has no minimal p_0-model. For instance, the p_0-model $\{a : [0.1, 0.1], b : [0.1, 0.1]\}$ is not minimal because $\{a : [0.05, 0.05], b : [0.05, 0.05]\}$ is a minor p_0-model.

	$\{\}$	$\{\vee^{\mathbf{h}}\}$	$\{\vee\}$
$p = -\infty$	YES	YES	NO
$p \in (-\infty, 0)$	YES	NO	NO
$p \in [0, +\infty]$	YES	YES	YES

Table 1

QDLP fragments generalizing DLP

Proposition 6 *Let $\mathcal{P}'$ be an arbitrary boolean quantitative disjunctive program. If M is a (non-boolean) p-model of $\mathcal{P}'$ then there exists a boolean p-model M' of $\mathcal{P}'$ such that $M' \leq M$, whatever is $p \in \overline{\mathbf{R}^+}$.* □

Theorem 16 *Let $\mathcal{P}$ be an arbitrary (non-HCF) disjunctive logic program, and $\mathcal{P}'$ be the quantitative version of $\mathcal{P}$. Each minimal model of $\mathcal{P}$ is a minimal p-model of $\mathcal{P}'$ iff $p \in \overline{\mathbf{R}^+}$.* □

Theorem 16 states that for each (non-HCF) disjunctive logic program $\mathcal{P}$ and its quantitative version $\mathcal{P}'$, the relation $\mathcal{MM}(\mathcal{P}) \subseteq \mathcal{MM}^{(p)}(\mathcal{P}')$ holds iff $p \in \overline{\mathbf{R}^+}$.

The following theorem proves that the minimal p-model semantics of the HCF (resp., non-HCF) disjunctive quantitative programs is a generalization of the minimal model semantics of the HCF (resp., non-HCF) disjunctive logic programs iff $p \in \{-\infty\} \cup \overline{\mathbf{R}^+}$ (resp., iff $p \in \overline{\mathbf{R}^+}$).

Theorem 17 *Let $\mathcal{P}$ be an arbitrary HCF (resp., non-HCF) disjunctive quantitative program, and $\mathcal{P}'$ its quantitative version. $\mathcal{MM}(\mathcal{P}) = \mathcal{MM}^{(p)}(\mathcal{P}')$ iff $p \in \{-\infty\} \cup \overline{\mathbf{R}^+}$ (resp., iff $p \in \overline{\mathbf{R}^+}$).* □

The results on generalizations are summarized in Table 1. Each column of the table collects the results for a specific class of programs for the T-norms induced by the values of p on the rows. The symbol $\vee^h$ refers to the head cycle free (HCF) disjunction.[6]

[6]The notion of Head Cycle Free Disjunction [3,4] is extended from traditional DLP to QDLP in a straightforward manner. Its formal definition is given in Appendix A.

A box of the table contains the answer YES if the class of quantitative programs given by the corresponding column header is a generalization for the values of p given by the header of the corresponding row of the table, and NO otherwise.

From the non-disjunctive programs class, every $p \in \overline{\mathbf{R}}$ induces a quantitative extension of DLP.

From the class of disjunctive programs, like in the non-disjunctive case, there are values of p which induce quantitative extensions of DLP, but unlike in the non-disjunctive case, where $p \in \overline{\mathbf{R}}$, p is reduced to $\{-\infty\} \cup [0, +\infty]$ for the HCF case and to $[0, +\infty]$ for the non-HCF case. The generalizations of the HCF and non-HCF programs are not supported by other values of p.

Intuitively, the fact that a fragment QF of QDLP is not a generalization of the corresponding fragment F of DLP is due to (i) the disjunctive rules' heads, and (ii) that some values of p induce T-conorms for which, when applied to a disjunction of atoms, it is not absolutely necessary that the certainty degree interval of all atoms be $[1, 1]$ or $[0, 0]$ in order to derive $[1, 1]$ as certainty degree interval for the disjunction. For these values of p, the quantitative version $\mathcal{P}'$ in QF of a program $\mathcal{P}$ in F has pure quantitative minimal p-models in QDLP which clearly cannot be accepted as minimal models in DLP for $\mathcal{P}$. Only the T-conorms and not also the T-norms corresponding to these values of p were reasons for not obtaining generalizations of DLP.

6. Conclusions

In this work we proposed an approach to define the representation, inference, and control of uncertain information in the framework of the disjunctive logic programming. Logic programming, in general, provides a model for rule-based reasoning in expert systems and our purpose was to define a model for rule-based reasoning with uncertainty. To do this, we extended both syntax and semantics of DLP by considering a multiple-valued logic where the truth value **false** is regarded as the real number 0, **true** as the real number 1, and the concept of truth value is extended to include all real numbers between 0 and 1. We denoted the extended DLP by Quantitative DLP (QDLP).

We singled out the fragments of QDLP which are generalizations of the corresponding fragments of DLP and we proposed algorithms for computing the minimal models of these fragments for each T-norm in the non-disjunctive case and for the T-norm T_3 for the disjunctive case.

Ongoing work concerns QDLP extended by negation and the analysis of the computational complexity. Preliminary results on these topics have been published in [26].

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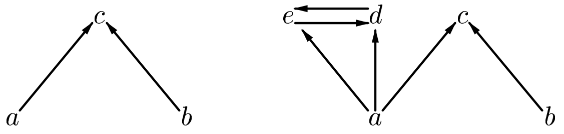

(a) (b)

Fig. 5. Dependency Graph ($DG_{\mathcal{P}}$)

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Appendix

A. Head Cycle Free QDLP

At every program $\mathcal{P}$, we associate a directed graph $DG_{\mathcal{P}} = (N, E)$, called the *dependency graph* of $\mathcal{P}$, in which (i) each predicate of $\mathcal{P}$ is a node in N, and (ii) there is an arc in E directed from a node a to a node b iff there is a rule in $\mathcal{P}$ such that b and a are the predicates of a positive literal appearing in $H(r)$ and $B(r)$, respectively. $DG_{\mathcal{P}}$ singles out the dependencies of the head predicates of a rule r from the positive predicates in its body.

Example 5 *Consider the program $\mathcal{P}_1$ consisting of the following rules:*[7]

$$a \vee b \overset{[0.6,0.6]}{\longleftarrow} . \qquad c \overset{[0.6,0.6]}{\longleftarrow} a. \qquad c \overset{[0.6,0.6]}{\longleftarrow} b.$$

$DG_{\mathcal{P}_1}$ is depicted in Figure 5a. (Note that, since the sample program is propositional, the nodes of the graph are atoms, as atoms coincide with predicates in this case.)

Consider now the program $\mathcal{P}_2$, obtained by adding to $\mathcal{P}_1$ the rules

$$d \vee e \overset{[0.8,0.8]}{\longleftarrow} a. \qquad d \overset{[0.4,0.4]}{\longleftarrow} e. \qquad e \overset{[0.5,0.5]}{\longleftarrow} d.$$

The dependency graph $DG_{\mathcal{P}_2}$ is shown in Figure 5b. □

The HCF quantitative programs are an important class of the quantitative programs with disjunction in the head and are defined in the classical way, as defined in [3,4].

A program $\mathcal{P}$ is HCF iff there is no clause r in $\mathcal{P}$ such that two predicates occurring in the head of r are in the same cycle of $DG_{\mathcal{P}}$. In this work, if a disjunctive program is not explicitly said to be HCF, it is assumed to be non-HCF.

---

[7] We point out that we use propositional programs for simplicity, but the results are valid for the general case of (function-free) programs with variables.

Example 6 *The dependency graphs given in Figure 5 reveal that program $\mathcal{P}_1$ of Example 5 is HCF and that $\mathcal{P}_2$ is not HCF, as rule $d \vee e \stackrel{[0.8,0.8]}{\longleftarrow} a$ contains in its head two predicates belonging to the same cycle of $DG_{\mathcal{P}_2}$.* □

B. Proofs

Theorem 1 *The evaluation of the T-norms and T-conorms at the extremes of the unity interval $[0, 1]$ satisfies the truth tables of the logical operators AND and OR, respectively.*

Proof. Suppose that the real values 0 and 1 correspond to the logical values **false** and **true**, respectively. We have to show that for each $a \in \{0, 1\}$, $b \in \{0, 1\}$ and $p \in \overline{\mathbf{R}}$ the following equalities are verified:

$$T^{(p)}(a, b) = AND(a, b)$$
$$S^{(p)}(a, b) = OR(a, b)$$

in particular, for each $p \in \overline{\mathbf{R}}$

$$T^{(p)}(0, 0) = AND(0, 0) = 0$$
$$T^{(p)}(0, 1) = AND(0, 1) = 0$$
$$T^{(p)}(1, 0) = AND(1, 0) = 0$$
$$T^{(p)}(1, 1) = AND(1, 1) = 1$$

$$S^{(p)}(0, 0) = OR(0, 0) = 0$$
$$S^{(p)}(0, 1) = OR(0, 1) = 1$$
$$S^{(p)}(1, 0) = OR(1, 0) = 1$$
$$S^{(p)}(1, 1) = OR(1, 1) = 1$$

These equalities are direct consequences of the boundary properties of the T-norms and T-conorms. However, the above equality relations result also from the definition of the parametrized family $T(a, b, p)$ (resp., $S(a, b, p)$) by substituting a and b by 0 and 1 in all possible ways and by variating the value of the parameter p in $\overline{\mathbf{R}}$. □

Theorem 2 *Let $\mathcal{P}$ be an arbitrary non-disjunctive quantitative program. The operator $\Phi_{\mathcal{P}}^{(p)}$ converges and computing $lfp(\Phi_{\mathcal{P}}^{(p)})$ is polynomial in the number of rules of $\mathcal{P}$, whatever is $p \in \overline{\mathbf{R}}$.*

Proof. The operator $\Phi_{\mathcal{P}}^{(p)}$ is strictly monotonic and since the T-norms and T-conorms are upper bounded by 1 whatever is $p \in \overline{\mathbf{R}}$, the operator $\Phi_{\mathcal{P}}^{(p)}$ converges whatever is $p \in \overline{\mathbf{R}}$. An upper bound of the least fixpoint $lfp(\Phi_{\mathcal{P}}^{(p)})$ is the quantitative interpretation which assigns $[1, 1]$ to all atoms from $B_{\mathcal{P}}$.

We show that at each iteration i of the fixpoint computation at least one atom $A \in B_{\mathcal{P}}$ is done (i.e., $I^i(A) = lfp(\Phi_{\mathcal{P}}^{(p)})(A)$).

The atoms which are done at each iteration of the fixpoint computation of $\Phi_{\mathcal{P}}^{(p)}$ are inserted progressively in a set S of atoms. Before the computation starts, S is empty. It is easy to see that the following is valid whatever is $p \in \overline{\mathbf{R}}$.

Clearly, at the first iteration all atoms which have no incoming arcs in the dependency graph[8] $DG_{\mathcal{P}}$ are done because they are defined only by facts and their values do not change at the following iterations. All atoms with no incoming arc are inserted in S.

Consider now a generic iteration i, $i > 1$, of the computation:

> Let $\Delta S = \{A \in B_{\mathcal{P}} \setminus S \mid \forall\, (B, A) \in DG_{\mathcal{P}} \Rightarrow B \in S\}$, where $(B, A) \in DG_{\mathcal{P}}$ means that there exists in $DG_{\mathcal{P}}$ an oriented arc from B to A.
> If $\Delta S \neq \emptyset$ then
>
> – Each atom $A \in \Delta S$ is done because the body of each rule defining A contains only atoms which are done.
> – Let $S = S \cup \Delta S$.
>
> Else (each atom of $B_{\mathcal{P}} \setminus S$ is in a cycle in $DG_{\mathcal{P}}$ consisting of only atoms of $B_{\mathcal{P}} \setminus S$)
>
> Let $\mathcal{P}' = \{r \in \mathcal{P} \mid B(r) \subseteq S\}$.
> If $\mathcal{P}' \neq \emptyset$ then
>
> – Let $r \in \mathcal{P}'$, $H(r) \stackrel{[x,y]}{\longleftarrow} B(r)$, such that for each rule $r' \in \mathcal{P}'$, $H(r') \stackrel{[x',y']}{\longleftarrow} B(r')$, the following holds:
>
> $$\mathcal{T}_2((I^{i-1})^{(p)}(B(r)), [x, y]) \geq \mathcal{T}_2((I^{i-1})^{(p)}(B(r')), [x', y'])$$
>
> – The atom $H(r)$ is done because the values that are derived for $H(r)$ along the cycles containing $H(r)$, due to the monotonicity property of the T-norms, cannot exceed the value inferred from r.
> – Let $S = S \cup \{H(r)\}$.
>
> Else (for each atom $A \in B_{\mathcal{P}} \setminus S$, each rule defining A contains at least one atom of $B_{\mathcal{P}} \setminus S$)
>
> – Each atom $A \in B_{\mathcal{P}} \setminus S$ is done and $lfp(\Phi_{\mathcal{P}}^{(p)}(A) = [0, 0]$.
> – The iteration i is the last iteration of the computation (i.e., $\Phi_{\mathcal{P}}^{(p)}$ reaches the fixpoint at iteration $i - 1$).

We have shown that at each iteration, at least one atom is done. Thus, the number of iterations cannot exceed the number of atoms in $B_{\mathcal{P}}$ which is not greater than the number of rules in $\mathcal{P}$ because $\mathcal{P}$ contains no undefined atom.

At each iteration of the computation, at most all rules of $\mathcal{P}$ are considered. The statement of the theorem now follows. □

Theorem 3 *Let $\mathcal{P}$ be an arbitrary non-disjunctive quantitative program. The least fixpoint $lfp(\Phi_{\mathcal{P}}^{(p)})$ is a minimal p-model of $\mathcal{P}$, whatever is $p \in \overline{\mathbf{R}}$.*

[8] See Appendix A for the definition of the dependency graph $DG_{\mathcal{P}}$ of a program $\mathcal{P}$.

Proof. Remember that $\Phi^{(p)}_{\mathcal{P}}$ is a monotonic function and this guarantees the existence of the least fixpoint of $\Phi^{(p)}_{\mathcal{P}}$. The fact that $lfp(\Phi^{(p)}_{\mathcal{P}})$ is a p-model for $\mathcal{P}$ is guaranteed by the definition of $\Phi^{(p)}_{\mathcal{P}}$.

Suppose now that $lfp(\Phi^{(p)}_{\mathcal{P}})$ is not minimal. Then there exists a p-model M for $\mathcal{P}$ such that $M < lfp(\Phi^{(p)}_{\mathcal{P}})$ and let S be the set of atoms $A \in B_{\mathcal{P}}$ such that $M(A) < lfp(\Phi^{(p)}_{\mathcal{P}})(A)$. Note that S cannot be empty.

Let A be an arbitrary atom from S. It is clear that $lfp(\Phi^{(p)}_{\mathcal{P}})(A) > [0,0]$. The body of each rule r_i, $A \xleftarrow{[x_i, y_i]} B(r_i)$, defining A such that

$$lfp(\Phi^{(p)}_{\mathcal{P}})(A) = \mathcal{T}_2(lfp(\Phi^{(p)}_{\mathcal{P}})^{(p)}(B(r_i)), [x_i, y_i])$$

must contain at least one atom of S, otherwise

$$lfp(\Phi^{(p)}_{\mathcal{P}})^{(p)}(B(r_i)) = M^{(p)}(B(r_i))$$

and this is not possible because the rule r_i would not be p-satisfied w.r.t. M. Let $\Delta R(A, S) = \{r \in \mathcal{P} | H(r) = A \text{ and } B(r) \cap S \neq \emptyset\}$ be the set of rules of $\mathcal{P}$ defining A whose bodies contain at least one atom of S and note that $\Delta R(A, S) \neq \emptyset$ for each $A \in S$. Note also that if

$$S' = \{\ A \in S \mid \exists\ A' \in S \text{ and } A \in B(r') \text{ for some } r' \in \Delta R(A', S)\ \},$$

S' cannot be empty and it contains all sets of atoms from S which are mutually recursive and let S'' be one of these sets. Since $lfp(\Phi^{(p)}_{\mathcal{P}})(A) > [0,0]$ for each $A \in S''$, it follows that there exists $A \in S''$ such that A is defined by a rule $r \notin \Delta R(A, S'')$, that is, r is a non-recursive rule, and r is a rule such that, at some iteration i the operator $\Phi^{(p)}_{\mathcal{P}}$ infers from r a value $[m, n]$ for A greater than $[0,0]$. Such a rule must exist because through the inference from the only recursive rules we can at most conserve the greatest of the initial values assigned by I_0 (i.e., $[0,0]$) to the mutually recursive atoms and we cannot increase this value. The value of A cannot decrease w.r.t. $[m, n]$ at any iteration $j > i$ because by definition, $\Phi^{(p)}_{\mathcal{P}}$ assigns to A the greatest value inferred from the rules defining A. Moreover, the value of A could increase w.r.t. $[m, n]$ at some iteration $j > i$ only through the inference from another non-recursive rule of A because, as noticed before, through the inference from the recursive rules, the initial value cannot increase (in this case the initial value is the last value inferred from a non-recursive rule). We obtained that the value $lfp(\Phi^{(p)}_{\mathcal{P}})(A)$ was inferred from a non-recursive rule $r \notin \Delta R(A, S'')$. Note that this does not exclude the existence of a recursive rule $r \in \Delta R(A, S'')$ from which the value $lfp(\Phi^{(p)}_{\mathcal{P}})(A)$ is inferred as well, but we proved that all rules defining A from which $lfp(\Phi^{(p)}_{\mathcal{P}})(A)$ is inferred must be recursive and this is a contradiction. $\square$

Theorem 4 *Every non-disjunctive quantitative program has exactly one minimal p-model, whatever is* $p \in \overline{\mathbf{R}}$.

Proof. According to Theorem 2, $lfp(\Phi^{(p)}_{\mathcal{P}})$ exists for every non-disjunctive quantitative program $\mathcal{P}$ and every $p \in \overline{\mathbf{R}}$. Moreover, $\Phi^{(p)}_{\mathcal{P}}$ has a unique least fixpoint, and according to Theorem 3, $lfp(\Phi^{(p)}_{\mathcal{P}})$ is a minimal p-model of $\mathcal{P}$. It follows that every non-disjunctive quantitative program $\mathcal{P}$ has exactly one minimal p-model (called the least p-model of $\mathcal{P}$), given by the least fixpoint $lfp(\Phi^{(p)}_{\mathcal{P}})$, whatever is $p \in \overline{\mathbf{R}}$. $\square$

Lemma 1 *The equation* $S^{(p)}(x, y) = 1$ *has a finite number of minimal solutions iff* $p \in \{-\infty\} \cup \overline{\mathbf{R}^+}$.

Proof. If $p = -\infty$, $S^{(p)}(x, y) = 1$ becomes (i) $max(x, y) = 1$ under the condition $min(x, y) = 0$, and (ii) $1 = 1$ under the condition $min(x, y) > 0$. From the case (i) we obtain the minimal solutions $\{x = 0, y = 1\}$ and $\{x = 1, y = 0\}$. For the case (ii), each $\{x, y\}$ such that $x > 0$ and $y > 0$ satisfies the equation $S^{(p)}(x, y) = 1$, but none of them is minimal because there exists always $\{x', y'\}$ such that $x' > 0$, $y' > 0$ and $x' < x$ and $y' < y$. It follows that the equation has the only minimal solutions $\{x = 0, y = 1\}$ and $\{x = 1, y = 0\}$.

If $p = +\infty$, $S^{(p)}(x, y) = 1$ becomes $max(x, y) = 1$. It is clear that the equation has the only minimal solutions $\{x = 0, y = 1\}$ and $\{x = 1, y = 0\}$.

If p has a finite value and $p < 0$, $S^{(p)}(x, y) = 1$ becomes (i) $(1-x)^{-p} + (1-y)^{-p} = 1$ if $(1-x)^{-p} + (1-y)^{-p} \geq 1$, and (ii) $1 = 1$ if $(1-x)^{-p} + (1-y)^{-p} \leq 1$. For simplicity, let $u = 1 - x$, $w = 1 - y$, and $q = -p$. It follows that $u, w \in [0,1]$, q is finite and $q > 0$. The equations become (i) $u^q + w^q = 1$ if $u^q + w^q \geq 1$, and (ii) $1 = 1$ if $u^q + w^q \leq 1$. Consider now the particular case $w = fu$, where $0 < f \leq 1$. Note that for each f, $0 < f \leq 1$, $\{u = \frac{1}{(1+f^q)^{\frac{1}{q}}}, w = \frac{f}{(1+f^q)^{\frac{1}{q}}}\}$ is a minimal solution for the case (i), hence there are infinitely many minimal solutions.[9]

If $p = 0$, $S^{(p)}(x, y) = 1$ becomes $(1-x)(1-y) = 0$, hence either x or y must be 1. It is obvious that $\{x = 0, y = 1\}$ and $\{x = 1, y = 0\}$ are the only minimal solutions.

If p has a finite value and $p > 0$, $S^{(p)}(x, y) = 1$ becomes $(1-x)^{-p} + (1-y)^{-p} = +\infty$, hence at least one of x and y must be assigned 1. It is obvious that $\{x = 0, y = 1\}$ and $\{x = 1, y = 0\}$ are the only minimal solutions. $\square$

Lemma 2 *The equation* $S^{(p)}(x, y) = m$*, where* m *is an arbitrary value such that* $0 < m < 1$*, has a finite number of minimal solutions iff* $p \in \{-\infty, +\infty\}$.

Proof. If $p = -\infty$, since $m < 1$, $S^{(p)}(x, y) = m$ becomes $max(x, y) = m$ under the condition $min(x, y) = 0$. It is clear that the equation has the only minimal solutions $\{x = 0, y = m\}$ and $\{x = m, y = 0\}$.

If $p = +\infty$, $S^{(p)}(x, y) = m$ becomes $max(x, y) = m$. It is clear that the equation has the only minimal solutions

[9] For two distinct values f_1 and f_2 of f such that $f_1 < f_2$ we obtain two solutions $\{u_1, w_1\}$ and $\{u_2, w_2\}$ such that $u_1 > u_2$ and $w_1 < w_2$; the two solutions are not comparable, hence they are minimal.

$\{x = 0, y = m\}$ and $\{x = m, y = 0\}$.

If p has a finite value and $p < 0$, since $m < 1$, $S^{(p)}(x, y) = m$ becomes $(1-x)^{-p} + (1-y)^{-p} = 1+(1-m)^{-p}$ under the condition $(1-x)^{-p}+(1-y)^{-p} \geq 1$. For simplicity, let $u = 1-x$, $w = 1-y$, and $q = -p$. It follows that $u, w \in [0, 1]$, q is finite and $q > 0$. The equation becomes $u^q + w^q = 1 + (1-m)^q$ under the condition $u^q + w^q \geq 1$. Consider now the particular case $w = fu$, where $0 < f \leq 1$. We obtain $u = (\frac{1+(1-m)^q}{1+f^q})^{\frac{1}{q}}$ under the condition $u \geq \frac{1}{(1+f^q)^{\frac{1}{q}}}$ which is clearly satisfied. Note that $\{u = (\frac{1+(1-m)^q}{1+f^q})^{\frac{1}{q}}, w = (\frac{1+(1-m)^q}{1+\frac{1}{f^q}})^{\frac{1}{q}}\}$ is a minimal solution for each $1-m \leq f \leq 1$, hence there are infinitely many minimal solutions.[10]

If $p = 0$, $S^{(p)}(x, y) = m$ becomes $x + y - xy = m$. Consider now the particular case $y = 1 - f(1-x)$, where $0 < f \leq 1$. We obtain $x = 1 - \sqrt{\frac{1-m}{f}}$. Note that $\{x = 1 - \sqrt{\frac{1-m}{f}}, y = 1 - \sqrt{f(1-m)}\}$ is a minimal solution for each $1 - m \leq f \leq 1$, hence there are infinitely many minimal solutions.

If p has a finite value and $p > 0$, $S^{(p)}(x, y) = m$ becomes $(1-x)^{-p} + (1-y)^{-p} = 1 + (1-m)^{-p}$. For simplicity, let $u = 1-x$ and $w = 1-y$. It follows that $u, w \in [0, 1]$ and the equation becomes $\frac{1}{u^p} + \frac{1}{w^p} = 1 + \frac{1}{(1-m)^p}$. Consider now the particular case $w = fu$, where $0 < f \leq 1$. We obtain $u = (\frac{1+\frac{1}{f^p}}{1+\frac{1}{(1-m)^p}})^{\frac{1}{p}}$. Note that $\{u = (\frac{1+\frac{1}{f^p}}{1+\frac{1}{(1-m)^p}})^{\frac{1}{p}}, w = (\frac{1+f^p}{1+\frac{1}{(1-m)^p}})^{\frac{1}{p}}\}$ is a minimal solution for each $1-m \leq f \leq 1$, hence there are infinitely many minimal solutions. □

Theorem 5 *Let $\mathcal{P}$ be an arbitrary disjunctive quantitative program. Each computation $\mathcal{C}$ of $\mathcal{P}$ is finite.*

Proof. Let $\mathcal{C} = \langle I_0, I_1, \ldots, I_n\rangle$ be an arbitrary computation of $\mathcal{P}$ and suppose that $\mathcal{C}$ does not finish in a finite number of iterations (i.e., $n \to +\infty$). The number of rules from $\mathcal{P}$ is finite and each rule contains a finite number of atoms and, since $\mathcal{P}$ is a function-free program, it follows that the Herbrand base $B_{\mathcal{P}}$ is finite. If $\mathcal{C}$ is not finite, then there must exist at least one atom which is updated by infinitely many interpretations in $\mathcal{C}$ and let $S \subseteq B_{\mathcal{P}}$ be the maximal set of atoms with this property. Note that S cannot be empty and its size is finite.

Let A' be an arbitrary atom from S. It follows that there exists an infinite sequence of indexes[11] $\{i_n\}_{n\in\mathcal{N}} = \langle i_1, i_2, \ldots\rangle$ such that $\mathcal{A}_{\mathcal{C}}(i_1) = \mathcal{A}_{\mathcal{C}}(i_2) = \ldots = A'$. Since A' is defined by finitely many rules, there must exist at least one rule r' defining A' such that there exists an infinite subsequence[12] $\{i'_n\}_{n\in\mathcal{N}} = \langle i'_1, i'_2, \ldots\rangle$ of $\{i_n\}_{n\in\mathcal{N}}$, and both $\mathcal{A}_{\mathcal{C}}(i'_1) = \mathcal{A}_{\mathcal{C}}(i'_2) = \ldots = A'$ and $\mathcal{R}_{\mathcal{C}}(i'_1) = \mathcal{R}_{\mathcal{C}}(i'_2) = \ldots = r'$ hold. The latter relation can hold iff the value inferred from the body of r' increases infinitely many times (i.e., r' becomes infinitely many times p-unsatisfied and can be chosen at iterations i'_1, i'_2, ...) and this is possible iff at least one atom of $B(r')$ is updated by infinitely many interpretations in $\mathcal{C}$ between two iterations given by two consecutive indexes i'_k and i'_{k+1} of the sequence $\{i'_n\}_{n\in\mathcal{N}}$. We obtained that $B(r')$ contains at least one atom from S.

Since A' was an arbitrary atom from S, it follows that there exists a (finite) sequence of atoms $A_1, A_2, \ldots, A_\tau$ from S, a (finite) sequence of rules $r_1, r_2, \ldots, r_\tau$ from $\mathcal{P}$, and an infinite sequence of indexes $\{i_n\}_{n\in\mathcal{N}} = \langle i_1, i_2, \ldots\rangle$ such that

(i) $A_k \in B(r_{1+k \bmod \tau}) \quad k = 1, 2, \ldots, \tau$

(ii) $\begin{cases} \mathcal{A}_{\mathcal{C}}(i_k) = A_{1+(k-1) \bmod \tau} \\ \mathcal{R}_{\mathcal{C}}(i_k) = r_{1+(k-1) \bmod \tau} \end{cases} \quad k = 1, 2, \ldots$

(iii) at iteration i_{k+1}, $k = 1, 2, \ldots, \tau$, the rule $r_{1+k \bmod \tau}$ was chosen because the atom $A_{1+(k-1) \bmod \tau} \in B(r_{1+k \bmod \tau})$ was updated at iteration i_k.

We show that a contradiction arises. Note first that if a rule r, $H(r) \xleftarrow{[x,y]} B(r)$, is such that $I^{(p)}(H(r)) = \mathcal{T}_2(I^{(p)}(B(r)), [x, y])$, where I is a quantitative interpretation, then $I(C) \geq I(A)$, whatever are $A \in H(r)$ and $C \in B(r)$, as a consequence of the monotonicity property of the T-norms and T-conorms. It is then immediate that the following relation holds:

$$I_{i_1}(A_\tau) \geq I_{i_1}(A_1) = I_{i_2}(A_1) \geq I_{i_2}(A_2) = I_{i_3}(A_2) \geq \ldots \geq I_{i_\tau}(A_\tau) = I_{i_{\tau+1}}(A_\tau) \geq \ldots$$

We obtained that $I_{i_1}(A_\tau) \geq I_{i_\tau}(A_\tau)$ and this is not possible because $i_1 < i_\tau$ and A_τ was updated at iteration i_1 and i_τ (see Definition 2). □

Theorem 6 *Let $\mathcal{P}$ be an arbitrary disjunctive quantitative program. The number of computations of $\mathcal{P}$ (i.e., the size of $\mathcal{F}_{\mathcal{P}}$) is finite.*

Proof. By virtue of Theorem 5, each computation of $\mathcal{P}$ is finite and this implies that the depth of the choice tree of $\mathcal{P}$ is finite. The number of children of each choice node of $\mathcal{CT}$ is limited by $\sum_{r\in\mathcal{P}} |H(r)|$, which is clearly finite because $\mathcal{P}$ has a finite number of rules and the head of each rule has a finite number of atoms. It follows that the breadth of $\mathcal{CT}$ is finite as well, hence the size of $\mathcal{F}_{\mathcal{P}}$ is finite. □

Theorem 7 *Given an arbitrary disjunctive quantitative program $\mathcal{P}$ as input, the algorithm of Figure 3 derives a superset of the minimal p-models of $\mathcal{P}$ for $p = +\infty$.*

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[10] For two distinct values f_1 and f_2 of f such that $f_1 < f_2$ we obtain two solutions $\{u_1, w_1\}$ and $\{u_2, w_2\}$ such that $u_1 > u_2$ and $w_1 < w_2$; the two solutions are not comparable, hence they are minimal.

[11] A sequence of indexes $\{i_n\}_{n\in\mathcal{N}}$ is a sequence of positive integers $\langle i_1, i_2, \ldots, i_k\rangle$ such that $i_u < i_w$ for each u, w such that $1 \leq u < w \leq k$.

[12] A sequence of indexes $\{j_n\}_{n\in\mathcal{N}} = \langle j_1, \ldots, j_k\rangle$ is a subsequence of another sequence of indexes $\{i_n\}_{n\in\mathcal{N}} = \langle i_1, \ldots, i_l\rangle$ iff $j_u \in \{i_n\}_{n\in\mathcal{N}}$ for each $j_u \in \{j_n\}_{n\in\mathcal{N}}$.

Proof. Every $last(C)$, where C is an element from $allComp$, is a p-model of $\mathcal{P}$ as all rules from $\mathcal{P}$ are p-satisfied w.r.t. $last(C)$ ($unsat^{(p)}(\mathcal{P}, last(C))$ is empty).

At the end of the "while" loop, the set $\{ \ last(C) \mid C \in allComp \ \}$ contains all minimal p-models "candidates", because any minimal p-model must be constructible by a computational sequence where at each step the interval value of an atom is incremented only if the new value is derivable from a rule which is not p-satisfied. □

Theorem 8 *Let $\mathcal{P}$ be an arbitrary disjunctive quantitative program. If M is a minimal p-model of $\mathcal{P}$, there exists a minimal computation deriving M.*

Proof. According to Theorem 7, there exists a (finite) computation $\mathcal{C} = \langle I_0, I_1, \ldots, I_n \rangle$ of $\mathcal{P}$ which derives M. Suppose that $n > |\{A \in M | M(A) > [0,0]\}|$. Then, there exists at least one atom whose interval value is updated more than once in the computation $\mathcal{C}$ and let k, $1 \leq k \leq n$, be the highest index such that the atom $\mathcal{A}_{\mathcal{C}}(k)$ has this property. That is, the following both conditions are satisfied:

(i) $I_i(\mathcal{A}_{\mathcal{C}}(j)) = [0,0] \quad \forall\, i, j$ s.t. $i \leq k < j$, and
(ii) $\exists\, m < k$ s.t. $\mathcal{A}_{\mathcal{C}}(m) = \mathcal{A}_{\mathcal{C}}(k) = A'$ and $\mathcal{A}_{\mathcal{C}}(t) \neq A' \quad \forall\, t$ s.t. $m < t < k$.

We show that if M is minimal, then we can obtain a computation of $\mathcal{P}$ which derives M, by reconstructing $\mathcal{C}$ and skipping the iteration m and eventually, some other iterations between m and k. We show that either the iteration m has no effect on the iterations following it (i.e., the choices of the rules $\mathcal{R}_{\mathcal{C}}(i)$, $i > m$, and the values inferred from their bodies do not depend on $I_m(A')$) , or some iterations between only m and k depend on the iteration m but we can skip them without to change the result.

Firstly, suppose that there exists a subsequence of indexes $\langle i_1 = m, i_2, i_3, \ldots, i_u \rangle$ of $\langle m, m+1, \ldots, k-1 \rangle$ such that $\mathcal{A}_{\mathcal{C}}(i_w) \in B(\mathcal{R}_{\mathcal{C}}(i_{w+1}))$ for each w such that $1 \leq w < u$.[13] If we skip the iteration m, it might happen that at some iterations from $\langle i_1 = m, i_2, i_3, \ldots, i_u \rangle$, some of the rules $\mathcal{R}_{\mathcal{C}}(i_2), \ldots, \mathcal{R}_{\mathcal{C}}(i_u)$ are not chosen any more or the values inferred from their bodies for $\mathcal{A}_{\mathcal{C}}(i_2), \ldots, \mathcal{A}_{\mathcal{C}}(i_u)$ are lower than $I_{i_2}(\mathcal{A}_{\mathcal{C}}(i_2)), \ldots,$
$I_{i_u}(\mathcal{A}_{\mathcal{C}}(i_u))$. In both cases, there must be other ways, which do not depend on A', to infer the values $M(\mathcal{A}_{\mathcal{C}}(i_2))$, ..., $M(\mathcal{A}_{\mathcal{C}}(i_u))$ and this must happen at iterations lower than k[14], otherwise M is not minimal because with the reconstructed computation we would derive a p-model which is less than M. In both cases, we can skip these iterations without changing the result and the number of the skipped iterations cannot be greater than the number of the atoms which are updated more than once.

Secondly, note that $I_m(A') < I_k(A')$, hence $A' \notin B(\mathcal{R}_{\mathcal{C}}(k))$ and $I_m(A')$ does not influence the value of any atom from $B(\mathcal{R}_{\mathcal{C}}(k))$.[15] Moreover, the rules $\mathcal{R}_{\mathcal{C}}(i)$ such that $i > k$ are influenced by $I_m(A')$ neither directly, because the value of A' was updated at iteration k by $I_k(A')$, nor indirectly, by means of atoms depending on A' which are updated at iterations between m and k, because these atoms must reach their values from M by other ways, not depending on A', before the iteration k (otherwise M is not minimal).

Note also that all rules which might become p-unsatisfied as a consequence of skipping iteration m are p-satisfied after iteration k and we can ignore them.[16]

We proved that we can reduce the computation $\mathcal{C}$ to another computation which derives M, by skipping at least one iteration but not more than the number of atoms which are updated more than once. Applying iteratively this technique as long as the reduced computation has a number of iterations greater than $|\{A \in M | M(A) > [0,0]\}|$, we reach a computation which derives M and has exactly $|\{A \in M | M(A) > [0,0]\}|$ iterations. □

Theorem 9 *Every disjunctive quantitative program has finitely many minimal p-models for $p = +\infty$.*

Proof. The statement follows from Theorems 6 and 8. □

Theorem 10 *Given an arbitrary disjunctive quantitative program $\mathcal{P}$ as input, the algorithm of Figure 4 terminates in a finite amount of time and returns the minimal p-models of $\mathcal{P}$ for $p = +\infty$.*

Proof. We have observed that the algorithm essentially generates the reduced family $\mathcal{F}'_{\mathcal{P}}$ of the computations of $\mathcal{P}$. Thus, Theorems 5 and 6 guarantee that the algorithm terminates in a finite amount of time; the minimal p-model checking performed by $isMinimal$ terminates in a finite amount of time as well, because $isMinimal$ is coNP, according to the results of [25]. Theorems 7 and 8 guarantee both the soundness and the completeness of the algorithm. □

Theorem 11 *The language proposed by van Emden is a particular case of the p-model semantics, where $p = +\infty$ (i.e., $T^{(p)} = T_3$).*

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[13] In other terms, there exists a path $P = \langle A', A_1, \ldots, A_j \rangle$ in the dependency graph of $\mathcal{P}$ such that $\mathcal{A}_{\mathcal{C}}(i_1) = A_1, \ldots, \mathcal{A}_{\mathcal{C}}(i_j) = A_j$ for some subsequence $\langle i_1 = m, i_2, i_3, \ldots, i_u \rangle$ of $\langle m, m+1, \ldots, k-1 \rangle$.

[14] This must happen at iterations lower than k because of the condition (i).

[15] In other terms, there is no path $P = \langle A', A_1, \ldots, A_j \rangle$ in the dependency graph of $\mathcal{P}$ such that $\mathcal{A}_{\mathcal{C}}(i_1) = A_1, \ldots, \mathcal{A}_{\mathcal{C}}(i_j) = A_j$ for some subsequence $\langle i_1, \ldots, i_j \rangle$ of $\langle m+1, \ldots, k-1 \rangle$, whatever is $A_j \in B(\mathcal{R}_{\mathcal{C}}(k))$.

[16] Some rules containing A' in the head might become p-unsatisfied because instead of $I_m(A')$ we must consider the previous value of A' which is lower than $I_m(A')$, but once we updated the value of A' at iteration k they become p-satisfied again because $I_k(A') > I_m(A')$.

Proof. The proof is straightforward. □

Proposition 1 *Let $\mathcal{P}$ be a (disjunctive) logic program and $\mathcal{P}'$ be the quantitative version of $\mathcal{P}$. If $M \in \mathcal{M}^{(p)}(\mathcal{P}')$ then $M \in \mathcal{M}(\mathcal{P})$, whatever is $p \in \overline{\mathbf{R}}$.*

Proof. Suppose that M is not a model for $\mathcal{P}$. Then there exists a rule $r \in \mathcal{P}$,

$$h_1 \vee \ldots \vee h_n \leftarrow b_1, \ldots, b_m \qquad n \geq 1, \;\; m \geq 0$$

which is not satisfied w.r.t. M, that is, (i) the head $H(r)$ of r is false w.r.t. M (i.e., $h_i \notin M$, $1 \leq i \leq n$), and (ii) the body $B(r)$ of r is true w.r.t. M (i.e., $b_j \in M$, $1 \leq j \leq m$). It follows that $M(h_i) = [0,0]$, $1 \leq i \leq n$, $M(b_j) = [1,1]$, $1 \leq j \leq m$. If r' is the rule in $\mathcal{P}'$ corresponding to r, then the following relation is satisfied for each $p \in \overline{\mathbf{R}}$:

$$M^{(p)}(H(r')) < \mathcal{T}_2(M^{(p)}(B(r')), [1,1])$$

It follows that r' is not p-satisfied w.r.t. M, hence M is not a p-model for $\mathcal{P}'$ and this is a contradiction. □

Proposition 2 *Let $\mathcal{P}$ be a (disjunctive) logic program and $\mathcal{P}'$ be the quantitative version of $\mathcal{P}$. If $M \in \mathcal{M}(\mathcal{P})$ then $M \in \mathcal{M}^{(p)}(\mathcal{P}')$, whatever is $p \in \overline{\mathbf{R}}$.*

Proof. Suppose that M is not a p-model for $\mathcal{P}'$. Then there exists a rule $r' \in \mathcal{P}'$,

$$h_1 \vee \ldots \vee h_n \xleftarrow{[1,1]} b_1, \ldots, b_m \qquad n \geq 1, \;\; m \geq 0$$

which is not p-satisfied w.r.t. M, that is, the following relation is satisfied:

$$M^{(p)}(H(r')) < \mathcal{T}_2(M^{(p)}(B(r')), [1,1]) \qquad (2)$$

Since for each atom $A \in B_{\mathcal{P}}$, $M(A)$ can only be $[0,0]$ or $[1,1]$, the values of $M^{(p)}(H(r'))$ and $M^{(p)}(B(r'))$ do not depend on p according to the boundary properties of the T-norms and T-conorms. Moreover, the relation 2 is satisfied only if $M^{(p)}(H(r')) = [0,0]$ and $M^{(p)}(B(r')) = [1,1]$ and this is possible iff $M(h_i) = [0,0]$, $1 \leq i \leq n$ and $M(b_j) = [1,1]$, $1 \leq j \leq m$. It follows that $h_i \notin M$, $1 \leq i \leq n$, $b_j \in M$, and $1 \leq j \leq m$, hence the rule $r \in \mathcal{P}$ corresponding to r' is not satisfied w.r.t. M, hence M is not a model for $\mathcal{P}$ and this is a contradiction. □

Proposition 3 *Let $\mathcal{P}'$ be an arbitrary non-disjunctive boolean quantitative program. The least p-model of $\mathcal{P}'$ is boolean, whatever is $p \in \overline{\mathbf{R}}$.*

Proof. Since $\mathcal{P}'$ is a non-disjunctive quantitative program, it has a unique minimal p-model M which is the least fixpoint $lfp(\Phi^{(p)}_{\mathcal{P}'})$, whatever is $p \in \overline{\mathbf{R}}$ (see Theorem 3). From the definition of the operator $\Phi^{(p)}_{\mathcal{P}}$ for non-disjunctive quantitative programs and since $\mathcal{P}'$ is a boolean quantitative program (the implication strength of each rule is assigned $[1,1]$), it follows that $lfp(\Phi^{(p)}_{\mathcal{P}'})(A)$ can only be $[0,0]$ or $[1,1]$ for each $A \in B_{\mathcal{P}'}$ and each $p \in \overline{\mathbf{R}}$. □

Proposition 4 *Let $\mathcal{P}'$ be an arbitrary non-disjunctive boolean quantitative program. For each p-model M of $\mathcal{P}'$ there exists a boolean p-model M' of $\mathcal{P}'$ such that $M' \leq M$, whatever is $p \in \overline{\mathbf{R}}$.*

Proof. Let S be the set of atoms A in $B_{\mathcal{P}'}$ such that $M(A) \notin \{[0,0],[1,1]\}$ and $p \in \overline{\mathbf{R}}$ be an arbitrary real value. We show that the boolean quantitative interpretation M' defined like $M'(A) = [0,0]$ for each $A \in S$ and $M'(A) = M(A)$ for each $A \in B_{\mathcal{P}'} \setminus S$ is a p-model for $\mathcal{P}'$. We consider an arbitrary atom $A \in S$ and an arbitrary rule r defining A and we show that r is p-satisfied w.r.t. M'; it is clear that the other rules whose head does not contain any atom from S are p-satisfied w.r.t. M'. There are two cases: (i) the body of r contains at least one atom from S, or (ii) the body of r contains no atom from S but M assigns $[0,0]$ to at least one of its atoms (otherwise r is not p-satisfied w.r.t. M). In both cases r is p-satisfied w.r.t. M'. Note that $M = M'$ when M is boolean and the value of p does not influence these results. □

Theorem 12 *Let $\mathcal{P}$ be an arbitrary non-disjunctive logic program and $\mathcal{P}'$ be the quantitative version of $\mathcal{P}$. The least model of $\mathcal{P}$ coincides with the least p-model of $\mathcal{P}'$, whatever is $p \in \overline{\mathbf{R}}$.*

Proof. Let M be the least model of $\mathcal{P}$ and $p \in \overline{\mathbf{R}}$ be an arbitrary real value. From Proposition 2 it follows that M is a p-model for $\mathcal{P}'$.

Suppose now that M is not the least p-model for $\mathcal{P}'$, that is, there exists a p-model N for $\mathcal{P}'$ such that $N < M$. According to Proposition 4, there exists a boolean p-model N' for $\mathcal{P}'$ (obtained from N) such that $N' \leq N$. From Proposition 1 it follows that N' is a model for $\mathcal{P}$. Since $N' \subset M$ it follows that M is not the least model for $\mathcal{P}$ and this is a contradiction. □

Theorem 13 *Let $\mathcal{P}$ be an arbitrary non-disjunctive logic program, and $\mathcal{P}'$ its quantitative version. $\mathcal{MM}(\mathcal{P}) = \mathcal{MM}^{(p)}(\mathcal{P}')$, whatever is $p \in \overline{\mathbf{R}}$.*

Proof. Each non-disjunctive logic program $\mathcal{P}$ has a unique minimal model, called the least model of $\mathcal{P}$, which is given by the fixpoint of the operator $T_{\mathcal{P}}$.

For each $p \in \overline{\mathbf{R}}$, each non-disjunctive quantitative program $\mathcal{P}'$ has a unique minimal p-model, called the least p-model of $\mathcal{P}'$, which is given by the fixpoint of the operator $\Phi^{(p)}_{\mathcal{P}'}$.

The result of the theorem follows from the above mentioned observations and from Theorem 12. □

Lemma 3 *The assertion "if $S^{(p)}(a,b) = 1$ and $min(a,b) \neq 0$ then $max(a,b) = 1$" is true iff $p \in \overline{\mathbf{R}^{+}}$.*

Proof. Recall that

$$S(a,b,p)=\begin{cases}1-((1-a)^{-p}+(1-b)^{-p}-1)^{-\frac{1}{p}}, \\ \quad \text{if } (1-a)^{-p}+(1-b)^{-p}\geq 1 \text{ when } p<0 \\ 1, \text{ if } (1-a)^{-p}+(1-b)^{-p}\leq 1 \text{ when } p<0 \\ lim_{p\to 0}S(a,b,p)=a+b-ab, \text{ when } p\to 0 \\ 1-((1-a)^{-p}+(1-b)^{-p}-1)^{-\frac{1}{p}}, \\ \quad \text{when } p>0\end{cases}$$

First, note that if $min(a,b)=0$ and $S^{(p)}(a,b)=1$ then $max(a,b)=1$, whatever is $p\in\overline{\mathbf{R}}$ (see the boundary property of the T-conorms).

Consider now that $min(a,b)\neq 0$. There are the following cases:

- $p=+\infty$
  $S^{(p)}(a,b)=1 \Leftrightarrow max(a,b)=1\Leftrightarrow a=1$ or $b=1$.
- $p>0$
  $S^{(p)}(a,b)=1\Leftrightarrow 1-((1-a)^{-p}+(1-b)^{-p}-1)^{-\frac{1}{p}}=1\Leftrightarrow((1-a)^{-p}+(1-b)^{-p}-1)^{-\frac{1}{p}}=0 \Leftrightarrow \frac{1}{(\frac{1}{(1-a)^p}+\frac{1}{(1-b)^p}-1)^{\frac{1}{p}}}=0\Leftrightarrow(\frac{1}{(1-a)^p}+\frac{1}{(1-b)^p}-1)^{\frac{1}{p}}\to\infty\Leftrightarrow(\frac{1}{(1-a)^p}+\frac{1}{(1-b)^p}-1)\to\infty\Leftrightarrow \frac{1}{(1-a)^p}\to\infty$ or $\frac{1}{(1-b)^p}\to\infty\Leftrightarrow a=1$ or $b=1$.
- $p=0$
  $S^{(p)}(a,b)=1\Leftrightarrow a+b-ab=1\Leftrightarrow(1-a)(1-b)=0 \Leftrightarrow a=1$ or $b=1$.
- $p<0$
  (i) $(1-a)^{-p}+(1-b)^{-p}\leq 1$
  $S^{(p)}(a,b)=1\Leftrightarrow 1=1\Rightarrow$ it is not absolutely necessary that $a=1$ or $b=1$.
  (ii) $(1-a)^{-p}+(1-b)^{-p}\geq 1$
  Let $p=-q,\ q>0$.
  $S^{(p)}(a,b)=1\Leftrightarrow 1-((1-a)^q+(1-b)^q-1)^{\frac{1}{q}}=1 \Leftrightarrow((1-a)^q+(1-b)^q-1)^{\frac{1}{q}}=0\Leftrightarrow(1-a)^q+(1-b)^q=1\Rightarrow$ it is not absolutely necessary that $a=1$ or $b=1$.

  For each $p<0$, we can find $a<1$ and $b<1$ such that $S^{(p)}(a,b)=1$.
- $p=-\infty$
  $S^{(p)}(a,b)=1$ for each $a,b\in(0,1]$, hence it is not necessary that $a=1$ or $b=1$. □

Proposition 5 *The assertion "for each disjunctive boolean quantitative program $\mathcal{P}'$, each minimal p-model M of $\mathcal{P}'$ is boolean" holds iff* $p\in\{-\infty\}\cup\overline{\mathbf{R}^+}$.

Proof. *If direction ($\Rightarrow$):* consider the assertion true and prove that $p\in\{-\infty\}\cup\overline{\mathbf{R}^+}$. Suppose that the assertion is true and $p\notin\{-\infty\}\cup\overline{\mathbf{R}^+}$, i.e., $p<0$ and p is finite. We show that the assertion is false, that is, there exists a disjunctive boolean quantitative program $\mathcal{P}'$ such that one of the minimal p-models of $\mathcal{P}'$ does not assign only $[0,0]$ or $[1,1]$ to all atoms of $B_{\mathcal{P}'}$ and this is a contradiction.

(i) For the HCF[17] case, consider the program $\mathcal{P}'=\{a\vee b\xleftarrow{[1,1]}c\}$. It is easy to see that if $p\in(-\infty,0)$ then a quantitative interpretation $M=\{a:[x_1,y_1],b:[x_2,y_2],c:[x_3,y_3]\}$ is a minimal p-model for $\mathcal{P}'$ iff $[x_3,y_3]=[1,1]$ and $\mathcal{S}^{(p)}([x_1,y_1],[x_2,y_2])=[1,1]$. From Lemma 3 it follows that the latter equation has solutions such that $[0,0]<[x_1,y_1]<[1,1]$ and $[0,0]<[x_2,y_2]<[1,1]$ for each $p<0$ such that p is finite.

(ii) For the non-HCF case, consider the program $\mathcal{P}'=\{a\vee b\xleftarrow{[1,1]}c,\ a\xleftarrow{[1,1]}b,\ b\xleftarrow{[1,1]}a\}$ and note that a quantitative interpretation $M=\{a:[x_1,y_1],b:[x_2,y_2],c:[x_3,y_3]\}$ is a minimal p-model for $\mathcal{P}'$ iff $[x_3,y_3]=[1,1]$, $[x_1,y_1]=[x_2,y_2]$ and $\mathcal{S}^{(p)}([x_1,y_1],[x_2,y_2])=[1,1]$. Note also that the latter equation has solutions such that $[0,0]<[x_1,y_1]=[x_2,y_2]<[1,1]$ for each $p<0$ such that p is finite.

Only if direction ($\Leftarrow$): consider $p\in\{-\infty\}\cup\overline{\mathbf{R}^+}$ and prove that the assertion is true. Suppose that $p\in\{-\infty\}\cup\overline{\mathbf{R}^+}$ and the assertion is false, i.e., there exists a disjunctive[18] boolean quantitative program $\mathcal{P}'$ which has a non-boolean minimal p-model M. Then there exists a non-empty set $S\subseteq B_{\mathcal{P}'}$ such that $[0,0]<M(A)<[1,1]$ for each $A\in S$ and $M(A)\in\{[0,0],[1,1]\}$ for each $A\in B_{\mathcal{P}'}\setminus S$.

We show that the quantitative interpretation M' defined like $M'(A)=M(A)$ for each $A\in B_{\mathcal{P}'}\setminus S$ and $M'(A)=[u,w]$ for each $A\in S$, where $[u,w]$ is any desired interval such that $[0,0]<[u,w]<min\{M(A)|A\in S\}$[19], is a p-model[20] for $\mathcal{P}'$ which means that M is not a minimal p-model for $\mathcal{P}'$ because $M'<M$ and this is a contradiction.

Since M' changes only the values of the atoms in S, it is clear that each rule r of $\mathcal{P}'$ whose head does not involve any atom of S is p-satisfied w.r.t. M' because $M^{(p)}(H(r))=M'^{(p)}(H(r))$ and $M^{(p)}(B(r))\geq M'^{(p)}(B(r))$ ($M>M'$ and the T-norms are monotone). The remaining rules define the atoms of S. We consider now an arbitrary atom $A\in S$ and an arbitrary rule r, $r\in\mathcal{P}'$, defining A and we show that r is p-satisfied w.r.t. M'. If r is non-disjunctive there are two possibilities: (i) the body of r contains at least one atom from S, or (ii) the body of r contains no atom from $S_i\cap S$ but M assigns $[0,0]$ to at least one of its atoms. In both cases r is p-satisfied w.r.t. M'. Suppose now that r is disjunctive. If the body of r contains at least one atom from S then it is obvious that r is p-satisfied w.r.t. M', else the body of r contains no atom from S and there are the following possibilities:

(i) $M^{(p)}(B(r))=[0,0]$ and it is obvious that r is p-

[17] See Appendix A for the definition of HCF programs.

[18] The results that follow are valid for both HCF and non-HCF cases.

[19] Such an interval $[u,w]$ exists because $[0,0]<M(A)<[1,1]$ for each $A\in S$.

[20] The premise that $M'(A)=M'(B)$ for each $A,B\in S$ is crucial.

satisfied w.r.t. M'

(ii) $M^{(p)}(B(r)) = [1,1]$. On the one hand, if $p \in \overline{\mathbf{R}^+}$, according to Lemma 3 the head of r must contain at least one atom A' such that $M(A') = [1,1]$ and it is obvious that r is p-satisfied w.r.t. M'. On the other hand, if $p = -\infty$, either (i) $M(A') = [1,1]$ for some $A' \in H(r)$, or (ii) there exist two distinct atoms $A', A'' \in H(r)$ such that $M(A') > [0,0]$ and $M(A'') > [0,0]$. In both cases r is p-satisfied w.r.t. M'.

□

Theorem 14 *Let $\mathcal{P}$ be an arbitrary disjunctive logic program and $\mathcal{P}'$ be the quantitative version of $\mathcal{P}$. Each minimal p-model of $\mathcal{P}'$ is a minimal model of $\mathcal{P}$ iff $p \in \{-\infty\} \cup \overline{\mathbf{R}^+}$.*

Proof. Let p be an arbitrary real value and M an arbitrary minimal p-model of $\mathcal{P}'$. We show first that M is a minimal model of $\mathcal{P}$ iff M is boolean (i.e., $M(A) \in \{[0,0],[1,1]\}$ for each $A \in B_{\mathcal{P}'}$).

If direction (⇒): consider $M \in \mathcal{MM}(\mathcal{P})$ and prove that M is boolean. It is obvious.

Only if direction (⇐): consider that M is boolean and prove that $M \in \mathcal{MM}(\mathcal{P})$. Since $M \in \mathcal{MM}^{(p)}(\mathcal{P}')$, from Proposition 1 it follows that M is a model for $\mathcal{P}$. Suppose now that M is not a minimal model of $\mathcal{P}$, i.e., there exists a non-empty set of atoms $S \subseteq B_{\mathcal{P}}$ such that $N = M \setminus S$ is still a model for $\mathcal{P}$. From Proposition 2 it follows that N is a p-model for $\mathcal{P}'$. Moreover, $N < M$ because $N(A) = [0,0] < M(A) = [1,1]$ for each $A \in S$ and $N(A) = M(A)$ for each $A \in B_{\mathcal{P}} \setminus S$. It follows that M is not a minimal p-model for $\mathcal{P}'$ and this is a contradiction.

According to Proposition 5 all minimal p-models of $\mathcal{P}'$ are boolean iff $p \in \{-\infty\} \cup \overline{\mathbf{R}^+}$ and the result of the theorem follows. □

Theorem 15 *Let $\mathcal{P}$ be an arbitrary disjunctive HCF logic program, and $\mathcal{P}'$ be the quantitative version of $\mathcal{P}$. If $M \in \mathcal{MM}(\mathcal{P})$ then $M \in \mathcal{MM}^{(p)}(\mathcal{P}')$, whatever is $p \in \overline{\mathbf{R}}$.*

Proof. From Proposition 2 it follows that M is a p-model for $\mathcal{P}'$, whatever is $p \in \overline{\mathbf{R}}$. Let p be an arbitrary real value. Suppose now that M is not a minimal p-model for $\mathcal{P}'$, i.e., there exists a non-empty set $S \subseteq B_{\mathcal{P}'}$ such that the quantitative interpretation N defined like $N(A) = M(A)$ for each $A \in B_{\mathcal{P}'} \setminus S$ and $N(A) < M(A)$ for each $A \in S$ is a p-model for $\mathcal{P}'$. Note that $M(A) = [1,1]$ and $N(A) < [1,1]$ for each $A \in S$. Let $S' \subseteq S$ be the set of atoms $A \in S$ such that $[0,0] < N(A) < [1,1]$.

If $S' = \emptyset$ (i.e., N is a boolean p-model), from Proposition 1 it follows that N is a model for $\mathcal{P}$ and since $N \subset M$ it follows that M is not a minimal model for $\mathcal{P}$ and this is a contradiction.

If $S' \neq \emptyset$, let N' be the quantitative boolean interpretation obtained from N such that $N'(A) = [0,0]$ for each $A \in S'$ and $N'(A) = N(A)$ for each $A \in B_{\mathcal{P}'} \setminus S'$. If there is no disjunctive rule r such that $N^{(p)}(B(r)) = [1,1]$ and $N(A) < [1,1]$ for each $A \in H(r)$, it is easy to see that N' is a boolean p-model leading to the contradiction that M is not a minimal model of $\mathcal{P}$. Assume now the existence of such a rule r. Then $H(r)$ contains at least two atoms $A_1, A_2 \in S'$ (note also that p must be negative) and let (i) $S_1 \subset S'$ be the set such that each atom of S_1 appears in the body of at least one rule defining an atom of S_1 and $A_1 \in S_1$, and (ii) $S_2 \subset S'$ be the set such that each atom of S_2 appears in the body of at least one rule defining an atom of S_2 and $A_2 \in S_2$. Since $\mathcal{P}$ is HCF, it follows that $S_1 \cap S_2 = \emptyset$. Now, observe that the interpretation N'' defined like $N''(A) = [0,0]$ for each $A \in S_1$ and $N''(A) = N(A)$ for each $A \in B_{\mathcal{P}'} \setminus S_1$ is a model for $\mathcal{P}$ and $N'' \subset M$. It follows that M is not a minimal model for $\mathcal{P}$ and this is a contradiction. □

Proposition 6 *Let $\mathcal{P}'$ be an arbitrary disjunctive boolean quantitative program. If M is a (non-boolean) p-model of $\mathcal{P}'$ then there exists a boolean p-model M' of $\mathcal{P}'$ such that $M' \leq M$, whatever is $p \in \overline{\mathbf{R}^+}$.*

Proof. Let S be the set of atoms A in $B_{\mathcal{P}'}$ such that $M(A) \notin \{[0,0],[1,1]\}$. We show that the boolean quantitative interpretation M' defined like $M'(A) = [0,0]$ for each $A \in S$ and $M'(A) = M(A)$ for each $A \in B_{\mathcal{P}'} \setminus S$ is a p-model for $\mathcal{P}'$ whatever is $p \in \overline{\mathbf{R}^+}$. We consider an arbitrary atom $A \in S$ and an arbitrary rule r defining A and we show that r is p-satisfied w.r.t. M'; it is clear that the other rules whose head does not contain any atom from S are p-satisfied w.r.t. M'. If r is non-disjunctive there are two possibilities: (i) the body of r contains at least one atom from S, or (ii) the body of r contains no atom from S but M assigns $[0,0]$ to at least one of its atoms. In both cases r is p-satisfied w.r.t. M'. Suppose now that r is disjunctive. If the body of r contains at least one atom from S it is obvious that r is p-satisfied w.r.t. M'. If the body of r contains no atom from S there are the following possibilities:

(i) $M^{(p)}(B(r)) = [0,0]$ and it is obvious that r is p-satisfied w.r.t. M', whatever is $p \in \overline{\mathbf{R}^+}$;

(ii) $M^{(p)}(B(r)) = [1,1]$. Whatever is $p \in \overline{\mathbf{R}^+}$, according to Lemma 3 the head of r must contain at least one atom A' such that $M(A') = [1,1]$ and it is obvious that r is p-satisfied w.r.t. M'.

Note that $M = M'$ when M is boolean. □

Theorem 16 *Let $\mathcal{P}$ be an arbitrary (non-HCF) disjunctive logic program, and $\mathcal{P}'$ be the quantitative version of $\mathcal{P}$. Each minimal model of $\mathcal{P}$ is a minimal p-model of $\mathcal{P}'$ iff $p \in \overline{\mathbf{R}^+}$.*

Proof. *If direction (⇒):* consider that each minimal model of $\mathcal{P}$ is a minimal p-model of $\mathcal{P}'$ and prove that $p \in \overline{\mathbf{R}^+}$. Suppose that $p \in \overline{\mathbf{R}^+}$ is not true, i.e., $p \in [-\infty, 0)$. We show that there exists a (non-HCF) disjunctive logic program $\mathcal{P}$ such that a minimal model of $\mathcal{P}$ is not a minimal p-model of $\mathcal{P}'$, whatever is $p \in [-\infty, 0)$, and this is a contradiction.

Consider the logic program $\mathcal{P} = \{a \vee b \leftarrow,\ a \leftarrow b,\ b \leftarrow a\}$ which has the minimal model $M = \{a, b\}$. For each $p \in [-\infty, 0)$, the quantitative version $\mathcal{P}'$ of $\mathcal{P}$ has a p-model M' such that $[0, 0] < M'(a) = M'(b) < [1, 1]$ (see the definition of $S^{(p)}$ for $p \in [-\infty, 0)$), hence $M' < M$ and M cannot be a minimal p-model for $\mathcal{P}'$.

Only if direction ($\Leftarrow$): consider $p \in \overline{\mathbf{R}^+}$ true and prove that each minimal model of $\mathcal{P}$ is a minimal p-model of $\mathcal{P}'$. Suppose that there exists a minimal model M of $\mathcal{P}$ such that M is not a minimal p-model of $\mathcal{P}'$. From Proposition 2 it follows that M is a p-model for $\mathcal{P}'$. If M is a p-model for $\mathcal{P}'$ but it is not minimal, there exists a non-empty set $S \subseteq B_{\mathcal{P}'}$ such that the quantitative interpretation N defined like $N(A) = M(A)$ for each $A \in B_{\mathcal{P}'} \setminus S$ and $N(A) < M(A)$ for each $A \in S$ is a p-model for $\mathcal{P}'$. Note that $M(A) = [1, 1]$ and $N(A) < [1, 1]$ for each $A \in S$. Since $p \in \overline{\mathbf{R}^+}$, according to Proposition 6 there exists a boolean p-model N' for $\mathcal{P}'$ such that $N' \leq N$. From Proposition 1 it follows that N' is a model for $\mathcal{P}$ and since $N' \subset M$ it results that M is not a minimal model for $\mathcal{P}$ and this is in contradiction with the hypothesis. $\square$

Theorem 17 *Let $\mathcal{P}$ be an arbitrary HCF (resp., non-HCF) disjunctive quantitative program, and $\mathcal{P}'$ its quantitative version. $\mathcal{MM}(\mathcal{P}) = \mathcal{MM}^{(p)}(\mathcal{P}')$ iff $p \in \{-\infty\} \cup \overline{\mathbf{R}^+}$ (resp., iff $p \in \overline{\mathbf{R}^+}$).*

Proof. The relation $\mathcal{MM}(\mathcal{P}) = \mathcal{MM}^{(p)}(\mathcal{P}')$ holds iff the following both relations (i) $\mathcal{MM}(\mathcal{P}) \subseteq \mathcal{MM}^{(p)}(\mathcal{P}')$ and (ii) $\mathcal{MM}(\mathcal{P}) \supseteq \mathcal{MM}^{(p)}(\mathcal{P}')$ hold.

For the HCF case, from Theorem 15 it follows that the relation (i) holds for each $p \in \overline{\mathbf{R}}$.

For the non-HCF case, from Theorem 16 it follows that the relation (i) holds iff $p \in \overline{\mathbf{R}^+}$.

For both HCF and non-HCF cases, from Theorem 14 it follows that the relation (ii) holds iff $p \in \{-\infty\} \cup \overline{\mathbf{R}^+}$.

The results of the theorem follow. $\square$