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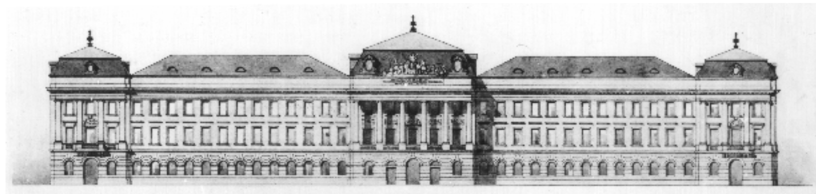

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Characteristics of Multiple Viewpoints in Abstract Argumentation

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Paul E. Dunne **Wolfgang Dvořák**
Thomas Linsbichler **Stefan Woltran**

Institut für Informationssysteme
Abteilung Datenbanken und
Artificial Intelligence
Technische Universität Wien
Favoritenstr. 9
A-1040 Vienna, Austria
Tel: +43-1-58801-18403
Fax: +43-1-58801-18493
sekret@dbai.tuwien.ac.at
www.dbai.tuwien.ac.at

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Characteristics of Multiple Viewpoints in Abstract Argumentation

Paul E. Dunne [1] **Wolfgang Dvořák** [2] **Thomas Linsbichler** [3] **Stefan Woltran** [4]

Abstract. The study of extension-based semantics within the seminal abstract argumentation model of Dung has largely focused on definitional, algorithmic and complexity issues. In contrast, matters relating to comparisons of representational limits, in particular, the extent to which given collections of extensions are expressible within the formalism, have been under-developed. As such, little is known concerning conditions under which a candidate set of subsets of arguments are "realistic" in the sense that they correspond to the extensions of some argumentation framework AF for a semantics of interest. In this paper we present a formal basis for examining extension-based semantics in terms of the sets of extensions that these may express within a single AF. We provide a number of characterization theorems which guarantee the existence of AFs whose set of extensions satisfy specific conditions and derive complexity results for decision problems that require such characterizations.

[1]University of Liverpool, UK. P.E.Dunne@liverpool.ac.uk
[2]University of Vienna, Austria. wolfgang.dvorak@univie.ac.at
[3]Vienna University of Technology, Austria. linsbich@dbai.tuwien.ac.at
[4]Vienna University of Technology, Austria. woltran@dbai.tuwien.ac.at

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1 Introduction

The last 15 years have seen an enormous effort to design, compare, and implement different semantics for Dung's abstract argumentation frameworks [18], AFs for short. As a result of this extensive study argumentation is now a significant topic within AI research [10, 28]. Surprisingly, systematic comparison of AF capabilities respecting *multiple* extensions, and thus their power in modelling multiple viewpoints within a single framework has, so far, been neglected. Nevertheless, an understanding of which extensions can, in principle, coexist when a framework is evaluated with respect to a semantics of interest not only clarifies the "strength" of that semantics but also is a crucial issue in several applications.

In this work, we address this issue by studying the *signatures* of argumentation semantics. A semantics σ maps each argumentation framework F to a set of extensions (i.e. a set of sets of arguments). The signature of σ is defined by

$$\Sigma_\sigma = \{\sigma(F) \mid F \text{ is an AF}\},$$

and gives the collection of all possible sets of extensions an AF can possess under semantics σ. We shall focus on several important semantics namely naive, preferred, semi-stable, stage, stable, and complete semantics [18, 29, 13] and aim at finding simple criteria to decide whether a set of extensions $\mathbb{S}$ is contained in Σ_σ. For instance, we will show that each $\mathbb{S} \in \Sigma_{\mathit{pref}}$ satisfies the condition that for each pair of distinct sets A and B from $\mathbb{S}$ there is at least one $a \in A$ and one $b \in B$ such that a and b do not occur together in any set in $\mathbb{S}$. Thus, for instance, $\mathbb{S} = \{\{a, b\}, \{c, d\}, \{a, c\}, \{b, d\}\}$ is part of the signature Σ_{pref} (the argumentation framework consisting of arguments a, b, c, and d and symmetric attacks between a and d, and b and c, respectively, has $\mathbb{S}$ as its preferred extensions) while neither $\mathbb{S} \cup \{\{a, d\}\}$ nor $\mathbb{S} \cup \{\{b, c\}\}$ are. This fact can be exploited in a search procedure for enumerating preferred extensions: assume for a given AF F, the extensions from $\mathbb{S}$ have already been calculated as preferred extensions of F. The procedure can now restrict the search space to find further extensions of F (if they exist) to sets with at least one argument different from a, b, c, and d.

The problem we study here is also essential in many other aspects.

First, our results are important for constructing AFs. Indeed, knowing whether a set $\mathbb{S}$ is contained in Σ_σ is a necessary condition which should be checked before actually looking for an AF F which *realizes* $\mathbb{S}$ under σ, i.e. $\sigma(F) = \mathbb{S}$. This is of high importance when dynamic aspects of argumentation are considered [27]. As an example, suppose a framework F possesses as its σ-extensions a set $\mathbb{S}$ and one asks for an adaptation of the framework F such that its σ-extensions are given by $\mathbb{S} \cup \{E\}$, i.e. one extension is to be added. The addition of E to $\mathbb{S}$ may, for instance, be desired by some agent on the grounds that E contains some subset of arguments which it wishes to be collectively accepted by other agents: no extension in $\mathbb{S}$, however, provides support for the subset of interest to be considered justifiable. Furthermore the agent wishing to add E is reluctant to jeopardize the chance of this happening if the modified AF is such that some existing element of $\mathbb{S}$ ceases to be an extension: such an outcome being likely to prejudice other agents against agreeing to changes which admit E. Before considering the adapted framework's structure, it is obviously crucial to know whether an appropriate framework exists at all, i.e. whether $S \cup \{E\} \in \Sigma_\sigma$. In a recent paper on revision of AFs [16], the authors circumvent this issue by allowing revision to result not only in a single AF, but in a set of AFs such that the union of their extensions yields the desired outcome. Our results provide exact conditions under which their approach admits a single AF as an outcome of a given revision.

Second, our work adds to the comparison of semantics (see, e.g., [3]) by means of different properties. So far such properties have largely focused on aspects of *single* extensions $S \in \mathbb{S}$ rather than on *sets* of such. An obvious exception being incomparability (the sets in $\mathbb{S}$ are not proper subsets of each other; this property is also known as I-maximality); as we will see, however, all of the standard semantics impose additional (yet distinct) requirements on $\mathbb{S}$ in order for containment in the signature to hold. Furthermore, our results add to the growing body of work considering *generic* treatments of argumentation semantics, that is with respect to shared properties rather than from the perspective of distinguishing features. For instance, we show that most semantics σ are closed under intersection of extensions (more formally, for all AFs F_1, F_2, there exists an AF F such that $\sigma(F) = \sigma(F_1) \cap \sigma(F_2)$, whenever $\sigma(F_1) \cap \sigma(F_2) \neq \emptyset$).

Previous examinations of abstract argumentation semantics have focused on "principle-based evaluation" methods as proposed by Baroni and Giacomin [3] and the characterization of semantics by means of equations by Besnard and Doutre [11]. The former paper introduces several properties for argumentation semantics and gives a systematic investigation which of the properties are satisfied by a given semantics, no matter which AF is given. The latter paper focuses on alternative and uniform characterizations for the extensions of a *given* AF, whereas in our work we shall characterize the set of *all* possible sets of extensions a semantics is capable to express, thus abstracting away from concrete AFs. Related work also includes studies on enforcing [6, 7] certain outcomes, where the task is to modify AFs in such a way that desired arguments become acceptable. However, the issue of multiple extensions is not covered. In fact, the work which is closest to our investigations are studies of intertranslatability issues [23, 25], where signatures of semantics are put in relation to each other. More precisely, if there is an exact translation such that θ-extensions of the transformed AF coincide with the σ-extensions of the original AF, then θ is at least as expressive as σ, that is $\Sigma_\sigma \subseteq \Sigma_\theta$ in our terms. These results, however, tell us little about the *contents* of Σ_σ and Σ_θ, but only relate Σ_σ to Σ_θ.

To summarize, the main contributions of our work are:

- We first identify necessary conditions any set of extensions under a given semantics σ satisfies, i.e. we identify sets of extensions Σ_σ^+ with $\Sigma_\sigma \subseteq \Sigma_\sigma^+$. This not only informs the exact characteristics for the signature of σ, but also determines those sets of extensions that are impossible to be jointly expressed with one AF.

- Then, we provide sufficient conditions for a set of extensions to be realizable under a given semantics σ. For any such set $\mathbb{S}$, we present constructions of canonical frameworks having $\mathbb{S}$ as their σ-extensions. In other words we identify sets of extensions Σ_σ^- with $\Sigma_\sigma^- \subseteq \Sigma_\sigma$. For all semantics σ under consideration (with the exception of complete extensions) our results hold with $\Sigma_\sigma^+ = \Sigma_\sigma^-$. We, thus, obtain exact characterizations of the signatures Σ_σ.

- We apply our results to study the aforementioned property of closure under intersection (of extension-sets). Moreover, we identify limits of disagreement the different semantics face.

While the capabilities of semantics differ in cases involving more than two extensions, it turns out that the maximal number of extensions (over n arguments) that can be captured is equal for many semantics.

- One particular application of our results is the problem of recasting, i.e. to decide if the σ-extensions of a given AF can be expressed via another semantics θ. The relevance of this problem is, for instance, given if θ is semantics for which we have faster systems available. It is important to note that such gains, if achieved, are likely to be with respect to *average-case* performance. As confirmed by our preliminary complexity results of the recasting problem there can be significant complexity barriers – extending up to Π_2^{P}-completeness – in exploiting the approach. In contrast, we show that the problem of checking if $\mathbb{S} \in \Sigma_\sigma$ for a given $\mathbb{S}$ is decidable in polynomial time for all exactly characterized semantics. Finally, we also give results for the case where $\mathbb{S}$ is not given explicitly but by the models of a propositional formula.

A preliminary version of this article has been published as [21].

2 Preliminaries

In what follows, we briefly recall the necessary background on abstract argumentation. For an excellent recent overview, we refer to [1].

Throughout the paper, we assume a countably infinite domain $\mathfrak{A}$ of arguments.

Definition 2.1 *An argumentation framework (*AF*) is a pair $F = (A, R)$ where $A \subseteq \mathfrak{A}$ is finite, and $R \subseteq A \times A$ is the attack relation. The collection of all* AF*s over $\mathfrak{A}$ is given as $AF_{\mathfrak{A}}$.*

We write $a \mapsto_R b$ for $(a, b) \in R$ and $S \mapsto_R a$ (resp. $a \mapsto_R S$) if $\exists s \in S$ such that $s \mapsto_R a$ (resp. $a \mapsto_R s$). We drop subscript R in $\mapsto_R$ if there is no ambiguity. For $S \subseteq A$, the *range* of S (w.r.t. R), denoted S_R^+, is the set $S \cup \{b \mid S \mapsto_R b\}$.

Definition 2.2 *Given $F = (A, R)$, an argument $a \in A$ is* defended *(in F) by a set $S \subseteq A$ if for each $b \in A$, such that $b \mapsto_R a$, also $S \mapsto_R b$. A set T of arguments is defended (in F) by S if each $a \in T$ is defended by S (in F).*

The following result is in the spirit of Dung's fundamental lemma and is used later.

Lemma 2.3 *Given an* AF *$F = (A, R)$ and two sets of arguments $S, T \subseteq A$. If S defends itself in F and T defends itself in F, then $S \cup T$ defends itself in F.*

Proof. To the contrary assume that $S \cup T$ does not defend itself in F. Then there exists a $b \in A$ with $b \mapsto (S \cup T)$ such that $(S \cup T) \mapsto b$ does not hold. Consider $b \mapsto S$. Since $(S \cup T) \mapsto b$ does not hold also $S \mapsto b$ does not hold. Therefore S does not defend itself in F which is a contradiction to the assumption. The case where $b \mapsto T$ behaves symmetrically. □

Next, we introduce the semantics we study in this work. These are the naive, stable, preferred, complete, grounded, stage, and semi-stable semantics, which we will abbreviate by *naive*, *stb*, *pref*, *com*, *grd*, *stage*, and *sem*, respectively. For a given semantics σ we denote $\sigma(F)$ as the set of extensions of F under σ.

Definition 2.4 *Given an* AF $F = (A, R)$*, a set* $S \subseteq A$ *is* conflict-free *(in* F*), if there are no arguments* $a, b \in S$*, such that* $(a, b) \in R$*. We denote the set of all conflict-free sets in* F *as* $cf(F)$*.* $S \in cf(F)$ *is called* admissible *(in* F*) if* S *defends itself. We denote the set of admissible sets in* F *as* $adm(F)$*.*

For a conflict-free set $S \in cf(F)$*, we say that*

- $S \in naive(F)$*, if there is no* $T \in cf(F)$ *with* $T \supset S$*,*
- $S \in stb(F)$*, if* $S \mapsto a$ *for all* $a \in A \setminus S$*,*
- $S \in pref(F)$*, if* $S \in adm(F)$ *and* $\nexists T \in adm(F)$ *s.t.* $T \supset S$*,*
- $S \in com(F)$*, if* $S \in adm(F)$ *and* $a \in S$ *for all* $a \in A$ *defended by* S*,*
- $S \in grd(F)$*, if* $S = \bigcap_{T \in com(F)} T$*,*
- $S \in stage(F)$*, if* $\nexists T \in cf(F)$ *with* $T_R^+ \supset S_R^+$*, and*
- $S \in sem(F)$*, if* $S \in adm(F)$ *and* $\nexists T \in adm(F)$ *s.t.* $T_R^+ \supset S_R^+$*.*

The objects of our interest are the *signatures* of semantics.

Definition 2.5 *The signature* Σ_σ *of a semantics* σ *is defined as*

$$\Sigma_\sigma = \{\sigma(F) \mid F \in AF_{\mathfrak{A}}\}.$$

For characterizing the signatures of the semantics of our interest we will make frequent use of the following concepts.

Definition 2.6 *Given* $\mathbb{S} \subseteq 2^{\mathfrak{A}}$*, we use*

- $Args_{\mathbb{S}}$ *to denote* $\bigcup_{S \in \mathbb{S}} S$*, and*
- $Pairs_{\mathbb{S}}$ *to denote* $\{(a, b) \mid \exists S \in \mathbb{S} : \{a, b\} \subseteq S\}$*.*

$\mathbb{S}$ *is called an* extension-set *(over* $\mathfrak{A}$*) if* $Args_{\mathbb{S}}$ *is finite. While* $|\mathbb{S}|$ *denotes the number of extensions in* $\mathbb{S}$*,* $\|\mathbb{S}\|$ *stands for* $|Args_{\mathbb{S}}|$*.*

Observe that for any $a \in Args_{\mathbb{S}}$, $(a, a) \in Pairs_{\mathbb{S}}$ holds for all extension-sets $\mathbb{S}$; also note that for all considered semantics σ each element $\mathbb{S} \in \Sigma_\sigma$ is an extension-set (since we are dealing with finite AFs).

3 Properties of Argumentation Semantics

Our ultimate goal is to characterize the signatures of the semantics under consideration. In this section, we provide necessary conditions for an extension-set $\mathbb{S}$ to be in the signature Σ_σ. To this end, we have to find common properties for $\sigma(F)$ which hold for any AF F. We do so by abstracting away from the syntactical structure of a given framework and focus on the sets of extensions.

Some properties are rather easy to see: $cf(F)$ satisfies the property that for any $S \in cf(F)$, also $S' \in cf(F)$ for any $S' \subseteq S$. Likewise, for all $\sigma \in \{naive, stb, stage, pref, sem\}$ it obviously holds that $\sigma(F)$ is incomparable for any AF. We define these two properties formally.

Definition 3.1 *Let $\mathbb{S} \subseteq 2^{\mathfrak{A}}$. The* downward-closure*, $dcl(\mathbb{S})$, of $\mathbb{S}$ is given by $\{S' \subseteq S \mid S \in \mathbb{S}\}$. We call $\mathbb{S}$*

- *downward-closed if $\mathbb{S} = dcl(\mathbb{S})$ and*
- *incomparable if all elements $S \in \mathbb{S}$ are pairwise incomparable, i.e. for each $S, S' \in \mathbb{S}$, $S \subseteq S'$ implies $S = S'$.*

However, extension-sets of all considered semantics σ enjoy additional properties. The following example indicates this fact.

Example 3.2 *Consider the incomparable extension-set $\mathbb{S} = \{\{a, b\}, \{a, c\}, \{b, c\}\}$. and a semantics σ that preserves conflict-freeness, i.e. $\sigma(F) \subseteq cf(F)$ for any AF F. Now suppose there exists an AF F with $\sigma(F) = \mathbb{S}$. Then F must not contain attacks between a and b, a and c, and respectively b and c. But then $\sigma(F)$ typically contains $\{a, b, c\}$.*

There are several ways to define the required property which excludes sets like $\mathbb{S}$ from above. To characterize stable, stage and naive extensions, we start with a rather strong restriction making use of the concept of $Pairs_{\mathbb{S}}$ (in Section 6, we will discuss an alternative yet equivalent definition in terms of a ternary operator on $\mathbb{S}$). For the other semantics, we will use a strictly weaker criterion.

Definition 3.3 *An extension-set $\mathbb{S} \subseteq 2^{\mathfrak{A}}$ is tight if for all $S \in \mathbb{S}$ and $a \in Args_{\mathbb{S}}$ it holds that if $S \cup \{a\} \notin \mathbb{S}$ then there exists an $s \in S$ such that $(a, s) \notin Pairs_{\mathbb{S}}$.*

For incomparable $\mathbb{S}$, the premiss of the condition, $S \cup \{a\} \notin \mathbb{S}$, is always fulfilled. Therefore the definition of $\mathbb{S}$ being tight reduces in that case to check whether for all $S \in \mathbb{S}$ and $a \in Args_{\mathbb{S}} \setminus S$ there is an $s \in S$ such that $(a, s) \notin Pairs_{\mathbb{S}}$. The idea behind the notion of being tight is that if an argument a does not occur in some extension S there must be a reason for that. The most simple reason one can think of is that there is a conflict between a and some $s \in S$, i.e. a and s do not occur jointly in any extension-set of $\mathbb{S}$ or, in other words, $(a, s) \notin Pairs_{\mathbb{S}}$. In a way, this limits the multitude of incomparable elements of an extension-set.

Example 3.4 *The extension-set $\mathbb{S} = \{\{a, b\}, \{a, c\}, \{b, c\}\}$ from Example 3.2 is not tight, since there is no reason to, for instance, exclude c from extension $\{a, b\}$ ((a, c) and (b, c) are both contained in $Pairs_{\mathbb{S}}$). On the other hand, the set $\{\{a, b\}, \{a, c\}, \{b, d\}, \{c, d\}\}$ is easily checked to be tight.*

Before stating our first results, we examine certain properties of tight sets which we exploit later.

Lemma 3.5 *For a tight extension-set $\mathbb{S} \subseteq 2^{\mathfrak{A}}$ it holds that*

1. *the $\subseteq$-maximal elements in $\mathbb{S}$ form a tight set, and*
2. *if $\mathbb{S}$ is incomparable then each $\mathbb{S}' \subseteq \mathbb{S}$ is tight.*

Proof. Consider some tight extension-set $\mathbb{S}$. (1) Let $\mathbb{S}'$ be the set of $\subseteq$-maximal elements in $\mathbb{S}$. Note that $\mathit{Pairs}_{\mathbb{S}} = \mathit{Pairs}_{\mathbb{S}'}$, $\mathit{Args}_{\mathbb{S}} = \mathit{Args}_{\mathbb{S}'}$ and $\mathbb{S}' \subseteq \mathbb{S}$. For each $S \in \mathbb{S}'$ and $a \in \mathit{Args}_{\mathbb{S}}$ it holds that if $S \cup \{a\} \notin \mathbb{S}'$ then also $S \cup \{a\} \notin \mathbb{S}$. Therefore we know, from $\mathbb{S}$ being tight, that $\exists s \in S : (a, s) \notin \mathit{Pairs}_{\mathbb{S}}$, hence $\mathbb{S}'$ is tight too. (2) Let $\mathbb{S}$ be incomparable, and consider some $\mathbb{S}' \subseteq \mathbb{S}$. Then $\mathit{Pairs}_{\mathbb{S}'} \subseteq \mathit{Pairs}_{\mathbb{S}}$ and, as $\mathbb{S}$ is incomparable, $S \cup \{a\} \notin \mathbb{S}$ iff $S \cup \{a\} \notin \mathbb{S}'$ for all $S \in \mathbb{S}$, $a \in \mathit{Args}_{\mathbb{S}}$. Thus, since $\mathbb{S}$ is tight by the hypothesis, $\mathbb{S}'$ is tight. □

Proposition 3.6 *For each* AF $F = (A, R)$*,*

1. *$\mathit{cf}(F)$ is non-empty, downward-closed and tight;*
2. *$\mathit{naive}(F)$ is non-empty, incomparable and its downward-closure is tight;*
3. *$\mathit{stage}(F)$ is non-empty, incomparable and tight;*
4. *$\mathit{stb}(F)$ is incomparable and tight.*

Proof. The properties of being non-empty and incomparable are clear. Likewise, it is easy to see that $\mathit{cf}(F) = \mathit{dcl}(\mathit{cf}(F))$, i.e. $\mathit{cf}(F)$ is downward-closed, since each subset of a conflict-free set is conflict-free too. To show that $\mathit{cf}(F)$ is tight, let $S \in \mathit{cf}(F)$ and $a \in \mathit{Args}_{\mathit{cf}(F)}$, such that $S \cup \{a\} \notin \mathit{cf}(F)$. It follows that $S \neq \emptyset$ (otherwise, $\{a\} \notin \mathit{cf}(F)$ and since we know $\mathit{cf}(F) = \mathit{dcl}(\mathit{cf}(F))$, this would yield $a \notin \mathit{Args}_{\mathit{cf}(F)}$). Moreover there exists an argument $s \in S$ such that $s \mapsto a$ or $a \mapsto s$. Then $\{a, s\} \notin \mathit{cf}(F)$ and since $\mathit{cf}(F) = \mathit{dcl}(\mathit{cf}(F))$, $\{a, s\} \not\subseteq T$ for any $T \in \mathit{cf}(F)$. It follows that $(a, s) \notin \mathit{Pairs}_{\mathit{cf}(F)}$. Since this applies to all S, a with $S \in \mathit{cf}(F)$ and $a \notin S$, tightness of $\mathit{cf}(F)$ follows. Next, observe that $\mathit{dcl}(\mathit{naive}(F)) = \mathit{cf}(F)$, by definition of the semantics. Thus $\mathit{dcl}(\mathit{naive}(F))$ is tight, and by Lemma 3.5, $\mathit{naive}(F)$ is tight. With the same lemma, we get that every $\mathbb{S} \subseteq \mathit{naive}(F)$ is tight. By the well-known fact that $\mathit{stb}(F) \subseteq \mathit{stage}(F) \subseteq \mathit{naive}(F)$, it follows that $\mathit{stb}(F)$ as well as $\mathit{stage}(F)$ is tight. □

Lemma 3.5 implies that if the downward-closure of an incomparable extension-set $\mathbb{S}$ is tight, then $\mathbb{S}$ itself is tight too. Therefore Proposition 3.6 assigns stricter conditions to naive extension-sets than to stable and stage extension-sets, respectively.

Example 3.7 *For the* AF F *in Figure 1, we have* $\mathbb{S} = \mathit{stb}(F) = \mathit{stage}(F) = \{\{a_1, b_2, b_3\}, \{a_2, b_1, b_3\}, \{a_3, b_1, b_2\}\}$ *and* $\mathit{naive}(F) = \mathbb{S} \cup \{\{b_1, b_2, b_3\}\}$*. One can check that* $\mathbb{S}$ *is indeed tight. Take, for instance,* $E = \{a_1, b_2, b_3\}$*; then for each argument* t *not in* E *(i.e.* b_1*,*

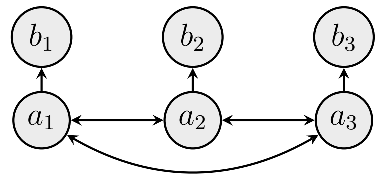

Figure 1: Argumentation framework F used in Example 3.7.

a_2, a_3) there is an argument $s \in E$ such $(s,t) \notin$ Pairs$_{\mathbb{S}}$. In this particular case a_1 plays this role for each t, since neither (a_1, b_1), (a_1, a_2), nor (a_1, a_3) is contained in Pairs$_{\mathbb{S}}$. The other two extensions behave in a symmetric way. However, dcl($\mathbb{S}$) is not tight. In fact, $\{b_2, b_3\} \in$ dcl($\mathbb{S}$) and now for b_1, $\{b_1, b_2, b_3\} \notin$ dcl($\mathbb{S}$) but (b_1, b_2) and (b_1, b_3) are contained in Pairs$_{dcl(\mathbb{S})}$ = Pairs$_{\mathbb{S}}$. Proposition 3.6 now gives evidence that an AF G with naive$(G) = \mathbb{S}$ cannot exist.

Turning to the semantics based on admissibility, we introduce another, weaker, property of extension-sets. It will turn out to be a very general property, which is fulfilled by extension-sets of every "reasonable" semantics for abstract argumentation.

Definition 3.8 *A set $\mathbb{S} \subseteq 2^{\mathfrak{A}}$ is called conflict-sensitive[1] if for each $A, B \in \mathbb{S}$ such that $A \cup B \notin \mathbb{S}$ it holds that $\exists a, b \in A \cup B : (a, b) \notin$ Pairs$_{\mathbb{S}}$.*

As the name suggests, the property checks whether the absence of the union of any pair of extensions in an extension-set $\mathbb{S}$ is justified by a conflict indicated by $\mathbb{S}$.

Note that for $a, b \in A$ (likewise $a, b \in B$), $(a, b) \in$ *Pairs*$_{\mathbb{S}}$ holds by definition. Thus the property of conflict-sensitivity is determined by arguments $a \in A \setminus B$, $b \in B \setminus A$, for $A, B \in \mathbb{S}$.

We next show that tight sets are also conflict-sensitive.

Lemma 3.9 *Every tight extension-set is also conflict-sensitive.*

Proof. Consider some tight extension-set $\mathbb{S}$ and let $A, B \in \mathbb{S}$. We need to show that in case $(a, b) \in$ *Pairs*$_{\mathbb{S}}$ for each $a, b \in A \cup B$, then also $A \cup B \in \mathbb{S}$. Let $B = \{b_1, \ldots, b_n\}$. As $\mathbb{S}$ is tight and $(a, b_1) \in$ *Pairs*$_{\mathbb{S}}$ for all $a \in A$, $A \cup \{b_1\} = A_1 \in \mathbb{S}$ must hold. We can do this for each $b_i \in B$ and get $A_{n-1} \cup \{b_n\} = (A \cup B) \in \mathbb{S}$. □

Observe that for incomparable $\mathbb{S}$ the property of being conflict-sensitive reduces to check for each $A, B \in \mathbb{S}$ ($A \neq B$), whether there exist $a, b \in A \cup B$ such that $(a, b) \notin$ *Pairs*$_{\mathbb{S}}$.

We can give results analogous to those of Lemma 3.5.

Lemma 3.10 *For a conflict-sensitive extension-set $\mathbb{S} \subseteq 2^{\mathfrak{A}}$,*

1. *the $\subseteq$-maximal elements in $\mathbb{S}$ form a conflict-sensitive set,*

[1] An equivalent property was called *adm-closed* in [21]. In the interest of adequacy to its content we stick to the term *conflict-sensitive*.

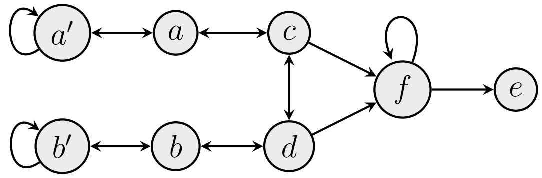

Figure 2: Argumentation framework F used in Example 3.12.

2. *if* $\mathbb{S}$ *is incomparable then each* $\mathbb{S}' \subseteq \mathbb{S}$ *is conflict-sensitive, and*

3. $\mathbb{S} \cup \{\emptyset\}$ *is conflict-sensitive.*

Proof. (1) basically follows from the fact that $Pairs_{\mathbb{S}} = Pairs_{\mathbb{S}'}$ where $\mathbb{S}'$ is the set of $\subseteq$-maximal elements in $\mathbb{S}$. For (2) recall that for incomparable $\mathbb{S}$, checking conflict-sensitivity reduces to check for each $A, B \in \mathbb{S}$ with $A \neq B$, whether there exist $a, b \in A \cup B$ such that $(a, b) \notin Pairs_{\mathbb{S}}$. It is easy to see that this property still holds for $\mathbb{S}' \subseteq \mathbb{S}$ (since then, $Pairs_{\mathbb{S}'} \subseteq Pairs_{\mathbb{S}}$). (3) holds, since $S \cup \emptyset \in \mathbb{S}$ for each $S \in \mathbb{S}$ ensures that $\mathbb{S} \cup \{\emptyset\}$ is conflict-sensitive by definition. □

The following proposition finally shows the role the property of being conflict-sensitive plays in terms of admissible, preferred and semi-stable semantics.

Proposition 3.11 *For each* AF $F = (A, R)$*,*

1. $adm(F)$ *is conflict-sensitive and contains* $\emptyset$*;*

2. $pref(F)$ *is non-empty, incomparable and conflict-sensitive;*

3. $sem(F)$ *is non-empty, incomparable and conflict-sensitive.*

Proof. (1) By definition, $\emptyset$ is always admissible. We show that $adm(F)$ is conflict-sensitive. Towards a contradiction, assume $B, C \in adm(F)$ such that $B \cup C \notin adm(F)$, but for all $b, c \in B \cup C$, $(b, c) \in Pairs_{adm(F)}$. From Lemma 2.3 we know that $B \cup C$ defends itself in F. So for $B \cup C \notin adm(F)$ there must be a conflict in $B \cup C$, i.e. there is an attack $(b, c) \in R$ such that $\{b, c\} \subseteq B \cup C$. But then, for all $D \in adm(F)$, $\{b, c\} \not\subseteq D$. Hence, $(b, c) \notin Pairs_{adm(F)}$, a contradiction.

(2) By definition, the preferred semantics yields at least one extension and for any $S, S' \in \sigma(F)$, $S \subseteq S'$ implies $S = S'$. By Lemma 3.10.1 and definition of preferred extensions, the fact that $pref(F)$ is conflict-sensitive for every AF F follows from the already shown property of $adm(F)$ being conflict-sensitive.

(3) Again, semi-stable extensions are non-empty and incomparable by definition. Since $sem(F) \subseteq pref(F)$ holds for all AFs F, Lemma 3.10.2 shows that $sem(F)$ is conflict-sensitive, as well. □

Example 3.12 *Consider the framework F in Figure 2 and let $A = \{a, b\}$, $B = \{a, d, e\}$, $C = \{b, c, e\}$, and $\mathbb{S} = \{A, B, C\}$. We have pref(F) = sem$(F) = \mathbb{S}$. $\mathbb{S}$ is conflict-sensitive, since for each pair of extensions, there exists a pair of arguments not contained in Pairs$_{\mathbb{S}}$: $b, d \in A \cup B$ and $(b, d) \notin$ Pairs$_{\mathbb{S}}$; $a, c \in A \cup C$ and $(a, c) \notin$ Pairs$_{\mathbb{S}}$; $c, d \in B \cup C$ and $(c, d) \notin$ Pairs$_{\mathbb{S}}$. However, we also observe that $\mathbb{S}$ is not tight, since $A \cup \{e\} \notin \mathbb{S}$ but both (a, e) and (b, e) are contained in Pairs$_{\mathbb{S}}$. This shows that the reverse of Lemma 3.9 does not hold, i.e. that conflict-sensitivity is a strictly weaker condition than tightness.*

All semantics considered so far, i.e. the naive, stable, stage, preferred and semi-stable semantics, share a common property, which is implicit in their definitions. That is, given an arbitrary AF F, between any two extensions of F there must be at least one conflict. If there was not, their union would be an extension instead. This property can be seen as a principle in the spirit of [3] which should be fulfilled by every reasonable semantics for abstract argumentation. The following proposition shows that extension-sets under such semantics are always conflict-sensitive.

Proposition 3.13 *Consider an arbitrary semantics $\sigma : AF_{\mathfrak{A}} \to 2^{2^{\mathfrak{A}}}$ such that for any $F = (A, R) \in AF_{\mathfrak{A}}$ it holds that $\sigma(F) \subseteq cf(F)$ and for all $S_1, S_2 \in \sigma(F)$ ($S_1 \neq S_2$) there exist $a, b \in S_1 \cup S_2$ with $(a, b) \in R$. Then for each AF F, $\sigma(F)$ is conflict-sensitive.*

Proof. Let $F = (A, R) \in AF_{\mathfrak{A}}$ and $S_1, S_2 \in \sigma(F)$. By assumption there exist w.l.o.g. $a \in S_1$ and $b \in S_2$ with $(a, b) \in R$. Now since $\sigma(F) \subseteq cf(F)$, there is no $T \in \sigma(F)$ with $T \supseteq \{a, b\}$, hence $(a, b) \notin Pairs_{\sigma(F)}$. Therefore $\sigma(F)$ is conflict-sensitive. □

Finally, we turn to the complete semantics. Compared to being conflict-sensitive, we need a further relaxation.

Definition 3.14 *Given an extension-set $\mathbb{S} \subseteq 2^{\mathfrak{A}}$ and $E \subseteq \mathfrak{A}$. We define the completion-sets $\mathbb{C}_{\mathbb{S}}(E)$ of E in $\mathbb{S}$ as the set of $\subseteq$-minimal sets $S \in \mathbb{S}$ with $E \subseteq S$.*

In words, completion-sets just give the "next" (in terms of supersets) elements contained in $\mathbb{S}$.

Definition 3.15 *A set $\mathbb{S} \subseteq 2^{\mathfrak{A}}$ is called* com-closed *if for each $\mathbb{T} \subseteq \mathbb{S}$ the following holds: if $(a, b) \in$ Pairs$_{\mathbb{S}}$ for each $a, b \in$ Args$_{\mathbb{T}}$, then Args$_{\mathbb{T}}$ has a unique completion-set in $\mathbb{S}$, i.e. $|\mathbb{C}_{\mathbb{S}}(Args_{\mathbb{T}})| = 1$. For a com-closed extension-set $\mathbb{S} \subseteq 2^{\mathfrak{A}}$ and $E \subseteq$ Args$_{\mathbb{S}}$, we denote the unique element of $\mathbb{C}_{\mathbb{S}}(E)$ by $C_{\mathbb{S}}(E)$.*

The intuitive meaning of com-closed is the following. Consider an extension-set $\mathbb{S}$ and elements $\mathbb{T}$ thereof. Now assume $\mathbb{S}$ gives no evidence of a conflict between arguments in *Args*$_{\mathbb{T}}$. Then, in contrast to the case when $\mathbb{S}$ is conflict-sensitive, not *Args*$_{\mathbb{T}}$ has to be in $\mathbb{S}$, but $\mathbb{S}$ has to contain a unique superset of *Args*$_{\mathbb{T}}$, the completion-set.

Lemma 3.16 *Each conflict-sensitive extension-set is com-closed.*

Proof. Consider a conflict-sensitive extension-set $\mathbb{S}$ and $\mathbb{T} \subseteq \mathbb{S}$. Then $(a, b) \in$ *Pairs*$_{\mathbb{S}}$ for each $a, b \in$ *Args*$_{\mathbb{T}}$ implies *Args*$_{\mathbb{T}} \in \mathbb{S}$, i.e. $\mathbb{C}_{\mathbb{S}}(Args_{\mathbb{T}}) = \{Args_{\mathbb{T}}\}$. □

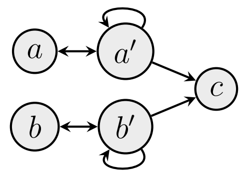

Figure 3: Argumentation framework F used in Example 3.18.

In case of incomparable sets, the notions conflict-sensitive and com-closed coincide. In anticipation of the following result, this coincidence reflects the fact that all preferred extensions are complete.

Proposition 3.17 *For each* AF F*,* $\mathit{com}(F)$ *is a non-empty, com-closed extension-set with* $(\bigcap_{S\in\ \mathit{com}(F)} S) \in \mathit{com}(F)$*.*

Proof. First note that there is always at least one complete extension, namely the grounded extension. Moreover the grounded extension is the unique $\subseteq$-minimal complete extension and hence $(\bigcap_{S\in \mathit{com}(F)} S) \in \mathit{com}(F)$. Finally consider a set of complete extensions $\mathbb{E} \subseteq \mathit{com}(F)$ such that $(a, b) \in \mathit{Pairs}_{\mathit{com}(F)}$ for each $a, b \in \mathit{Args}_{\mathbb{E}}$. By Lemma 2.3, $\mathit{Args}_{\mathbb{E}}$ is an admissible set and thus can be extended to a unique complete extension $E' \supseteq \mathit{Args}_{\mathbb{E}}$ by iteratively adding all defended arguments. Therefore $\mathit{com}(F)$ is com-closed. $\Box$

Example 3.18 *Consider the* AF F *in Figure 3. We have* $\mathit{com}(F) = \{\emptyset, \{a\}, \{b\}, \{a, b, c\}\}$*, which is com-closed, in particular, as* $\mathbb{C}_{\mathit{com}(F)}(\{a\} \cup \{b\}) = \{\{a, b, c\}\}$*. Observe that since* $\{a, b\} \notin \mathit{com}(F)$*, but* $(a, b) \in \mathit{Pairs}_{\mathit{com}(F)}$*,* $\mathit{com}(F)$ *is not conflict-sensitive. This also shows that the reverse direction of Lemma 3.16 does not hold.*

4 Realizability

In the previous section we have given necessary characteristics for the extension-sets $\mathbb{S} \in \Sigma_\sigma$, where $\sigma \in \{\mathit{cf}, \mathit{adm}, \mathit{naive}, \mathit{stb}, \mathit{stage}, \mathit{pref}, \mathit{sem}, \mathit{com}\}$. Except for the complete semantics, we will now show that these characteristics are also sufficient.

To this end, we need the concept of *realizability*: an extension-set $\mathbb{S} \subseteq 2^{\mathfrak{A}}$ is realizable under σ (or just σ-realizable) if there is an AF $F \in \mathit{AF}_{\mathfrak{A}}$, such that $\sigma(F) = \mathbb{S}$. We start with the following concept of a canonical argumentation framework, which will underlie all subsequent results on realizability.

Definition 4.1 *Given an extension-set* $\mathbb{S}$*, we define the canonical argumentation framework for* $\mathbb{S}$ *as*

$$F^{\mathit{cf}}_{\mathbb{S}} = \big(\mathit{Args}_{\mathbb{S}}, (\mathit{Args}_{\mathbb{S}} \times \mathit{Args}_{\mathbb{S}}) \setminus \mathit{Pairs}_{\mathbb{S}}\big).$$

The idea behind the framework is simple: we draw a relation between two arguments iff they do *not* occur jointly in any set $S \in \mathbb{S}$. Thus, for any $\mathbb{S}$, $F^{cf}_{\mathbb{S}}$ is symmetric and has no self-attacking arguments (recall that for each $a \in \mathit{Args}_{\mathbb{S}}$, $(a, a) \in \mathit{Pairs}_{\mathbb{S}}$). As an example, consider $\mathbb{T} = \{\{a_1, b_2, b_3\}, \{a_2, b_1, b_3\}, \{a_3, b_1, b_2\}, \{b_1, b_2, b_3\}\}$. $F^{cf}_{\mathbb{T}}$ has the same structure as the AF from Figure 1, but with all attacks being symmetric. We obtain $\mathit{stb}(F^{cf}_{\mathbb{T}}) = \mathit{naive}(F^{cf}_{\mathbb{T}}) = \mathbb{T}$. When we consider $\mathbb{S} = \{\{a_1, b_2, b_3\}, \{a_2, b_1, b_3\}, \{a_3, b_1, b_2\}\}$, i.e. $\mathbb{S} = \mathbb{T} \setminus \{\{b_1, b_2, b_3\}\}$, we obtain the same framework $F^{cf}_{\mathbb{S}} = F^{cf}_{\mathbb{T}}$, since $\mathit{Args}_{\mathbb{S}} = \mathit{Args}_{\mathbb{T}}$ and $\mathit{Pairs}_{\mathbb{S}} = \mathit{Pairs}_{\mathbb{T}}$. In terms of naive semantics, this is not problematic, since $\mathbb{S}$ (as discussed in Example 3.7) cannot be realized under naive semantics at all. However, this observation readily suggests that realizing $\mathbb{S}$ with, say, stable semantics, requires additional concepts.

Proposition 4.2 *For each extension-set $\mathbb{S} \neq \emptyset$, which is downward-closed and tight, $\mathit{cf}(F^{cf}_{\mathbb{S}}) = \mathbb{S}$.*

Proof. Let $\mathbb{S}$ be a downward-closed and tight extension-set and consider $F^{cf}_{\mathbb{S}} = (\mathit{Args}_{\mathbb{S}}, R_{\mathbb{S}})$. $\mathit{cf}(F^{cf}_{\mathbb{S}}) \subseteq \mathbb{S}$: Observe that for each $E \in \mathit{cf}(F^{cf}_{\mathbb{S}})$, $(a, b) \in \mathit{Pairs}_{\mathbb{S}}$ for all $a, b \in E$ by construction of $R_{\mathbb{S}}$. Now suppose there exists $E' \in \mathit{cf}(F^{cf}_{\mathbb{S}})$ such that $E' \notin \mathbb{S}$. W.l.o.g. let E' be $\subseteq$-minimal with this property. Then $E' = S \cup \{c\}$ for some $S \in \mathbb{S}$. As $\mathbb{S}$ is tight there is an $s \in S$ such that $(s, c) \notin \mathit{Pairs}_{\mathbb{S}}$, a contradiction to the above observation. $\mathit{cf}(F^{cf}_{\mathbb{S}}) \supseteq \mathbb{S}$: Follows immediately by the fact that $\forall (a, b) \in \mathit{Pairs}_{\mathbb{S}} : (a, b) \notin R_{\mathbb{S}}$. □

Proposition 4.3 *For each incomparable extension-set $\mathbb{S} \neq \emptyset$, where $\mathit{dcl}(\mathbb{S})$ is tight, $\mathit{naive}(F^{cf}_{\mathbb{S}}) = \mathbb{S}$.*

Proof. First, note that $F^{cf}_{\mathit{dcl}(\mathbb{S})} = F^{cf}_{\mathbb{S}}$, since $\mathit{Args}_{\mathit{dcl}(\mathbb{S})} = \mathit{Args}_{\mathbb{S}}$ and $\mathit{Pairs}_{\mathit{dcl}(\mathbb{S})} = \mathit{Pairs}_{\mathbb{S}}$. Since $\mathit{dcl}(\mathbb{S})$ is downward-closed, and by assumption, tight and non-empty, $\mathit{cf}(F^{cf}_{\mathbb{S}}) = \mathit{dcl}(\mathbb{S})$ by Proposition 4.2. By construction of $\mathit{dcl}(\mathbb{S})$, the $\subseteq$-maximal sets in $\mathit{dcl}(\mathbb{S})$ are given by $\mathbb{S}$ and as naive sets are just $\subseteq$-maximal conflict-free, $\mathit{naive}(F^{cf}_{\mathbb{S}}) = \mathbb{S}$ follows. □

We proceed with stable and stage semantics. Stable semantics are the only semantics that can realize $\mathbb{S} = \emptyset$. Note that $\mathbb{S} = \emptyset$ is easily *stb*-realizable, for instance with the framework $(\{a\}, \{(a, a)\})$. In Proposition 3.6 the only difference between stable and stage semantics was the case $\mathbb{S} = \emptyset$. The next result will show that this indeed is the only difference between the signatures for stable and stage semantics.

The idea of the construction below is to suitably extend the canonical framework from Definition 4.1 such that undesired sets are excluded. Coming back to our example with $\mathbb{S} = \{\{a_1, b_2, b_3\}, \{a_2, b_1, b_3\}, \{a_3, b_1, b_2\}\}$, recall that $F^{cf}_{\mathbb{S}}$ had one such undesired naive (and stable) extension, $E = \{b_1, b_2, b_3\}$. To get rid of it we add a new argument which is attacked by all other sets from $\mathbb{S}$ but not by E, see Figure 4 for illustration. The next definition generalizes this idea which is inspired by a translation in [23].

Definition 4.4 *Given an extension-set $\mathbb{S}$ and its canonical framework $F^{cf}_{\mathbb{S}} = (A^{cf}_{\mathbb{S}}, R^{cf}_{\mathbb{S}})$, let $\mathbb{X} = \mathit{stb}(F^{cf}_{\mathbb{S}}) \setminus \mathbb{S}$. We define*

$$
\begin{aligned}
F^{st}_{\mathbb{S}} = (&A^{cf}_{\mathbb{S}} \cup \{\bar{E} \mid E \in \mathbb{X}\},\\
&R^{cf}_{\mathbb{S}} \cup \{(\bar{E}, \bar{E}), (a, \bar{E}) \mid E \in \mathbb{X}, a \in \mathit{Args}_{\mathbb{S}} \setminus E\}).
\end{aligned}
$$

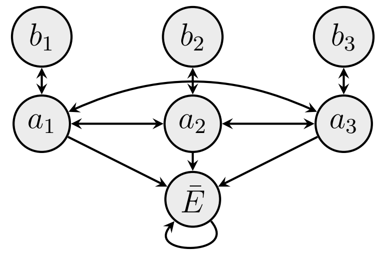

Figure 4: Excluding $\{b_1, b_2, b_3\}$ from $\mathit{stb}(F^{\mathit{cf}}_{\mathbb{S}})$.

Proposition 4.5 *For each non-empty, incomparable and tight extension-set* $\mathbb{S}$*,* $\mathit{stb}(F^{\mathit{st}}_{\mathbb{S}}) = \mathit{stage}(F^{\mathit{st}}_{\mathbb{S}}) = \mathbb{S}$*.*

Proof. Since $\mathbb{S}$ is non-empty, it is sufficient to show $\mathit{stb}(F^{\mathit{st}}_{\mathbb{S}}) = \mathbb{S}$ (for each F with $\mathit{stb}(F) \neq \emptyset$, $\mathit{stb}(F) = \mathit{stage}(F)$ holds).

Moreover, for each incomparable and tight extension-set $\mathbb{S}$, it holds that $\mathbb{S} \subseteq \mathit{stb}(F^{\mathit{cf}}_{\mathbb{S}})$: Towards a contradiction, assume $S \in \mathbb{S}$ such that $S \notin \mathit{stb}(F^{\mathit{cf}}_{\mathbb{S}})$. By construction of $F^{\mathit{cf}}_{\mathbb{S}}$, $S \in \mathit{cf}(F^{\mathit{cf}}_{\mathbb{S}})$. Thus $\exists a \in \mathit{Args}_{\mathbb{S}} : S \not\mapsto a$. But then by construction, $(a, s) \in \mathit{Pairs}_{\mathbb{S}}$, for all $s \in S$; this contradicts $\mathbb{S}$ being tight.

Now let $\mathbb{X} = \mathit{stb}(F^{\mathit{cf}}_{\mathbb{S}}) \setminus \mathbb{S}$ and consider $F^{\mathit{st}}_{\mathbb{S}}$. We show that $\mathit{stb}(F^{\mathit{st}}_{\mathbb{S}}) = \mathbb{S}$. $\mathit{stb}(F^{\mathit{st}}_{\mathbb{S}}) \subseteq \mathbb{S}$: Let $E \in \mathit{stb}(F^{\mathit{st}}_{\mathbb{S}})$. Then also $E \in \mathit{stb}(F^{\mathit{cf}}_{\mathbb{S}}) = \mathbb{X} \cup \mathbb{S}$. But if $E \in \mathbb{X}$, by Definition 4.4, $E \not\mapsto \bar{E}$ and also $\bar{E} \notin E$, thus $E \notin \mathit{stb}(F^{\mathit{st}}_{\mathbb{S}})$. Hence, $E \in \mathbb{S}$. $\mathit{stb}(F^{\mathit{st}}_{\mathbb{S}}) \supseteq \mathbb{S}$: Let $E \in \mathbb{S}$. We already know that $E \in \mathit{stb}(F^{\mathit{cf}}_{\mathbb{S}})$. Now consider $F^{\mathit{st}}_{\mathbb{S}}$. E still attacks all arguments in $\mathit{Args}_{\mathbb{S}} \setminus E$. Now consider an arbitrary argument $\bar{E}'$ for $E' \in \mathbb{X}$. $\bar{E}'$ is attacked by all arguments $a \in \mathit{Args}_{\mathbb{S}} \setminus E'$ and as E and E' are both stable in $F^{\mathit{cf}}_{\mathbb{S}}$ (and thus incomparable) at least one of these arguments must be contained in E. Hence $E \in \mathit{stb}(F^{\mathit{st}}_{\mathbb{S}})$. □

Towards a suitable canonical AF for admissibility-based semantics we introduce the following technical concept.

Definition 4.6 *Given an extension-set* $\mathbb{S}$*, the defense-formula* $\mathcal{D}^{\mathbb{S}}_a$ *of an argument* $a \in \mathit{Args}_{\mathbb{S}}$ *in* $\mathbb{S}$ *is defined as*

$$\bigvee_{S \in \mathbb{S} \text{ s.t. } a \in S} \; \bigwedge_{s \in S \setminus \{a\}} s.$$

$\mathcal{D}^{\mathbb{S}}_a$ *given as (a logically equivalent) CNF is called CNF-defense-formula* $\mathcal{CD}^{\mathbb{S}}_a$ *of* a *in* $\mathbb{S}$*.*

Intuitively, $\mathcal{D}^{\mathbb{S}}_a$ describes the conditions for the argument a being in an extension. The variables coincide with the arguments. Each disjunct represents a set of arguments which jointly allows a to “join” an extension, i.e. represents a collection of arguments defending a. Further note that $\mathcal{D}^{\mathbb{S}}_a \equiv \top$ (i.e. $\mathcal{D}^{\mathbb{S}}_a$ is a tautology) if and only if $\{a\} \in \mathbb{S}$.

Example 4.7 *Consider the extension-set* $\mathbb{T} = \{\emptyset, \{a\}, \{b, c\}, \{a, c, d\}\}$. *Then* $\mathcal{D}_a^{\mathbb{T}} = \top \vee (c \wedge d) \equiv \top$, $\mathcal{D}_b^{\mathbb{T}} = c$, $\mathcal{D}_c^{\mathbb{T}} = b \vee (a \wedge d)$ *and* $\mathcal{D}_d^{\mathbb{T}} = a \wedge c$. *The corresponding CNF-defense-formulas*[2] *are* $\mathcal{CD}_a^{\mathbb{T}} = \{\}$, $\mathcal{CD}_b^{\mathbb{T}} = \{\{c\}\}$, $\mathcal{CD}_c^{\mathbb{T}} = \{\{a, b\}, \{b, d\}\}$, *and* $\mathcal{CD}_d^{\mathbb{T}} = \{\{a\}, \{c\}\}$.

The following lemma shows that the (CNF-)defense-formula for any argument a captures the intuition of describing which arguments it takes for a in order to join an element of the given extension-set.

Lemma 4.8 *Given an extension-set* $\mathbb{S}$ *and an argument* $a \in \mathit{Args}_{\mathbb{S}}$, *then for each* $S \subseteq \mathit{Args}_{\mathbb{S}}$ *with* $a \in S$: $(S \setminus \{a\})$ *is a model of* $\mathcal{D}_a^{\mathbb{S}}$ *(resp.* $\mathcal{CD}_a^{\mathbb{S}}$*) iff there exists an* $S' \subseteq S$ *with* $a \in S'$ *such that* $S' \in \mathbb{S}$.

Proof. The if-direction follows straight forwardly by definition of $\mathcal{D}_a^{\mathbb{S}}$, since the conjunction of the elements of $S' \setminus \{a\}$ forms a term of $\mathcal{D}_a^{\mathbb{S}}$ for each $S' \in \mathbb{S}$ with $a \in S'$.

To show the only-if-direction consider some $S \subseteq \mathit{Args}_{\mathbb{S}}$ with $a \in S$ where $S \setminus \{a\}$ is a model of $\mathcal{D}_a^{\mathbb{S}}$. If $\mathcal{D}_a^{\mathbb{S}} \equiv \top$ then it holds that $\{a\} \in \mathbb{S}$. For $S \setminus \{a\}$ to be a model of $\mathcal{D}_a^{\mathbb{S}} \not\equiv \top$, there must be some term $\tau \in \mathcal{D}_a^{\mathbb{S}}$, whose elements form a subset of $S \setminus \{a\}$. Consider such a term $\tau \in \mathcal{D}_a^{\mathbb{S}}$. Then by construction of $\mathcal{D}_a^{\mathbb{S}}$ there is some $S' \in \mathbb{S}$ with $a \in S'$, where $S' \setminus \{a\}$ coincides with the elements of τ. So $S' \subseteq S$. □

Having at hand a formula for each argument, such that the models coincide with the sets of arguments that should defend that argument, we can give the following construction.

Definition 4.9 *Given an extension-set* $\mathbb{S}$, *the* canonical defense-argumentation-framework $F_{\mathbb{S}}^{\mathit{def}} = (A_{\mathbb{S}}^{\mathit{def}}, R_{\mathbb{S}}^{\mathit{def}})$ *extends the canonical* AF $F_{\mathbb{S}}^{\mathit{cf}} = (\mathit{Args}_{\mathbb{S}}, R_{\mathbb{S}}^{\mathit{cf}})$ *as follows:*

$$A_{\mathbb{S}}^{\mathit{def}} = \mathit{Args}_{\mathbb{S}} \cup \bigcup_{a \in \mathit{Args}_{\mathbb{S}}} \{\alpha_{a\gamma} \mid \gamma \in \mathcal{CD}_a^{\mathbb{S}}\}, \text{ and}$$
$$R_{\mathbb{S}}^{\mathit{def}} = R_{\mathbb{S}}^{\mathit{cf}} \cup \bigcup_{a \in \mathit{Args}_{\mathbb{S}}} \{(b, \alpha_{a\gamma}), (\alpha_{a\gamma}, \alpha_{a\gamma}), (\alpha_{a\gamma}, a) \mid \gamma \in \mathcal{CD}_a^{\mathbb{S}}, b \in \gamma\}.$$

$F_{\mathbb{S}}^{\mathit{def}}$ consists of all arguments given in the extension-set plus a certain amount of additional arguments. First of all each of these new arguments attacks itself in order not to be taken into account when it comes to figuring out the admissible sets of the framework. Further each $\alpha_{a\gamma}$ attacks argument a and is attacked by all arguments occurring as literals in clause γ of the CNF-defense-formula of a. So in $F_{\mathbb{S}}^{\mathit{def}}$ for a to be defended from $\alpha_{a\gamma}$ it takes at least one argument of these occurring as atoms in clause γ of $\mathcal{CD}_a^{\mathbb{S}}$.

Example 4.10 *Again consider extension-set* $\mathbb{T} = \{\emptyset, \{a\}, \{b, c\}, \{a, c, d\}\}$. *We have given the CNF-defense-formulas in Example 4.7.* $F_{\mathbb{T}}^{\mathit{def}}$ *is given by the* AF *in Figure 5. Considering, for instance, argument* c, *where* $\mathcal{CD}_c^{\mathbb{T}} = \{\{a, b\}, \{b, d\}\}$, *one can see that in* $F_{\mathbb{T}}^{\mathit{def}}$ *it takes* a *or* b *in order to defend* c *from* $\alpha_{c\{a,b\}}$, *and* b *or* d *in order to defend* c *from* $\alpha_{c\{b,d\}}$.

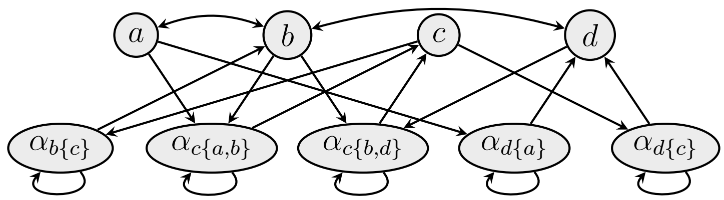

Figure 5: AF $F^{def}_{\mathbb{T}}$ used in Example 4.10.

Proposition 4.11 *For each conflict-sensitive extension-set $\mathbb{S}$ where $\emptyset \in \mathbb{S}$, it holds that $adm(F^{def}_{\mathbb{S}}) = \mathbb{S}$.*

Proof. $\mathbb{S} \subseteq adm(F^{def}_{\mathbb{S}})$: Let $S \in \mathbb{S}$. If $S = \emptyset$, the assertion trivially holds. If $S = \{a\}$, then $\mathcal{CD}^{\mathbb{S}}_a$ is the empty set of clauses. By definition of $R^{def}_{\mathbb{S}}$, a is then defended in $F^{def}_{\mathbb{S}}$ and thus $S \in adm(F^{def}_{\mathbb{S}})$. Thus let $S \in \mathbb{S}$ contain at least two arguments. By construction, S is conflict-free in $F^{def}_{\mathbb{S}}$. It remains to show that each $s \in S$ is defended by S in $F^{def}_{\mathbb{S}}$. Let $s \in S$. First, we know that s defends itself from all $t \in Args_{\mathbb{S}}$, since $F^{def}_{\mathbb{S}}$ extends $F^{cf}_{\mathbb{S}}$, which is symmetric. Moreover, we know from Lemma 4.8 that $S \setminus \{s\}$ is a model of $\mathcal{CD}^{\mathbb{S}}_s$. Hence, each clause $\gamma \in \mathcal{CD}^{\mathbb{S}}_s$ contains at least one variable $t_\gamma \in S \setminus \{s\}$. Thus, by construction of $R^{def}_{\mathbb{S}}$, $(S \setminus \{s\}) \mapsto \alpha_{s\gamma}$ for each $\gamma \in \mathcal{CD}^{\mathbb{S}}_s$, i.e. S defends s in $F^{def}_{\mathbb{S}}$.

$adm(F^{def}_{\mathbb{S}}) \subseteq \mathbb{S}$: Let $S \in adm(F^{def}_{\mathbb{S}})$. For $S = \emptyset$, $S \in \mathbb{S}$ by definition. If $S = \{a\}$, then $a \mapsto b$ whenever some argument b attacks a. Hence, $\mathcal{CD}^{\mathbb{S}}_a = \{\}$, i.e. $\mathcal{D}^{\mathbb{S}}_a \equiv \top$ must hold, therefore $S \in \mathbb{S}$. Now assume S contains at least two arguments. As S is conflict-free in $F^{def}_{\mathbb{S}}$, the construction of $R^{def}_{\mathbb{S}}$ guarantees that for all $a, b \in S$, $(a, b) \in Pairs_{\mathbb{S}}$. Let $s \in S$ with $\{s\} \notin adm(F^{def}_{\mathbb{S}})$. Then we have $\alpha_{s\gamma} \mapsto s$ for each $\gamma \in \mathcal{CD}^{\mathbb{S}}_s$. Since s is defended by S, for each $\gamma \in \mathcal{CD}^{\mathbb{S}}_s$, $\exists t_\gamma \in (S \setminus \{s\}) : t_\gamma \mapsto \alpha_{s\gamma}$. By definition of $F^{def}_{\mathbb{S}}$, thus t_γ occurs in the clause γ. It follows that $T = \{t_\gamma \mid \gamma \in \mathcal{CD}^{\mathbb{S}}_s\}$ is a model of $\mathcal{CD}^{\mathbb{S}}_s$ and $\mathcal{D}^{\mathbb{S}}_s$. Then by Lemma 4.8 there is some $S'_s \subseteq T \cup \{s\}$ (note that also $S'_s \subseteq S$ as $s \in S$ and for each $t_\gamma \in T$ also $t_\gamma \in S$) with $s \in S'_s$ such that $S'_s \in \mathbb{S}$. Recall also that in case $\{s\} \in adm(F^{def}_{\mathbb{S}})$, we know that $\{s\} \in \mathbb{S}$ (say $S'_s = \{s\}$). Now since $(a, b) \in Pairs_{\mathbb{S}}$ for all $a, b \in S$ and since $\mathbb{S}$ is conflict-sensitive, it must hold that $S'_{s_1} \cup S'_{s_2} \in \mathbb{S}$ for any $s_1, s_2 \in S$. Hence $S = \left(\bigcup_{s_i \in S} S'_{s_i}\right) \in \mathbb{S}$. □

The construction also works for the preferred semantics. In order to realize a non-empty, incomparable and conflict-sensitive extension-set $\mathbb{S}$ under the semi-stable semantics we can also make use of $F^{def}_{\mathbb{S}}$, but with an additional self-attacking argument a' for each argument a in $Args_{\mathbb{S}}$ together with an attack from a to a'. For details we refer to the translation from preferred to semi-stable semantics in [25].

Proposition 4.12 *For each non-empty, incomparable and conflict-sensitive extension-set $\mathbb{S}$ there exist F, F' with $pref(F) = \mathbb{S}$ and $sem(F') = \mathbb{S}$.*

[2]In what follows, we write CNF formulas in clause form.

Proof. By Proposition 4.11 and Lemma 3.10.3, $adm(F^{def}_{\mathbb{S}}) = \mathbb{S} \cup \{\emptyset\}$. Since $\mathbb{S}$ is incomparable, $pref(F^{def}_{\mathbb{S}}) = \mathbb{S}$. Moreover, as shown by Dvořák and Woltran [25], for each AF F there is an AF F' such that $pref(F) = sem(F')$. □

Each extension-set $\mathbb{S}$ realizable under the admissible semantics, can also be realized under the complete semantics: extend $F^{def}_{\mathbb{S}}$ for each $a \in Args_{\mathbb{S}}$ by a self-attacking argument with an attack from and to a. As Example 3.18 already suggests, the complete semantics is however capable of realizing more diverse extension-sets. However, the property of being com-closed (cf. Definition 3.15) so far only gives an approximation of the sets which can realized.

Example 4.13 *Let* $\mathbb{S} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b,c\}, \{a,d,e\}, \{b,d,f\}, \{x,c\}, \{x,d\}\}$. $\mathbb{S}$ *is com-closed and and satisfies* $(\bigcap_{S \in \mathbb{S}} S) = \emptyset \in \mathbb{S}$. *We argue that* $\mathbb{S}$ *is not com-realizable. Towards a contradiction consider an* AF F *such that* $com(F) = \mathbb{S}$. *First as* $\{x,c\}, \{x,d\} \in \mathbb{S}$ *and no superset of* $\{x,c,d\}$ *is in* $\mathbb{S}$*, there must be a conflict between* c *and* d*. In case* $(c,d) \in R$ *and* $(d,c) \notin R$*,* (x,d) *is not admissible (as* $(x,c) \in Pairs_{\mathbb{S}}$*). Similar for* $(d,c) \in R$ *and* $(c,d) \notin R$*. Thus* $(c,d), (d,c) \in R$*. Now consider* $\{a,b\}$ *which must be admissible as* $\{a\}, \{b\}$ *are admissible and* $(a,b) \in Pairs_{\mathbb{S}}$*. But as* $\{a,b\} \notin \mathbb{S}$ *it has to defend argument* c *(*$C_{\mathbb{S}}(\{a,b\}) = \{a,b,c\}$*). But we have* $(d,c) \in R$ *and* $(a,d), (b,d) \in Pairs_{\mathbb{S}}$*. Hence* $\{a,b\}$ *cannot defend* c*.*

5 Signatures & Limits of Disagreement

We now state our main result, i.e. exact characterizations of the signatures for the semantics dealt with in this paper.

Theorem 5.1 *The signatures for the considered semantics are given by the following collections of extension-sets.*

$$\begin{aligned}
\Sigma_{cf} &= \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \textit{ is downward-closed and tight}\}\\
\Sigma_{naive} &= \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \textit{ is incomparable and } dcl(\mathbb{S}) \textit{ is tight}\}\\
\Sigma_{stb} &= \{\mathbb{S} \mid \mathbb{S} \textit{ is incomparable and tight}\}\\
\Sigma_{stage} &= \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \textit{ is incomparable and tight}\}\\
\Sigma_{adm} &= \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \textit{ is conflict-sensitive and contains } \emptyset\}\\
\Sigma_{pref} &= \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \textit{ is incomparable and conflict-sensitive}\}\\
\Sigma_{sem} &= \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \textit{ is incomparable and conflict-sensitive}\}\\
\Sigma_{grd} &= \{\mathbb{S} \mid |\mathbb{S}| = 1\}
\end{aligned}$$

The characterizations of the theorem are immediate from results in the previous section and the corresponding characteristics from Propositions 3.6 and 3.11. The result for the grounded semantics follows directly from the facts that every AF F has $|grd(F)| = 1$ and every extension-set $\mathbb{S}$ with $|\mathbb{S}| = 1$ is realized by the AF $(Args_{\mathbb{S}}, \emptyset)$ under the grounded semantics. Moreover, $\Sigma_{com} \subset \{\mathbb{S} \neq \emptyset \mid \mathbb{S}$ is com-closed and $(\bigcap_{S \in \mathbb{S}} S) \in \mathbb{S}\}$, cf. Proposition 3.17 and Example 4.13.

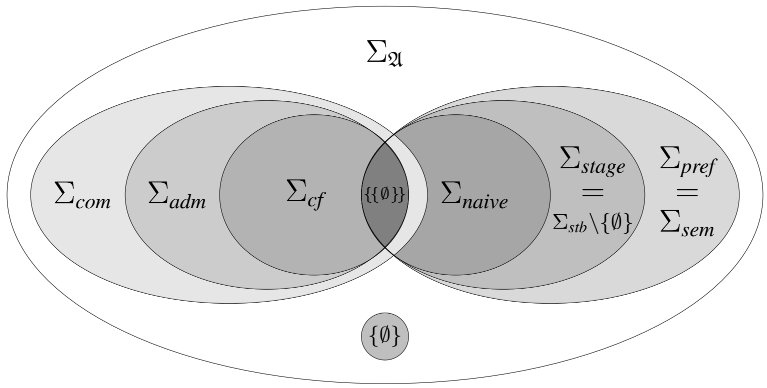

Figure 6: Relations between signatures.

By inspecting the respective properties, we can now immediately put the signatures of different semantics in relation to each other. For illustration see Figure 6, where $\Sigma_{\mathfrak{A}}$ just denotes the set of all extension-sets over $\mathfrak{A}$, and the singleton $\{\emptyset\}$, i.e. the extension-set containing no extension, belongs to Σ_{stb}. The right side of Figure 6 shows signatures of these semantics providing only incomparable extension-sets. The only extension-set they have in common with the signatures of conflict-free and admissible sets is the one only containing the empty extension. On the other hand, the intersection with Σ_{com}, which exactly coincides with Σ_{grd}, contains all extension-sets $\mathbb{S}$ with $|\mathbb{S}| = 1$ in addition.

Theorem 5.2 *The following relations hold*

$$\Sigma_{naive} \subset \Sigma_{stage} \subset \Sigma_{sem} = \Sigma_{pref}, \qquad \Sigma_{stb} = \Sigma_{stage} \cup \{\emptyset\}$$
$$\Sigma_{cf} \subset \Sigma_{adm} \subset \Sigma_{com}$$
$$\{dcl(\mathbb{S}) \mid \mathbb{S} \in \Sigma_{naive}\} = \Sigma_{cf}, \quad \Sigma_{adm} \supset \{\mathbb{S} \cup \{\emptyset\} \mid \mathbb{S} \in \Sigma_{pref}\}$$

Proof. In what follows, we make implicit use of the results from Theorem 5.1. First, by Lemma 3.5.1, $\Sigma_{naive} \subseteq \Sigma_{stage}$; $\Sigma_{naive} \neq \Sigma_{stage}$ is witnessed by Example 3.7, where an extension-set $\mathbb{S} \notin \Sigma_{naive}$ is realized under the stage semantics. Relations $\Sigma_{stage} \subseteq \Sigma_{sem}$ and $\Sigma_{cf} \subseteq \Sigma_{adm}$ follow from Lemma 3.9. $\Sigma_{stage} \neq \Sigma_{sem}$ is by Example 3.12. $\Sigma_{cf} \neq \Sigma_{adm}$ is seen by extension-set $\{\emptyset, \{a, b\}\}$ which is conflict-sensitive but not downward-closed. The fact that *adm*-realizability implies *com*-realizability and Example 3.18 show $\Sigma_{adm} \subset \Sigma_{com}$. The relations in the last line follow from the definition of $dcl(\cdot)$ and Lemma 3.10.3 respectively. □

Let us next compare this landscape of expressiveness illustrated in Figure 6, with those given by the research on intertranslatability. In [25], only translations that are efficiently computable are considered and thus the results are hardly comparable to ours (since we have not put any restrictions on the size of AFs when realizing a set of extensions). Another work [23] studies translations without computational bounds by considering so-called (weakly) exact translations. More precisely, a translation is called exact for $\sigma \Rightarrow \theta$ if the θ-extensions of the transformed AF coincide with the σ-extensions of the original AF. We have that there is an exact translation for $\sigma \Rightarrow \theta$ iff $\Sigma_\sigma \subseteq \Sigma_\theta$. Thus, the landscape drawn in [23] is almost identical to ours with the exception of stable semantics.[3] However, we recall that the results in [23] do *not* tell us how the signatures exactly look like.

The characterization of the signatures of preferred and semi-stable semantics also shows that they enjoy maximal expressiveness within reasonable semantics as defined in Proposition 3.13. That is, no semantics which always guarantees a conflict between two different extensions can express more than preferred and semi-stable semantics, respectively.

Theorem 5.3 *Consider an arbitrary semantics $\sigma : AF_{\mathfrak{A}} \to 2^{2^{\mathfrak{A}}}$ such that for any $F = (A, R) \in AF_{\mathfrak{A}}$ it holds that $\sigma(F) \subseteq cf(F)$ and for all $S_1, S_2 \in \sigma(F)$ ($S_1 \neq S_2$) there exist $a, b \in S_1 \cup S_2$ with $(a, b) \in R$. It holds that $\Sigma_\sigma \subseteq \Sigma_{pref}$ and $\Sigma_\sigma \subseteq \Sigma_{sem}$.*

Proof. Follows directly by Proposition 3.13 and Theorem 5.1. □

With our results on signatures at hand, we now provide some interesting implications. The first result examines the question whether the intersection of two extension-sets under a given semantics σ, can always be realized. With the exact characterizations of signatures in Theorem 5.1 at hand we can answer this question positively for all semantics except complete.

Theorem 5.4 *For each $\sigma \in \{cf, adm, naive, stb, stage, pref, sem\}$ it holds that for any* AF*s F_1, F_2 there exists an* AF *F such that $\sigma(F) = \sigma(F_1) \cap \sigma(F_2)$ if $\sigma(F_1) \cap \sigma(F_2) \neq \emptyset$.*[4]

Proof. We have to show that $\sigma(F_1) \cap \sigma(F_2)$ satisfies the properties according to the signature of the particular semantics. Existence of an AF F with the desired extensions is then a consequence of Theorem 5.1.

cf: Let $\mathbb{S} = cf(F_1) \cap cf(F_2)$. It is easy to see that $\mathbb{S}$ is downward-closed since $cf(F_1)$ and $cf(F_2)$ are downward-closed. So assume $\mathbb{S}$ is not tight, i.e. there is some $S \in \mathbb{S}$ and $a \in Args_{\mathbb{S}}$ with $S \cup \{a\} \notin \mathbb{S}$ but $\forall s \in S : (a, s) \in Pairs_{\mathbb{S}}$. This means that $S \in cf(F_1)$ and $S \in cf(F_2)$, but there is an $i \in \{1, 2\}$ such that $S \cup \{a\} \notin cf(F_i)$. Since $Pairs_{cf(F_i)} \supseteq Pairs_{\mathbb{S}}$, $cf(F_i)$ is not tight, a contradiction to Proposition 3.6.

adm: Towards a contradiction, assume $\mathbb{S} = adm(F_1) \cap adm(F_2)$ is not conflict-sensitive, i.e. there are $A, B \in \mathbb{S}$ such that $(A \cup B) \notin \mathbb{S}$, but for all $a, b \in (A \cup B)$, $(a, b) \in Pairs_{\mathbb{S}}$. Then there is some $i \in \{1, 2\}$, such that $A, B \in adm(F_i)$ but $(A \cup B) \notin adm(F_i)$. On the other hand, $Pairs_{adm(F_i)} \supseteq Pairs_{\mathbb{S}}$, hence $\forall a, b \in (A \cup B) : (a, b) \in Pairs_{adm(F_i)}$. Therefore $adm(F_i)$ is not conflict-sensitive, a contradiction to Proposition 3.11.

naive: Let $\mathbb{S} = naive(F_1) \cap naive(F_2)$ and assume that $dcl(\mathbb{S})$ is not tight, i.e. there is some $S \in dcl(\mathbb{S})$ and $a \in Args_{\mathbb{S}}$ with $S \cup \{a\} \notin dcl(\mathbb{S})$ but $\forall s \in S : (a, s) \in Pairs_{\mathbb{S}}$. This means that

[3] In fact, stable semantics can be translated to admissible, and stage (and thus also to complete, preferred and semi-stable) but none of the semantics can be translated to stable. This is for two (technical) reasons: First, the landscape of expressiveness there is drawn w.r.t. weakly exact translations, which generalize exact translations by allowing to exclude certain extensions from the transformed AF. Second, in [23] AFs are required to be non-empty and thus there is no way for stable to realize $\{\emptyset\}$.

[4] $\sigma(F_1) \cap \sigma(F_2) \neq \emptyset$ is not a necessary condition for $\sigma = stb$.

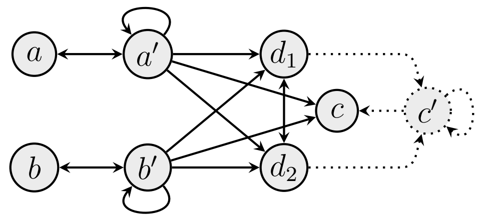

Figure 7: AFs F_1,F_2 with $(com(F_1) \cap com(F_2)) \notin \Sigma_{com}$.

there exists an $S' \supseteq S$ with $S' \in naive(F_1)$ and $S' \in naive(F_2)$ and therefore $S \in dcl(naive(F_1))$ and $S \in dcl(naive(F_2))$. Moreover, for some $i \in \{1, 2\}$ it holds that $\forall T \supseteq (S \cup \{a\}) : T \notin naive(F_i)$ and therefore $(S \cup \{a\}) \notin dcl(naive(F_i))$. Finally, since $Pairs_{naive(F_i)} \supseteq Pairs_{\mathbb{S}}$, $\forall s \in S : (a, s) \in Pairs_{naive(F_i)}$. These observations sum up to $dcl(naive(F_i))$ not being tight, a contradiction to Proposition 3.6.

The result for *stb* and *stage* follows from Lemma 3.5.2, just as the result for *pref* and *sem* is immediate by Lemma 3.10.2. □

Interestingly, the complete semantics turns out to be the only semantics not closed under this form of intersection. Consider the extension-sets $\mathbb{S} = \{\emptyset, \{a\}, \{b\}, \{a, b, c, d_1\}, \{a, b, c, d_2\}\}$, $\mathbb{S}_1 = \mathbb{S} \cup \{\{a, b\}\}$, and $\mathbb{S}_2 = \mathbb{S} \cup \{\{a, b, c\}\}$. $\mathbb{S}_1$ and $\mathbb{S}_2$ are realizable under the complete semantics. Corresponding AFs are depicted in Figure 7: $\mathbb{S}_1$ are the complete extensions of the entire AF, and $\mathbb{S}_2$ the ones of the AF without the dotted part. However, $\mathbb{S} = \mathbb{S}_1 \cap \mathbb{S}_2$ is not com-closed (since $\mathbb{C}_{\mathbb{S}}(\{a, b\}) = \{\{a, b, c, d_1\}, \{a, b, c, d_2\}\}$ does not provide a unique completion-set) and therefore, by Proposition 3.17, no AF F exists such that $com(F) = \mathbb{S}$.

Our next results concern limits in expressing multiple extensions. Our first result is positive in a sense that as long as two (maximal) sets of arguments are involved, all semantics satisfying incomparability of extensions are capable to deliver such extensions. Recall that three sets (e.g. $\mathbb{S} = \{\{a, b\}, \{a, c\}, \{b, c\}\}$) are already problematic. Under admissible and complete semantics, any extension-set containing two arbitrary sets of arguments together with $\emptyset$ is realizable.

Proposition 5.5 *For any extension-set $\mathbb{S}$ with $|\mathbb{S}| \leq 2$,*

1. *$\mathbb{S} \in \Sigma_\sigma$ for $\sigma \in \{naive, stb, stage, pref, sem\}$ if $\mathbb{S}$ is incomparable and $\mathbb{S} \neq \emptyset$,*
2. *$\mathbb{S} \cup \{\emptyset\} \in \Sigma_\theta$ for $\theta \in \{adm, com\}$.*

Proof.

1. By Theorem 5.2, it suffices to show the claim for naive semantics. By Theorem 5.1, we need to show that $dcl(\mathbb{S})$ is tight, which trivially holds for $|\mathbb{S}| = 1$. For $\mathbb{S} = \{S_1, S_2\}$, let $S \in dcl(\mathbb{S})$ and $a \in Args_{\mathbb{S}}$ such that $S \cup \{a\} \notin dcl(\mathbb{S})$. W.l.o.g. assume $S \subseteq S_1$. Then, $a \in S_2 \setminus S_1$ and $S \not\subseteq S_2$, i.e. there is some $s \in S \setminus S_2$. Since $|\mathbb{S}| = 2$, $(a, s) \notin Pairs_{\mathbb{S}}$.

2. We show that $\mathbb{T} = \mathbb{S} \cup \{\emptyset\}$ is conflict-sensitive as long as $|\mathbb{S}| \leq 2$. This trivially holds for $|\mathbb{S}| \leq 1$, since then for all $A, B \in \mathbb{T}$, $A \cup B \in \mathbb{T}$. So let $\mathbb{S} = \{S_1, S_2\}$ with $S_1 \neq S_2$ and w.l.o.g. $S_1, S_2 \neq \emptyset$. If $S_1 \subset S_2$ or $S_2 \subset S_1$ then $S_1 \cup S_2 \in \mathbb{T}$, hence $\mathbb{T}$ is conflict-sensitive. On the other hand, if S_1 and S_2 are incomparable there is an $a \in S_1$ and a $b \in S_2$ such that $(a, b) \notin \mathit{Pairs}_{\mathbb{T}}$, again showing conflict-sensitivity. Therefore $\mathbb{T} \in \Sigma_{\mathit{adm}}$. The result for *com* follows from the fact that $\Sigma_{\mathit{adm}} \subset \Sigma_{\mathit{com}}$ (cf. Theorem 5.2).

□

As a final result in this section, we address the question how many extensions can maximally be achieved by an AF under a semantics σ? Such insights ease checking $\mathbb{S} \in \Sigma_\sigma$ whenever the cardinality of $\mathbb{S}$ exceeds a certain number.

Research in this direction has recently been initiated by Baumann and Strass [9]. They proposed a function giving the maximal number of stable extensions an AF with n arguments can have. In accordance to realizability, we are interested in the number of extensions as a function of a fixed amount of arguments occurring in any extension.

While our results on signatures give insights about the extent of structural diversity a semantics can express, the maximal number of extensions gives, to some degree, an answer to how much quantitative disagreement a semantics can express.

Recall that for some extension-set $\mathbb{S}$, we denote the number of extensions in $\mathbb{S}$ as $|\mathbb{S}|$, and the number of arguments occurring in some extension of $\mathbb{S}$ as $\|\mathbb{S}\|$.

Definition 5.6 *Given a semantics σ, we define the diversity-function*

$$\Delta_\sigma(n) = \max_{F \in AF_{\mathfrak{A}}, \|\sigma(F)\| = n} |\sigma(F)|.$$

It is easy to see that $\Delta_\sigma(n) = 2^n$ for $\sigma \in \{\mathit{cf}, \mathit{adm}, \mathit{com}\}$. Given an extension-set $\mathbb{S}$ with $\|\mathbb{S}\| = n$, the AF $(\mathit{Args}_{\mathbb{S}}, \emptyset)$ has 2^n conflict-free and admissible sets and the AF $(\{a, a' \mid a \in \mathit{Args}_{\mathbb{S}}\}, \{(a, a'), (a', a), (a', a') \mid a \in \mathit{Args}_{\mathbb{S}}\})$ has 2^n complete extensions. For the other semantics, we first give a technical lemma.

Lemma 5.7 *For any* AF *F, there is a symmetric* AF *F^{sym} with $\|\mathit{pref}(F)\| = \|\mathit{pref}(F^{\mathit{sym}})\|$ and $|\mathit{pref}(F)| \leq |\mathit{pref}(F^{\mathit{sym}})|$.*

Proof. In order to get the symmetric AF F^{sym} we transform $F = (A, R)$ by (1) removing all arguments $a \notin \mathit{Args}_{\mathit{pref}(F)}$ together with adjacent attacks, and (2) adding (b, a) to R if $(a, b) \in R$. Obviously conflict-freeness and defense is preserved, i.e. any set admissible in F is admissible in F^{sym}. Moreover, as only attacks to and from arguments not occurring in any preferred extension of F are removed, any conflict between two preferred extensions $E_1, E_2 \in \mathit{pref}(F)$ survives the translation, therefore there must be two $E'_1, E'_2 \in \mathit{pref}(F^{\mathit{sym}})$ with $E_1 \subseteq E'_1$ and $E_2 \subseteq E'_2$. Hence $|\mathit{pref}(F)| \leq |\mathit{pref}(F^{\mathit{sym}})|$. As $\mathit{Args}_{\mathit{pref}(F^{\mathit{sym}})}$ coincides with the arguments of F^{sym} (by symmetry of F^{sym}), it follows that $\|\mathit{pref}(F)\| = \|\mathit{pref}(F^{\mathit{sym}})\|$. □

Recent results by Baumann and Strass [9] provide a function mapping numbers of arguments n to the maximal number of stable extensions an AF with n arguments can have. Their main result is as follows.

Proposition 5.8 *For any natural number n, it holds that*

$$\max_{F=(A,R)\in AF_{\mathfrak{A}}, |A|=n} |\mathit{stb}(F)| = \Lambda(n)$$

with

$$\Lambda_n = \begin{cases} 1, & \text{if } n = 1 \\ 3^s, & \text{if } n \geq 2 \wedge n = 3s \\ 4 \cdot 3^{s-1}, & \text{if } n \geq 2 \wedge n = 3s+1 \\ 2 \cdot 3^s, & \text{if } n \geq 2 \wedge n = 3s+2. \end{cases}$$

In contrast to Δ_σ, Baumann and Strass are interested in the maximal number of stable extensions which can be achieved by an AF with n arguments, no matter how many of these arguments occur in some extension. The AF giving the maximum number of extension is basically composed of connected components of size 3 (or 2) with each component being a clique (see [9] for details). The following result shows that the values of the function Λ carry over to the function Δ_σ for all incomparable semantics σ we consider in this paper. Informally, this means that additional arguments do not allow for a greater maximal number of extensions.

Theorem 5.9 *For $\sigma \in \{\mathit{naive}, \mathit{stb}, \mathit{stage}, \mathit{pref}, \mathit{sem}\}$ and any natural number n, it holds that*

$$\Delta_\sigma(n) = \Lambda(n).$$

Proof. Consider a semantics $\sigma \in \{\mathit{naive}, \mathit{stb}, \mathit{stage}, \mathit{pref}, \mathit{sem}\}$, a natural number n and an AF F with $\|\sigma(F)\| = n$. Moreover assume that F has maximal diversity, i.e. $|\sigma(F)| = \Delta_\sigma(n)$. Since $\Sigma_\sigma \subseteq \Sigma_{\mathit{pref}}$, we can find an AF F' with $\mathit{pref}(F') = \sigma(F)$, therefore $|\sigma(F)| = |\mathit{pref}(F')|$ and $\|\sigma(F)\| = \|\mathit{pref}(F')\|$. Moreover, by Lemma 5.7, we can find a symmetric AF $F^{\mathit{sym}} = (A^{\mathit{sym}}, R^{\mathit{sym}})$ such that $\|\mathit{pref}(F^{\mathit{sym}})\| = \|\mathit{pref}(F')\|$ and $|\mathit{pref}(F^{\mathit{sym}})| \geq |\mathit{pref}(F')|$. In this symmetric AF it holds that $\mathit{pref}(F^{\mathit{sym}}) = \sigma(F^{\mathit{sym}}) = \mathit{stb}(F^{\mathit{sym}})$. Moreover, each argument occurs in at least one σ-extension, i.e. $\|\sigma(F^{\mathit{sym}})\| = |A^{\mathit{sym}}|$. Therefore it follows by Proposition 5.8 that $|\sigma(F^{\mathit{sym}})| \leq \Lambda(n)$. Since we assumed F having maximal diversity, it follows that $\Delta_\sigma(n) \leq \Lambda(n)$.

Finally consider the fact that for all AFs $F = (A, R)$ with $|A| = n$ having $|\mathit{stb}(F)| = \Lambda(n)$ according to [9], it holds that each argument occurs in at least one stable extension, i.e. $\|stb(F)\| = |A|$. Moreover, F is symmetric, hence $\sigma(F) = \mathit{stb}(F)$. Therefore F is an AF with $\|\sigma(F)\| = n$ and $|\sigma(F)| = \Lambda(n)$, hence $\Delta_\sigma(n) = \Lambda(n)$. $\square$

6 Complexity

In this section we consider the computational complexity of several decision problems concerning realizability for semantics σ, σ'. (i) Given an extension-set, deciding whether it can be realized with σ. (ii) Given a propositional formula φ, deciding whether the set of models of φ can be realized with σ. (iii) Given an AF F, deciding whether $\sigma'(F)$ can be realized with a semantics σ.

We assume the reader is familiar with standard complexity concepts, as P, NP and completeness. Nevertheless we briefly recapitulate the concept of NP-oracle machines and the related complexity class Σ_2^{P}. By an NP-oracle machine we mean a Turing machine which can access an oracle that decides a given sub-problem from NP (or coNP) within one step. The class Σ_2^{P} (sometimes also denoted by $\mathsf{NP}^{\mathsf{NP}}$), contains the problems which can be decided in polynomial time by a nondeterministic NP-oracle machine. The complementary class of Σ_2^{P} is the class Π_2^{P} (sometimes also denoted by $\mathsf{coNP}^{\mathsf{NP}}$). Finally, a problem is in the class DP iff it can be characterized as the intersection of a problem in NP and a problem in coNP.

6.1 Realizability of an Extension-Set

In this section, we first consider the problem of checking realizability, i.e. given an extension-set $\mathbb{S}$ and a semantics σ, is there an AF F with $\sigma(F) = \mathbb{S}$. This is equivalent with checking membership in a signature for semantics σ, so whether $\mathbb{S} \in \Sigma_\sigma$ holds. For most of the semantics, it is not hard to see that this can be done in polynomial time in the size of $\mathbb{S}$. For instance, to check that $\mathbb{S} \in \Sigma_{stb}$, it is sufficient to check for incomparability via a double loop over $\mathbb{S}$ and for being tight one loops over all $S \in \mathbb{S}$ and in each such loop, another loop is required for each $a \in \mathit{Args}_{\mathbb{S}} \setminus S$. The only exception is the naive semantics, since the characterization in Theorem 5.1 makes use of $\mathit{dcl}(\mathbb{S})$ which is not polynomially bounded in the size of $\mathbb{S}$.

We provide an alternative characterization based on the ternary majority operator $\mathit{maj_3}$: given three sets $S_1, S_2, S_3 \subseteq \mathfrak{A}$, the majority of these sets is defined as $\mathit{maj_3}(S_1, S_2, S_3) = (S_1 \cap S_2) \cup (S_2 \cap S_3) \cup (S_1 \cap S_3)$. That is $s \in \mathit{maj_3}(S_1, S_2, S_3)$ iff s appears in at least two of the sets.

Proposition 6.1 *For an incomparable extension-set $\mathbb{S}$ it holds that $\mathit{dcl}(\mathbb{S})$ is tight iff for all $S_1, S_2, S_3 \in \mathbb{S}$ there is an $S \in \mathbb{S}$, such that $\mathit{maj_3}(S_1, S_2, S_3) \subseteq S$.*

Proof. First suppose that for all $S_1, S_2, S_3 \in \mathbb{S}$ there is some $S \in \mathbb{S}$ such that $\mathit{maj_3}(S_1, S_2, S_3) \subseteq S$. Towards a contradiction assume that the downward-closure of $\mathbb{S}$ is not tight, i.e. there exist $S' \in \mathit{dcl}(\mathbb{S})$ and $a \in \mathit{Args}_{\mathbb{S}} = \mathit{Args}_{\mathit{dcl}(\mathbb{S})}$, such that $(S' \cup \{a\}) \notin \mathit{dcl}(\mathbb{S})$ and for all $s \in S'$, $(a, s) \in \mathit{Pairs}_{\mathbb{S}} = \mathit{Pairs}_{\mathit{dcl}(\mathbb{S})}$. Assume $|S'| = 1$, i.e. $S' = \{s\}$. As $(a, s) \in \mathit{Pairs}_{\mathbb{S}}$ by assumption, there is a $T \in \mathbb{S}$ with $\{a, s\} \subseteq T$, a contradiction to $S' \cup \{a\} \notin \mathit{dcl}(\mathbb{S})$. Hence $|S'| > 1$, i.e. $S' = \{s_1, \ldots, s_n\}$ with $n > 1$. By assumption, $\{s_1, \ldots, s_n, a\} \notin \mathbb{S}$, but $(a, s_i) \in \mathit{Pairs}_{\mathbb{S}}$ for each $s_i \in S'$. Hence, for each $s_i \in S'$ there is some $S_i \in \mathbb{S}$ with $\{a, s_i\} \subseteq S_i$ for $i = 1 \ldots n$. Moreover $\mathit{maj_3}(S_i, S_j, S') \supseteq \{s_i, s_j, a\}$ for each $i, j \in \{1, \ldots, n\}$ since $s_i \in S' \cap S_i$, $s_j \in S' \cap S_j$, and $a \in S_i \cap S_j$. Therefore there is some $S_{ij} \in \mathbb{S}$ with $S_{ij} \supseteq \{s_i, s_j, a\}$. Now for some $k \in \{1, \ldots, n\}$, we get $S_{ijk} = \mathit{maj_3}(S_{ij}, S_k, S') \supseteq \{s_i, s_j, s_k, a\}$ and $S_{ijk} \in \mathbb{S}$. Following this procedure for all $1 \ldots n$ yields a $T \in \mathbb{S}$ with $T \supseteq \{s_1, \ldots, s_n, a\}$, a contradiction to $S' \cup \{a\} \notin \mathit{dcl}(\mathbb{S})$.

To show the only-if-direction consider some extension-set $\mathbb{S}$ where $\mathit{dcl}(\mathbb{S})$ is tight and assume, towards a contradiction, sets $S_1, S_2, S_3 \in \mathbb{S}$ such that $\mathit{maj_3}(S_1, S_2, S_3) \not\subseteq S$, for all $S \in \mathbb{S}$. Now, consider the $\subseteq$-maximal $S' \in \mathit{dcl}(\mathbb{S})$ with $S' \subset \mathit{maj_3}(S_1, S_2, S_3)$. It holds that $\exists a \in \mathit{maj_3}(S_1, S_2, S_3) \setminus S'$ such that $S' \cup \{a\} \notin \mathit{dcl}(\mathbb{S})$. As $S' \cup \{a\} \subseteq \mathit{maj_3}(S_1, S_2, S_3)$, all

$s \in (S' \cup \{a\})$ are also contained in $((S_1 \cap S_2) \cup (S_2 \cap S_3) \cup (S_1 \cap S_3))$. Thus for each $s \in S'$ there is some $S_i \in \{S_1, S_2, S_3\}$ with $\{s, a\} \subseteq S_i$. Hence, $(s, a) \in \mathit{Pairs}_{\mathbb{S}}$ for each $s \in S'$, which is, together with the facts that $S' \in \mathit{dcl}(\mathbb{S})$ and $S' \cup \{a\} \notin \mathit{dcl}(\mathbb{S})$, a witness that $\mathit{dcl}(\mathbb{S})$ is not tight, and therefore a contradiction to the assumption. □

Testing the majority criterion is done in polynomial time by looping over all triples (S_1, S_2, S_3) stemming from $\mathbb{S}$. Together with the above observations we obtain the following theorem.

Theorem 6.2 *For semantics $\sigma \in \{\mathit{cf}, \mathit{naive}, \mathit{stb}, \mathit{stage}, \mathit{adm}, \mathit{pref}, \mathit{sem}\}$, given $\mathbb{S}$, testing $\mathbb{S} \in \Sigma_\sigma$ is in polynomial time.*

Of course one difficulty with Theorem 6.2 is that one may be concerned with deciding realizability of collections of sets, $\mathbb{S}$, with such collections having size superpolynomial in $|\mathit{Args}_{\mathbb{S}}|$. In such cases it would be more realistic to encode $\mathbb{S}$ in a more compact form. We observe that there are a number of ways in which such "compact encodings" may be treated:

1. As the models of a given propositional formula.
2. As the extensions of an AF under another argumentation semantics.

To begin with, we consider the former case.

6.2 Realizing Models of Propositional Formulas

Here we study the problem of realizing the set of models of a propositional formula as extension-set for an argumentation semantics.

Definition 6.3 *A propositional formula φ over atoms X, denoted $\varphi(X)$, encodes the extension-set $\mathbb{S}$ if $\mathit{Args}_{\mathbb{S}} \subseteq X$ and S is a model of φ iff $S \in \mathbb{S}$.*[5] *We denote the extension-set encoded by $\varphi(X)$ as $\mathbb{S}_\varphi$ and the propositional function described by $\varphi(X)$ as f_φ.*

The next proposition gives complexity results for checking the characteristic properties of signatures, given an extension-set as models of a propositional formula.

Proposition 6.4 *Given a propositional formula $\varphi(X)$, it holds that*

1. *deciding if $\mathbb{S}_\varphi$ is non-empty is* NP*-complete,*
2. *deciding if $\mathbb{S}_\varphi$ is downward-closed is* coNP*-complete,*
3. *deciding if $\mathbb{S}_\varphi$ is incomparable is* coNP*-complete,*
4. *deciding if $\mathbb{S}_\varphi$ is tight is* coNP*-complete, and*

[5] We understand a set of arguments as the interpretation where the atoms with corresponding arguments in the set are assigned true and all other atoms are assigned false.

	cf	*adm*	*stb*	*stage*	*pref*	*sem*
$\varphi(X)$	coNP-c	coNP-c	coNP-c	DP-c	DP-c	DP-c

Table 1: Complexity of the σ-realizability problem for propositional formulas, $\varphi(X)$.

5. *deciding if* $\mathbb{S}_\varphi$ *is conflict-sensitive is* coNP-*complete.*

Proof Sketch. The proof is based on the following ideas: (1) Deciding whether $\mathbb{S}_\varphi$ is non-empty is equivalent to deciding whether φ is satisfiable; (2) Concerning $\mathbb{S}_\varphi$ is downward-closed we have to test whether the boolean function f_φ is anti-monotone. This can be falsified by finding a model M and a variable x such that $M \setminus \{x\}$ is not a model of φ; (3) To falsify incomparability of $\mathbb{S}_\varphi$ we can guess two models that are in $\subset$-relation; (4) To falsify that $\mathbb{S}_\varphi$ is tight we need a model S of φ, an argument x such that $S \cup \{x\} \notin \mathbb{S}_\varphi$, and a linear number of $T_i \in \mathbb{S}_\varphi$ such that for each pair $(s,x), s \in S$ there is one T_i with $\{s,x\} \subseteq T_i$; (5) To falsify that $\mathbb{S}_\varphi$ is conflict-sensitive we need two models S,T of φ, such that $S \cup T \notin \mathbb{S}_\varphi$, and a linear number of $T_i \in \mathbb{S}_\varphi$ such that for each pair $(s,t), s \in S, t \in T$ there is one T_i with $\{s,t\} \subseteq T_i$. The full proofs are provided in A.1. $\square$

With these results at hand, we are able to present upper and lower bounds on the complexity of deciding whether the models of a given propositional formula φ can be realized under the conflict-free, admissible, stable, stage, preferred and semi-stable semantics. We summarize the results in Table 1. An entry for column σ gives the complexity of deciding whether $\varphi(X) \in \Sigma_\sigma$ and $\mathcal{C}$-c abbreviates completeness for class $\mathcal{C}$.

Theorem 6.5 *The cf-realizability problem in which instances are propositional formulas, $\varphi(X)$, is* coNP-*complete.*

Proof. Recall that a $\mathbb{S}$ is *cf*-realizable, if $\mathbb{S} \neq \emptyset$, $\mathbb{S}$ is downward-closed, i.e. $dcl(\mathbb{S}) = \mathbb{S}$, and $\mathbb{S}$ is tight (cf. Theorem 5.1). The coNP membership is thus immediate from Proposition 6.4. Notice that we do not have to check $\mathbb{S} \neq \emptyset$ explicitly. This follows from downward-closure, since for a downward-closed set $\mathbb{S}$, $\mathbb{S} \neq \emptyset$ iff $\emptyset \in \mathbb{S}$. Testing $\emptyset \in \mathbb{S}_\varphi$ just additionally requires checking whether $\varphi(\langle \bot, \ldots, \bot \rangle)$ evaluates to true, which is easy.

To establish coNP-hardness, we use a straightforward reduction from SAT to the complementary problem. Given an instance $\psi(X)$ of SAT let $\varphi(X,y)$ be the propositional formula

$$\varphi(X,y) = (y \wedge \psi(X)) \vee \bigwedge_{x \in X \cup \{y\}} \neg x$$

where y is a new variable. Without loss of generality we assume that the empty set is not a model of $\psi(X)$. We claim $\psi(X)$ is satisfiable if and only if $\mathbb{S}_\varphi$ is not *cf*-realizable. Suppose that $\psi(X)$ is satisfied by a model M. Then $M \cup \{y\}$ is a model of $\varphi(X,y)$ while M is not. Hence, $\mathbb{S}_\varphi$ is not downward closed and thus not *cf*-realizable. Conversely suppose $\psi(X)$ is unsatisfiable. Then the empty set is the only model of $\varphi(X,y)$, i.e. $\mathbb{S}_\varphi = \{\emptyset\}$. Thus, $\mathbb{S}_\varphi$ is downward-closed and tight and thus *cf*-realizable.

We deduce that $\psi(X)$ is accepted as an instance of SAT if and only if $\varphi(X,y)$ is not accepted as an instance of *cf*-realizability and, hence, the *cf*-realizability problem is coNP-complete. $\square$

Theorem 6.6 *The adm-realizability problem in which instances are propositional formulas, $\varphi(X)$, is* coNP*-complete.*

Proof. First observe that deciding, given $\varphi(X)$ whether $\mathbb{S}_\varphi$ is *adm*-realizable just involves checking, that $\varphi(\langle\perp,\ldots,\perp\rangle)$ evalutates to true and that $\mathbb{S}_\varphi$ is conflict-sensitive (cf. Theorem 5.1). The first condition is trivially checked in polynomial time, the second we have shown to be in coNP in Proposition 6.4.

To prove that *adm*-realizability is coNP-hard we reduce from deciding conflict-sensitivity, which we showed to be coNP-hard (cf. Proposition 6.4). Let $\psi(X)$ be an instance of deciding whether $\mathbb{S}_\psi$ is conflict-sensitive, and consider the instance, $\varphi(X)$ of the *adm*-realizability problem given by,

$$\varphi(X) \;=\; \psi(X) \vee \left(\bigwedge_{x\in X} \neg x\right)$$

Then $\mathbb{S}_\varphi = \mathbb{S}_\psi \cup \{\emptyset\}$. As the empty set has no effect on conflict-sensitivity of an extension-set we have that $\mathbb{S}_\varphi$ is conflict-sensitive iff $\mathbb{S}_\psi$ conflict-sensitive. Further, as always $\emptyset \in \mathbb{S}_\varphi$, we finally have that $\mathbb{S}_\varphi$ is conflict-sensitive iff $\mathbb{S}_\psi$ is *adm*-realizable and, hence, the *adm*-realizability problem is coNP-complete. □

Theorem 6.7 *The stb-realizability problem in which instances are propositional formulas, $\varphi(X)$, is* coNP*-complete.*

Proof. The coNP membership is immediate from Proposition 6.4.

To show that deciding whether $\varphi(X)$ is *stb*-realizable is coNP-hard, we use a reduction from UNSAT. Given an instance $\psi(X)$ of SAT, form $\varphi(X,y,z)$ as an instance of comparability with

$$\varphi(X,y,z) \;=\; \psi(X) \;\wedge\; \left(\bigwedge_{x\in X} (x \rightarrow (y \vee z))\right)$$

Here y and z are new variables. We claim $\psi(X)$ is satisfiable if and only if $\mathbb{S}_\varphi$ is not *stb*-realizable.

Suppose that $\psi(X)$ is satisfiable using α and let $S_\alpha \subseteq X$ be the corresponding subset of X indicated by α. Then,

$$T \;=\; S_\alpha \cup \{y\} \in \mathbb{S}_\varphi$$

and

$$U \;=\; S_\alpha \cup \{y,z\} \in \mathbb{S}_\varphi$$

Clearly $T \subset U$ so that $\varphi(X,y,z)$ is comparable, and thus $\mathbb{S}_\varphi$ is not *stb*-realizable.

Conversely suppose $\psi(X)$ is unsatisfiable. Then also $\varphi(X,y,z)$ is unsatisfiable, i.e. $\mathbb{S}_\varphi = \{\}$. Hence, $\mathbb{S}_\varphi$ is *stb*-realizable.

We deduce that deciding if $\mathbb{S}_\varphi$ is *stb*-realizable is coNP-complete. □

Theorem 6.8 *The σ-realizability problem in which instances are propositional formulas, $\varphi(X)$, is in* DP *for $\sigma \in \{pref, stage, sem\}$.*

Proof. For $\sigma \in \{pref, stage, sem\}$ we have, with the exception of $\mathbb{S}_\sigma \neq \emptyset$, shown that the relevant characterizing conditions can be decided in coNP. Letting $\chi(\sigma)$ denote these, e.g. $\chi(pref) = \{$conflict-sensitive, incomparable$\}$, we see that, $\mathbb{S}_\varphi \in \Sigma_\sigma$ if and only if

$$\varphi(X) \ \in \ \{ \ \psi(X) \ : \ \psi(X) \text{ is satisfiable}\} \cap \{ \ \psi(X) \ : \ \mathbb{S}_\psi \text{ satisfies } \chi(\sigma) \ \}$$

So that σ-realizability is in DP for these three cases. □

There are also matching lower bounds for preferred, stage, and semi-stable semantics, given in the next theorem.

Theorem 6.9 *The σ-realizability problem in which instances are propositional formulas, $\varphi(X)$, is* DP*-hard $\sigma \in \{pref, stage, sem\}$.*

Proof Sketch. The proof roughly proceeds as follows (a full proof is given in A.2). First we show that σ-realizability is coNP-hard. Second we show that σ-realizability is NP-hard. Finally we show that σ-realizability has the property AND_2, i.e. the problem of deciding whether two instances of the σ-realizability problem are both true can be reduced to a single instance, and then apply a result from Chang and Kadin [15] to deduce DP-hardness. □

Notice that for *naive* semantics, for which *dcl*($\mathbb{S}$) being tight is a condition for naive realizability we have the following easy result.

Lemma 6.10 *Let dcl-member be the following decision problem. Given $\langle S, \varphi(X)\rangle$ with $S \subseteq X$, decide whether $S \in dcl(\mathbb{S}_\varphi)$ holds. dcl-member is* NP*-complete.*

Proof. To see that *d*cl-member is in NP, given an instance $\langle S, \varphi(X)\rangle$ it suffices to guess $T \subseteq X$ and verify $(S \subseteq T) \wedge (\varphi(\alpha_T) = \top)$, where α_T is the interpretation corresponding to T.

For NP-hardness, we use a reduction from SAT. Given an instance $\psi(X)$ of SAT simply form the instance $\langle \emptyset, \psi\rangle$ of *d*cl-member. If $\psi(X)$ is satisfied by α, then S_α (the corresponding subset of X) is in $\mathbb{S}_\psi$ and, trivially, $\emptyset \in dcl(\mathbb{S}_\psi)$ so $\langle \emptyset, \psi\rangle$ is accepted as an instance of *d*cl-member. Conversely, if $\emptyset \in dcl(\mathbb{S}_\psi)$, then there is some $T \subseteq X$ for which $T \in \mathbb{S}_\psi$, so that $\psi(\alpha_T) = \top$, i.e. ψ is satisfiable. □

6.3 Recasting Argumentation Semantics

Turning to the second compact encoding of a collection of subsets mentioned earlier, i.e. $\mathbb{S}$ is presented as the σ-extensions of a given AF, we introduce the problem of *recasting*: given an AF $F_1 \in AF_{\mathfrak{A}}$ and two semantics σ_1, σ_2, decide whether there exists an $F_2 \in AF_{\mathfrak{A}}$, such that $\sigma_1(F_1) = \sigma_2(F_2)$. By the very nature of signatures, this is equivalent to test $\sigma_1(F_1) \in \Sigma_{\sigma_2}$. If there is an exact translation in the sense of [23], i.e. a function Tr such that $\sigma_1(F_1) = \sigma_2(Tr(F))$ for

	cf	*adm*	*com*
cf	-	trivial	trivial
adm	coNP-c	-	trivial
com	coNP-c	coNP-c	-

Table 2: Complexity of the recasting problem (simple semantics).

each AF F, (or equivalently $\Sigma_{\sigma_1} \subseteq \Sigma_{\sigma_2}$) the answer is trivially yes. Here we are interested in combinations of semantics σ_1, σ_2 where no such universal translation exists and we ask whether there is a "translation" for a concrete AF. Table 2 and Table 3 summarize our results. Note that recasting only makes sense between semantics which are (resp. are not) I-maximal (i.e. the extensions are (resp. are not) incomparable); thus we separated the semantics under consideration. An entry for row σ_1 and column σ_2 gives the complexity of deciding whether $\sigma_1(F) \in \Sigma_{\sigma_2}$. $\mathcal{C}$-h abbreviates hardness for class $\mathcal{C}$; $\mathcal{C}$-c abbreviates completeness for class $\mathcal{C}$; "trivial" means that each instance is a "Yes"-instance. For both tables, these "trivial" entries are immediate from the relations between the signatures given in Theorem 5.2.

Theorem 6.11 *The complexity results depicted in Table 2 hold.*

Proof. The "trivial" results are immediate by the fact that $\Sigma_{cf} \subset \Sigma_{adm} \subset \Sigma_{com}$ (cf. Theorem 5.2). For coNP-membership (of the non-trivial problems) we provide non-deterministic algorithms disproving, given an AF F, $\sigma(F) \in \Sigma_{\sigma'}$.

(a) Recasting from $\sigma_1 \in \{adm, com\}$ to *cf* semantics. If $\sigma_1(F)$ is not *cf*-realizable then either (i) $\sigma_1(F)$ is not downward-closed or (ii) $\sigma_1(F)$ is not tight (cf. Theorem 5.1). To check (i) we just have to guess two sets B, C such that $B \subset C$, $C \in \sigma_1(F)$ and $B \notin \sigma_1(F)$. To check (ii) we have to guess a set E, an argument a and sets S_s for each $s \in E$ such that $E \in \sigma_1(F)$, $E \cup \{a\} \notin \sigma_1(F)$ and $\{a, s\} \subseteq S_s$ for each s. As verifying that a set is in $\sigma_1(F)$ is in polynomial time this gives coNP-procedures for testing whether $\sigma_1(F)$ is *cf*-realizable.

(b) Recasting from *com* to *adm* semantics. Here, we check (i) $\emptyset \notin com(F)$ or (ii) $com(F)$ is not conflict-sensitive. Clearly (i) can be done in polynomial time. For (ii) notice that if $com(F)$ is not conflict-sensitive then there exist sets $A, B \in com(F)$ such that $A \cup B \in cf(F)$ but $A \cup B \notin com(F)$. But then there exists a set C with $A \cup B \subset C \in com(F)$. So to disprove $com(F)$ being conflict-sensitive we guess sets A, B, C and test whether $A, B, C \in com(F)$, $A \cup B \subset C$ but $A \cup B \notin com(F)$, which is a coNP-procedure for testing whether $com(F)$ is *adm*-realizable.

For coNP-hardness, we first give the result for $\sigma_1 \in \{adm, com\}$ and $\sigma_2 = cf$. We use a standard reduction from CNF formulas $\varphi(X) = \bigwedge_{c \in C} c$ with each clause $c \in C$ a disjunction of literals from X to the AF $F_\varphi = (A_\varphi, R_\varphi)$ given as

$$\begin{aligned} A_\varphi &= \{\varphi, \bar{\varphi}\} \cup C \cup X \cup \bar{X} \\ R_\varphi &= \{(c, \varphi) \mid c \in C\} \cup \{(x, \bar{x}), (\bar{x}, x) \mid x \in X\} \cup \\ &\quad \{(x, c) \mid x \text{ occurs in } c\} \cup \{(\bar{x}, c) \mid \neg x \text{ occurs in } c\} \cup \\ &\quad \{(\varphi, \bar{\varphi}), (\bar{\varphi}, \bar{\varphi})\} \cup \{(\bar{\varphi}, x), (\bar{\varphi}, \bar{x}) \mid x \in X\}. \end{aligned}$$

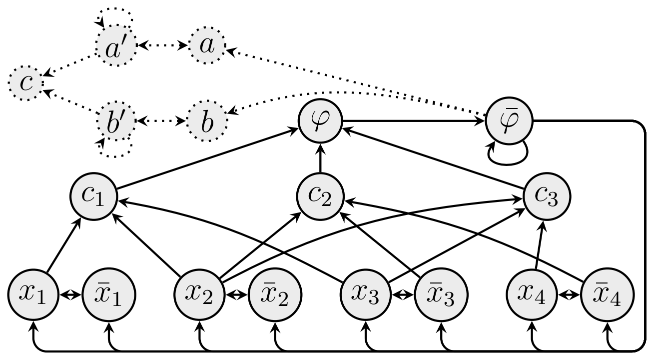

Figure 8: F_φ (without dotted part) and F'_φ (including dotted part) for the CNF-formula $\varphi = (x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee \neg x_4) \wedge (x_2 \vee x_3 \vee x_4)$.

We illustrate the framework F_φ (ignore the dashed part for the moment) for the CNF-formula $\varphi = (x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee \neg x_4) \wedge (x_2 \vee x_3 \vee x_4)$ in Figure 8.

It holds that φ is satisfiable iff there is an $S \neq \emptyset$ in $\sigma_1(F_\varphi)$ iff there is an $S \in \sigma_1(F_\varphi)$ with $\varphi \in S$ (see [17]). Now if $adm(F_\varphi) = com(F_\varphi) = \{\emptyset\}$ then $\sigma_1(F_\varphi)$ is clearly *cf*-realizable. On the other hand $\{\varphi\} \notin adm(F)$ (and thus $\{\varphi\} \notin com(F)$). So if there is an $S \in \sigma_1(F_\varphi)$ with $\varphi \in S$ then $\sigma_1(F_\varphi)$ is not downward-closed and thus not *cf*-realizable (cf. Theorem 5.1). Thus we obtain that φ is satisfiable iff $\sigma_1(F_\varphi)$ is not *cf*-realizable.

The result for $\sigma_1 = com$ and $\sigma_2 = adm$ can be shown via an extension of above reduction to $F'_\varphi = (A'_\varphi, R'_\varphi)$ with

$$\begin{aligned} A'_\varphi &= A_\varphi \cup \{a, a', b, b', c\} \\ R'_\varphi &= R_\varphi \cup \{(a, a'), (a', a), (a', a'), (a', c)\} \cup \\ &\quad \{(b, b'), (b', b), (b', b'), (b', c)\} \cup \{(\bar{\varphi}, a), (\bar{\varphi}, b)\}. \end{aligned}$$

The AF F'_φ for the example CNF is illustrated in Figure 8 including the dashed part. Clearly if φ is not accepted then none of the new arguments is accepted. So if φ is unsatisfiable we again have $com(F'_\varphi) = \{\emptyset\}$ and thus the extension-set is *adm*-realizable. Otherwise, for each model M of φ we have three complete extensions $M \cup \overline{X \setminus M} \cup \{\varphi, a\}$, $M \cup \overline{X \setminus M} \cup \{\varphi, b\}$ and $M \cup \overline{X \setminus M} \cup \{\varphi, a, b, c\}$. Thus, every pair of arguments of the union of the first two extensions $M \cup \overline{X \setminus M} \cup \{\varphi, a, b\}$ is contained in $Pairs_{com(F'_\varphi)}$, but $M \cup \overline{X \setminus M} \cup \{\varphi, a, b\} \notin com(F'_\varphi)$. This violates conflict-sensitivity of $com(F'_\varphi)$. We get that φ is satisfiable iff $com(F'_\varphi)$ is not *adm*-realizable (cf. Theorem 5.1). □

Theorem 6.12 *The complexity results depicted in Table 3 hold.*

The "trivial" results are immediate by the relations between the signatures given in Theorem 5.2 and the remaining entries are by the following lemmas.

Lemma 6.13 *Recasting from stb to $\sigma \in \{stage, pref, sem\}$ is* NP*-complete.*

	naive	*stb*	*stage*	*pref*	*sem*
naive	-	trivial	trivial	trivial	trivial
stb	coNP-h	-	NP-c	NP-c	NP-c
stage	coNP-h	trivial	-	trivial	trivial
pref	coNP-h	Π_2^{P}-c	Π_2^{P}-c	-	trivial
sem	coNP-h	Π_2^{P}-c	Π_2^{P}-c	trivial	-

Table 3: Complexity of the recasting problem (I-maximal semantics).

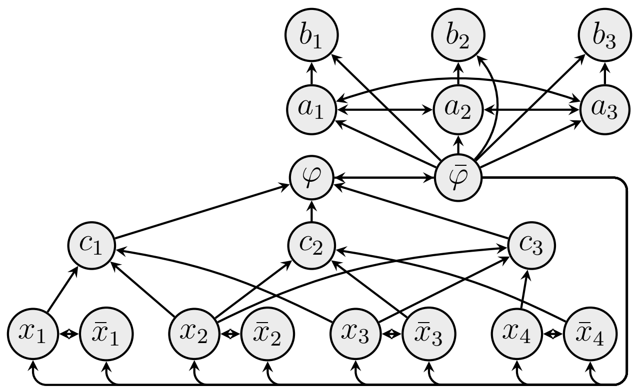

Figure 9: F^*_φ for the CNF-formula $\varphi = (x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee \neg x_4) \wedge (x_2 \vee x_3 \vee x_4)$.

Proof. For the complexity of recasting stable to $\sigma \in \{\mathit{stage}, \mathit{pref}, \mathit{sem}\}$ we use $\Sigma_{\mathit{stb}} = \Sigma_{\mathit{stage}} \cup \{\emptyset\}$, $\Sigma_{\mathit{stage}} \subset \Sigma_{\mathit{sem}} = \Sigma_{\mathit{pref}}$ from Theorem 5.2. Thus, $\mathit{stb}(F) \in \Sigma_\sigma$ iff $\mathit{stb}(F) \neq \emptyset$. Deciding whether an AF has a stable extension is well-known to be NP-complete [17]. □

Lemma 6.14 *Recasting from $\sigma \in \{\mathit{stb}, \mathit{stage}, \mathit{pref}, \mathit{sem}\}$ to naive is* coNP*-hard.*

Proof. We show coNP-hardness of recasting to naive semantics by a reduction from unsatisfiability of CNF formulas. Starting from a formula $\varphi(X) = \bigwedge_{c \in C} c$ with each clause $c \in C$ a disjunction of literals from X, we extend F_φ from the proof of Theorem 6.11 and define $F^*_\varphi = (A^*_\varphi, R^*_\varphi)$ using an additional gadget (borrowed from Figure 1) via

$$\begin{aligned}
A^*_\varphi =& A_\varphi \cup \{a_1, a_2, a_3, b_1, b_2, b_3\}, \text{ and} \\
R^*_\varphi =& \big(R_\varphi \setminus \{(\bar{\varphi}, \bar{\varphi})\}\big) \cup \{(\bar{\varphi}, a_i), (\bar{\varphi}, b_i) \mid 1 \leq i \leq 3\} \cup \{(\bar{\varphi}, \varphi)\} \cup \\
& \{(a_i, a_j), (a_i, b_i) \mid 1 \leq i, j \leq 3, i \neq j\}.
\end{aligned}$$

In Figure 9 we illustrate the framework F^*_φ for the CNF-formula $\varphi = (x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee \neg x_4) \wedge (x_2 \vee x_3 \vee x_4)$.

It is known that φ is satisfiable iff $\{\bar{\varphi}\} \cup C$ is not the unique preferred extension of F^*_φ [19] (the additional part in the AF does not affect the claim due to directionality of preferred semantics [3]). As first $\{\bar{\varphi}\} \cup C$ is also a stable extension and second the preferred extensions corresponding to the models of φ are also stable extensions this result extends to *stb*, *sem* and *stage* semantics. Let $\sigma \in \{\mathit{stb}, \mathit{pref}, \mathit{sem}, \mathit{stage}\}$. If φ is unsatisfiable then $\{\bar{\varphi}\} \cup C$ is the only $S \in \sigma(F^*_\varphi)$, hence $\sigma(F^*_\varphi) \in \Sigma_{\mathit{naive}}$. Otherwise, for each model M of φ, there exist $S_1, S_2, S_3 \in \sigma(F^*_\varphi)$, such that $\mathit{maj}_3(S_1, S_2, S_3) = M \cup \overline{X \setminus M} \cup \{\varphi, b_1, b_2, b_3\}$. That are:

- $S_1 = M \cup \overline{X \setminus M} \cup \{\varphi, a_1, b_2, b_3\}$
- $S_2 = M \cup \overline{X \setminus M} \cup \{\varphi, a_2, b_1, b_3\}$
- $S_3 = M \cup \overline{X \setminus M} \cup \{\varphi, a_3, b_1, b_2\}$

The set $\mathit{maj}_3(S_1, S_2, S_3)$ is neither admissible (b_1, b_2, b_3 are not defended against attacks from a_1, a_2, a_3) nor having full range (missing a_1, a_2 and a_3) and thus is not in $\sigma(F^*_\varphi)$. Moreover, any set $S' \supset \mathit{maj}_3(S_1, S_2, S_3)$ is not conflict-free in F^*_φ, hence also $S' \notin \sigma(F^*_\varphi)$. By Proposition 6.1 and Theorem 5.1, $\sigma(F^*_\varphi) \notin \Sigma_{\mathit{naive}}$. Hence φ is satisfiable iff $\sigma(F^*_\varphi)$ is not *naive*-realizable. □

Lemma 6.15 *Recasting from $\sigma_1 \in \{\mathit{pref}, \mathit{sem}\}$ to $\sigma_2 \in \{\mathit{stb}, \mathit{stage}\}$ is Π^{P}_2-complete.*

Proof. Since $\sigma_1(F) \neq \emptyset$ for any AF F, and $\Sigma_{\mathit{stb}} = \Sigma_{\mathit{stage}} \cup \{\emptyset\}$, we can stick to $\sigma_2 = \mathit{stb}$. Membership is by an algorithm that, given an $F = (A, R)$, disproves $\sigma_1(F) \in \Sigma_{\mathit{stb}}$, that is it looks for counter examples (violating tightness) from $\sigma_1(F)$ as follows: guess sets $S \subseteq A$, $\{A_s \subseteq A \mid s \in S\}$ and $a \in A \setminus S$; check $S \in \sigma_1(F)$ and $S \cup \{a\} \notin \sigma_1(F)$, and for all $s \in S$ check $A_s \in \sigma_1(F)$ and $\{a, s\} \subseteq A_s$ (for all checks, an NP-oracle is sufficient [20, 24]). For sets passing all the checks we have that S and a contradict tightness as $S \cup \{a\} \notin \sigma_1(F)$ but for each $s \in S$, $(a, s) \in \mathit{Pairs}_{\sigma_1(F)}$. By Theorem 5.1, $\sigma_1(F) \notin \Sigma_{\mathit{stb}}$.

We show Π^{P}_2-hardness for $\sigma_1 = \mathit{pref}$ (*pref* semantics can be efficiently reduced to *sem* semantics [25], the result for $\sigma_1 = \mathit{sem}$ thus follows): Given QBF $\Phi = \forall Y \exists Z \varphi(Y, Z)$, where φ is a CNF $\bigwedge_{c \in C} c$ with each c a disjunction of literals from $X = Y \cup Z$, let $F_\Phi = (A_\Phi, R_\Phi)$ with

$$
\begin{aligned}
A_\Phi &= \{\varphi, g\} \cup C \cup X \cup \bar{X} \cup \{a, b, c, d, e, f\}, \text{and} \\
R_\Phi &= \{(c, \varphi) \mid c \in C\} \cup \{(x, \bar{x}), (\bar{x}, x) \mid x \in X\} \cup \\
&\quad \{(x, c) \mid x \text{ occurs in } c\} \cup \{(\bar{x}, c) \mid \neg x \text{ occurs in } c\} \cup \\
&\quad \{(\varphi, g), (g, g)\} \cup \{(g, z), (g, \bar{z}) \mid z \in Z\} \cup \\
&\quad \{(a, d), (d, a), (b, c), (c, b), (c, d), (d, c), (c, f), \\
&\quad (d, f), (f, e), (f, f), (\varphi, f)\}
\end{aligned}
$$

We illustrate F_Φ for the QBF $\Phi = \forall y_1, y_2 \exists z_3, z_4 \big((y_1 \vee y_2 \vee z_3) \wedge (y_2 \vee \neg z_3 \vee \neg z_4) \wedge (y_2 \vee z_3 \vee z_4)\big)$ in Figure 10. In fact, F_Φ links the reduction from [20] with an AF similar to Figure 2.

We show that Φ is valid iff $\mathit{pref}(F_\Phi) \in \Sigma_{\mathit{stb}}$. To this end let F'_Φ be F_Φ without arguments $\{a, b, c, d, e, f\}$ and without attacks involving these arguments (the continuous part in Figure 10). By [20] we know that φ is contained in each $E \in \mathit{pref}(F'_\Phi)$ iff Φ is valid. Therefore it remains to show that $\mathit{pref}(F_\Phi) \in \Sigma_{\mathit{stb}}$ iff φ is contained in all $E \in \mathit{pref}(F'_\Phi)$. First recall that since arguments $\{a, b, c, d, e, f\}$ do not attack the F'_Φ-part we know from the splitting theorem in [5] that each

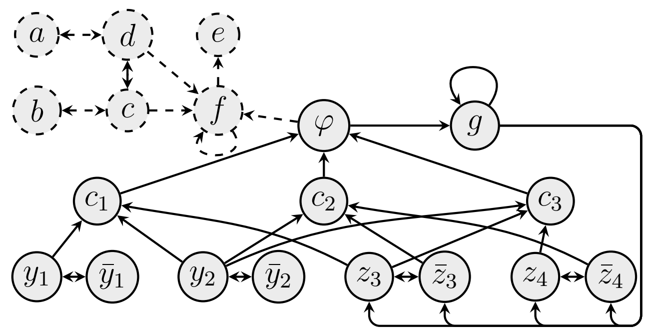

Figure 10: F_Φ for the QBF $\Phi = \forall y_1, y_2 \exists z_3, z_4 \big((y_1 \vee y_2 \vee z_3) \wedge (y_2 \vee \neg z_3 \vee \neg z_4) \wedge (y_2 \vee z_3 \vee z_4)\big)$

preferred extension of F_Φ contains a preferred extension of F'_Φ and each preferred extension of F'_Φ is contained in some preferred extension of F_Φ. Now suppose φ is contained in each $E \in$ *pref*(F'_Φ). Then E defends e in F_Φ and therefore *pref*$(F_\Phi) = \{\{a, b, e\} \cup E, \{a, c, e\} \cup E, \{b, d, e\} \cup E \mid E \in$ *pref*$(F'_\Phi)\}$. Note that for each $E \in F_\Phi$, all arguments of F_Φ not contained in E are attacked by E. In other words, *pref*$(F_\Phi) =$ *stb*(F_Φ), and thus *pref*$(F_\Phi) \in \Sigma_{stb}$. Now let us assume there is a preferred extension E of F'_Φ such that $\varphi \notin E$. Then, the preferred extensions of F_Φ containing E are exactly $\{a, b\} \cup E$ (e is neither defended by E nor by $\{a, b\}$), $\{a, c, e\} \cup E$, and $\{b, d, e\} \cup E$. Generalising the argument from Example 3.12, we observe that *pref*(F_Φ) is not tight. In fact, we have $E \cup \{a, b, e\} \notin$ *pref*(F_Φ) but for all $s \in E \cup \{a, b\}$, we have $(s, e) \in Pairs_{pref(F_\Phi)}$ (as witnessed by the extensions $\{a, c, e\} \cup E$ and $\{b, d, e\} \cup E$). By Theorem 5.1, *pref*$(F_\Phi) \notin \Sigma_{stb}$. $\square$

7 Discussion

In this work, we tackled a novel problem in the area of abstract argumentation. We initiated a study on the characteristics the sets of extensions w.r.t. a given semantics satisfy. For the semantics naive, stable, stage, preferred, and semi-stable we have an exact picture fully describing their signatures. These results also tell about the limits of global disagreement (a notion introduced in [12]) that can be modelled within AFs, e.g. our results show that preferred and semi-stable semantics are able to express more disagreement than stage semantics: $\Sigma_{stage} \subset \Sigma_{pref} = \Sigma_{sem}$.

Besides an exact characterization of the signature of complete semantics, future work includes an investigation of other important semantics, in particular cf2-semantics [4] and resolution-based grounded (RBG) [2]; for the latter semantics recent results [22] show a quite different behavior. More specifically, these results show that the signature of RBG is a proper subset of Σ_{pref} (and thus of Σ_{sem}) but incomparable to Σ_{stb} (resp. to Σ_{stage}). Moreover, one can easily show that it is a proper superset of Σ_{naive}. However, an exact characterization is still open. We also plan an according extension of our complexity analysis. Furthermore, exact bounds for the some of the decision problems considered are still missing for the case of naive extensions. Since we have viewed semantics here only in an extension-based way, it would also be of high interest to extend our studies to labelling-based semantics [14]. Preliminary work in this direction has recently been done by Dyrkolbotn [26].

Another interesting direction is to restrict the concepts of signatures and realizability in such a way that no additional arguments are allowed (recall that we made heavy use of such arguments, for instance in the the canonical defense-argumentation-framework, cf. Definition 4.9). A first step towards this direction has recently been undertaken in [8]. That paper also presents alternative characterizations for some of the signatures we have presented here. Finally, we also want to explore how our results allow for pruning the search space in algorithms for abstract argumentation.

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A Proofs from Section 6.

A.1 Proof of Proposition 6.4.

The proof of Proposition 6.4 is by the following lemmas.

Lemma A.1 *Given a propositional formula φ, it holds that deciding if $\mathbb{S}_\varphi$ is non-empty is* NP-*complete.*

Proof. Trivial reduction from SAT. Given an instance, φ of SAT, if $\mathbb{S}_\varphi = \emptyset$ then $f_\varphi \equiv \bot$, i.e. $\varphi(X)$ is unsatisfiable. Hence $\varphi(X)$ encodes a non-empty set if and and only if $\varphi(X)$ is satisfiable. □

Lemma A.2 *Given a propositional formula φ, it holds that deciding if $\mathbb{S}_\varphi$ is downward-closed is* coNP-*complete.*

Proof. Let $\varphi(X)$ be any propositional formula with $\mathbb{S}_\varphi$ the system of subsets of *Args*$_{\mathbb{S}}$ it encodes. For membership in coNP, note that in order for $\mathbb{S}_\varphi$ to be downward-closed, f_φ must be *anti-monotone*, i.e.

$$\forall\, x_i \text{ and assignments } \alpha \text{ to } \langle x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\rangle$$

$f_\varphi(\alpha, x_i := \top) \leq f_\varphi(\alpha, x_i := \bot)$ (where the ordering $\bot < \top$ is assumed).[6] Thus $\mathbb{S}_\varphi$ is *not* downward-closed system if and only if

$$\exists(\alpha \in \langle \bot, \top\rangle^n, i) : (\varphi(\alpha // x_i := \top) \wedge (\neg\varphi(\alpha // x_i := \bot))) = \top$$

Here $(\alpha // x_i := a)$ denotes the assignment obtained from α by substituting a for the value of x_i and leaving other values unchanged. The test that $\mathbb{S}_\varphi$ is not downward-closed is easily seen to be achievable by an NP algorithm, thus verifying that $\mathbb{S}_\varphi$ *is* downward-closed can be decided in coNP.

To establish coNP-hardness, we use a straightforward reduction from SAT to the complementary problem. Given an instance $\psi(X)$ of SAT let $\varphi(X, y)$ be the propositional formula $y \wedge \psi(X)$ where y is a new variable. We claim $\psi(X)$ is satisfiable if and only if $\varphi(X, y)$ does not define an anti-monotone function. Suppose that $\psi(X)$ is satisfied by α. Then $\varphi(\alpha, y) \equiv y$ which fails to be anti-monotone. Conversely suppose $\varphi(X, y)$ is not anti-monotone this being witnessed by some pair $\langle \beta, i\rangle$. We must have $\beta_y = \top$ in such an assignment (otherwise $\varphi(\beta) = \bot$). But then since $\varphi(\beta // x_i := \top) = \top$ and $\beta_y = \top$ it follows that the projection of β on X yields an assignment, α, satisfying $\psi(X)$. □

Lemma A.3 *Given a propositional formula φ, it holds that deciding if $\mathbb{S}_\varphi$ is incomparable is* coNP-*complete.*

[6] It is well-known that $f(X)$ is an anti-monotone propositional function if and only if it can be represented using a propositional formula, $\varphi(X)$ over the binary basis $\{\wedge, \vee\}$ in which only negated literals are used.

Proof. Let $\varphi(X)$ be any propositional formula with $\mathbb{S}_\varphi$ the system of subsets of $Args_{\mathbb{S}}$ it encodes. Again we proceed by showing the complementary problem is decidable by an NP algorithm. Given an assignment, α to X, let $S_\alpha \subseteq Args_{\mathbb{S}}$ be the set $\{ x_i \ : \ \alpha_i = \top\}$. Similarly, α_S denotes the assignment in which $\langle \alpha_i = \top \ : \ x_i \in S\rangle \cup \langle \alpha_i = \bot \ : \ x_i \notin S\rangle$. In order for $f_\varphi(X)$ to describe a system containing comparable sets there must be assignments, α and β, for which

$$\varphi(\alpha) \wedge \varphi(\beta) \wedge (S_\alpha \subset S_\beta)$$

Checking any given pair (α, β) is polynomial time decidable, hence comparability is in NP as claimed.

To show that deciding comparability of $\varphi(X)$ is NP-hard, we use a reduction from SAT. Given an instance $\psi(X)$ of SAT, form $\varphi(X, y, z)$ as an instance of comparability with

$$\varphi(X, y, z) \ = \ \psi(X) \ \wedge \ \left(\bigwedge_{x \in X} (x \rightarrow (y \vee z)) \right)$$

Here y and z are new variables.

Suppose that $\psi(X)$ is satisfiable using α and let $S_\alpha \subseteq X$ be the corresponding subset of X indicated by α. Then,

$$T \ = \ S_\alpha \cup \{y\} \in \mathbb{S}_\varphi$$

and

$$U \ = \ S_\alpha \cup \{y, z\} \in \mathbb{S}_\varphi$$

Clearly $T \subset U$ so that $\varphi(X, y, z)$ is comparable.

Conversely suppose $\varphi(X, y, z)$ is comparable and this property is witnessed by sets S_α, $S_\beta \subseteq X \cup \{y, z\}$ arising from assignments, α and β. Let α^X be the projection of α onto X, then from $S_\alpha \in \mathbb{S}_\varphi$

$$\varphi(\alpha^X, \alpha_y, \alpha_z) \ = \ \psi(\alpha^X) \ \wedge \ \left(\bigwedge_{i=1}^{n} (\alpha_i \rightarrow (\alpha_y \vee \alpha_z)) \right) \ = \ \top$$

so that $\psi(\alpha^X) = \top$, i.e $\psi(X)$ is satisfiable as witnessed by the assignment α_X. We deduce that deciding if $\varphi(X)$ describes an incomparable function is coNP-complete. □

Lemma A.4 *Given a propositional formula φ, it holds that deciding if $\mathbb{S}_\varphi$ is tight is* coNP-*complete.*

Proof. Let $\varphi(X)$ be any propositional formula with $\mathbb{S}_\varphi$ the system of subsets of $Args_{\mathbb{S}}$ it encodes. Again we consider the complementary problem, calling $\mathbb{S}$ a *loose* set whenever

$$\exists (S, x) \ : \ ((S \in \mathbb{S}) \ \wedge \ (S \cup \{x\} \notin \mathbb{S})) \rightarrow (\forall \ s \in S \ : \ (x, s) \in Pairs_{\mathbb{S}})$$

We may rewrite this as,

$$\exists (S, x, T_1, T_2, \ldots, T_n) \; : \; (S \notin \mathbb{S}) \;\vee\; (S \cup \{x\} \in \mathbb{S}) \;\vee\; \left(\bigwedge_{x_i \in S} \{x, x_i\} \cup T_i \in \mathbb{S} \right)$$

Corresponding to the polynomial time test on

$$\neg\varphi(\alpha_S) \vee \varphi(\alpha_{S\cup\{x\}}) \vee \left(\bigwedge_{x_i \in S} \varphi(\alpha_{T_i \cup \{x, x_i\}}) \right) \;=\; \top$$

Hence whether $\varphi(X)$ describes a tight function can be accomplished with coNP.

To show it is also coNP-hard we present a reduction from the following variant of UNSAT

$$\text{UNSAT}^{+} \;=\; \{\, \psi(X) \;:\; \forall\, \alpha\, \psi(\alpha) = \top \;\Rightarrow\; \alpha = \langle \top, \top, \ldots, \top \rangle \}$$

By treating an instance, $\psi(X)$ of UNSAT as the instance $(\neg y) \wedge \psi(X)$ of UNSAT^{+} it is easy to see that UNSAT^{+} is coNP-complete.

We now give a reduction from UNSAT^{+} to deciding tightness, showing coNP-completeness of the latter.

Let $\psi(X)$ be an instance of UNSAT^{+}. Without loss of generality we assume that $\psi(X)$ is such that $|X| \geq 2$.

Consider the instance, $\varphi(X, y)$ of deciding whether $\mathbb{S}_\varphi$ is tight (in which y is a new variable not occurring in X) given by,

$$\varphi(X, y) \;=\; \left[\left(\bigvee_{x \in X} \neg x \right) \;\wedge\; \psi(X) \right] \;\oplus\; y$$

Here $\oplus$ is the Boolean *exclusive-or* operator for which $x \oplus y = \top$ if and only if the values of x and y differ.[7]

Suppose that $\psi(X)$ is a positive instance of UNSAT^{+}. In this case,

$$\mathbb{S}_\varphi \;=\; \{\; \{ y \cup T \;:\; T \subseteq X \} \;\}$$

That $\mathbb{S}_\varphi$ is tight is immediate from the fact $\{y\} \in \mathbb{S}_\varphi$ (the case $T = \emptyset$) and

$$\forall\, S \in \mathbb{S}_\varphi,\; x \in X \;\; S \cup \{x\} \in \mathbb{S}_\varphi$$

Notice that in the event of $\psi(\langle \top, \ldots, \top \rangle) = \top$, $\varphi(\langle \top, \ldots \top \rangle, y) \equiv y$. Thus $\psi(X) \in$ UNSAT^{+} implies that $\varphi(X, y)$ describes a tight set $\mathbb{S}_\varphi$.

[7]For $|X| = 1$, instances of UNSAT^{+} are either equivalent to constant functions – $\bot$ or $\top$ (eg. $x \vee \neg x$) or one of the functions $\{x, \neg x\}$. Thus $\varphi(x, y)$ is one of $\{y, (\neg x) \oplus y\}$. These respectively give $\mathbb{S}_\varphi$ as $\{\{y\}, \{x, y\}\}$, or $\{\emptyset, \{x, y\}\}$. The first of these is tight and the source cases for $\psi(X)$ – $\psi(X) = \bot$, $\psi(X) = x$ – are positive instances of UNSAT^{+}. The second case, $\{\emptyset, \{x, y\}\}$ fails to be tight (neither $\{x\} = \emptyset \cup \{x\}$ nor $\{y\} = \emptyset \cup \{y\}$ belong to $\mathbb{S}_\varphi$), however neither of the possible sources, $\psi(x) = \top$, $\psi(x) = \neg x$ are positive instances of UNSAT^{+}.

For the converse direction suppose that $\psi(X)$ is a negative instance of UNSAT$^+$ and $M \subset X$ is a model of $\psi(X)$. Then M is also a model of $\varphi(X, y)$, i.e. $M \in \mathbb{S}_\varphi$, and thus $M \cup \{y\} \notin \mathbb{S}_\varphi$. But as still $X \cup \{y\} \in \mathbb{S}_\varphi$ we have that $(y, s) \in \mathit{Pairs}_{\mathbb{S}_\varphi}$ for each $s \in M$. Hence, M is a loose set for $\mathbb{S}_\varphi$.

We deduce that $\psi(X)$ is accepted as an instance of UNSAT$^+$ if and only if $\mathbb{S}_\varphi$ is tight and, hence, deciding tightness of $\mathbb{S}_\varphi$ is coNP-complete. □

Lemma A.5 *Given a propositional formula φ, it holds that deciding if $\mathbb{S}_\varphi$ is conflict-sensitive is* coNP-*complete.*

Proof. Let $\varphi(X)$ be any propositional formula with $\mathbb{S}_\varphi$ the system of subsets of $\mathit{Args}_\mathbb{S}$ it encodes. Again dealing with the complementary problem, we call $\mathbb{S}$ *conflict-insensitive* when there are distinct sets, S and T in $\mathbb{S}$ for which $S \cup T \notin \mathbb{S}$ but every $\{x, y\} \subseteq S \cup T$ satisfies $(x, y) \in \mathit{Pairs}_\mathbb{S}$. Given $\varphi(X)$, the corresponding function, f_φ is conflict-insensitive if there exist $(S, T, U_{1,2}, U_{1,3}, \ldots, U_{n-1,n})$ for which

$$\varphi(\alpha_S) \wedge \varphi(\alpha_T) \wedge \neg\varphi(\alpha_{S\cup T}) \wedge \left(\bigwedge_{x_i \in S} \bigwedge_{x_j \in T} \varphi(\alpha_{U_{i,j} \cup \{x_i, x_j\}}) \right) = \top$$

The test described being polynomial time computable we deduce that $\varphi(X)$ defining a conflict-insensitive system is decidable in coNP.

To show it is also coNP-hard we present a reduction from UNSAT$^+$ (cf. proof of Lemma A.4). Let $\psi(X)$ be an instance of UNSAT$^+$. Without loss of generality we assume that $\psi(X)$ is such that $|X| \geq 2$.

Consider the instance, $\varphi(X, y, z)$ of the problem of deciding whether $\mathbb{S}_\varphi$ is conflict-sensitive (in which y, z are new variables not occurring in X) given by,

$$\varphi(X, y, z) = (\psi(X) \wedge \neg y \wedge z) \vee \left(y \wedge z \wedge \bigwedge_{x \in X} x \right) \vee \left(y \wedge \neg z \wedge \bigwedge_{x \in X} \neg x \right)$$

Suppose that $\psi(X)$ is a positive instance of UNSAT$^+$. In this case either, (i) $\mathbb{S}_\varphi = \{X \cup \{z\}, X \cup \{y, z\}, \{y\}\}$ if X is model of $\psi(X)$ or (ii) $\mathbb{S}_\varphi = \{X \cup \{y, z\}, \{y\}\}$ otherwise. In both cases $\mathbb{S}_\varphi$ is conflict-sensitive.

For the converse direction suppose that $\psi(X)$ is a negative instance of UNSAT$^+$ and $M \subset X$ is a model of $\psi(X)$. In this case we have $M \cup \{z\} \in \mathbb{S}_\varphi$, $\{y\} \in \mathbb{S}_\varphi$ and that $(M \cup \{z\}) \cup \{y\} \notin \mathbb{S}_\varphi$. Now as $X \cup \{y, z\} \in \mathbb{S}_\varphi$ we also have that $(y, s) \in \mathit{Pairs}_{\mathbb{S}_\varphi}$ for each $s \in M \cup \{z\}$, and thus $\mathbb{S}_\varphi$ is not conflict-sensitive.

We deduce that $\psi(X)$ is accepted as an instance of UNSAT$^+$ if and only if $\mathbb{S}_\psi$ is conflict-sensitive. Hence, deciding whether the models of a propositional formula are conflict-sensitive is coNP-complete. □

A.2 Proof of Theorem 6.9

First we rephrase a theorem from Chang and Kadin [15] that will be the basis for the proof.

Theorem A.6 ([15]) *A problem A is* DP*-hard iff all of the following hold:*

1. *A is* NP*-hard.*
2. *A is* coNP*-hard.*
3. *A has AND_2.*

We say a problem A has AND_2 if the problem of deciding whether two instances of A are both true can be reduced to a single instance of A.

This gives the structure for the following proof of Theorem 6.9 (with $\sigma \in \{\mathit{pref}, \mathit{stage}, \mathit{sem}\}$. In Lemma A.7 we show that σ-realizability is coNP-hard (here, we also show the result for $\sigma = \mathit{naive}$) In Lemma A.8 we show that σ-realizability is NP-hard. Finally, in Lemma A.9 we show that σ-realizability has the property AND_2,

Lemma A.7 *The σ-realizability problem in which instances are propositional formulas, $\varphi(X)$, is* coNP*-hard for $\sigma \in \{\mathit{pref}, \mathit{stage}, \mathit{sem}, \mathit{naive}\}$.*

Proof. To show that deciding σ-realizability of $\varphi(X)$ is coNP-hard, we use a reduction from UNSAT. Given an instance $\psi(X)$ of SAT, form $\varphi(X, y, z)$ as an instance of σ-realizability with

$$\varphi(X, y, z) \;=\; \left(\psi(X) \;\wedge\; \left(\bigwedge_{x \in X} (x \rightarrow (y \vee z))\right)\right) \vee (y \wedge z \wedge \bigwedge_{x \in X} \neg x)$$

Here y and z are new variables. We claim $\psi(X)$ is satisfiable if and only if $\mathbb{S}_\varphi$ is not σ-realizable.

Suppose that $\psi(X)$ is satisfiable using α and let $S_\alpha \subseteq X$ be the corresponding subset of X indicated by α. Then,

$$T \;=\; S_\alpha \cup \{y\} \in \mathbb{S}_\varphi$$

and

$$U \;=\; S_\alpha \cup \{y, z\} \in \mathbb{S}_\varphi$$

Clearly $T \subset U$ so that $\varphi(X, y, z)$ is comparable. and thus $\mathbb{S}_\varphi$ is not σ-realizable.

Conversely suppose $\psi(X)$ is unsatisfiable. Then $\{y, z\}$ is the only model of $\varphi(X, y, z)$, i.e. $\mathbb{S}_\varphi = \{\{y, z\}\}$. Thus, $\mathbb{S}_\varphi$ is incomparable, tight, conflict-sensitive and also $\mathit{dcl}(\mathbb{S}_\varphi)$ is tight. Hence, $\mathbb{S}_\varphi$ is σ-realizable for each $\sigma \in \{\mathit{pref}, \mathit{stage}, \mathit{sem}, \mathit{naive}\}$ (cf. Theorem 5.1).

We deduce that deciding if $\mathbb{S}_\varphi$ is σ-realizable is coNP-hard. $\square$

Lemma A.8 *The σ-realizability problem in which instances are propositional formulas, $\varphi(X)$, is* NP*-hard for $\sigma \in \{pref, stage, sem\}$.*

Proof. To show that deciding σ-realizability of $\varphi(X)$ is NP-hard, we use a reduction from SAT. Given an instance $\psi(X)$ of SAT, form $\varphi(X, y, z)$ as an instance of the σ-realizability problem with

$$\varphi(X, \bar{X}) \;=\; \psi(X) \;\wedge\; \bigwedge_{x \in X} (x \oplus \bar{x})$$

Here $\bar{X}$ is a set of new variables $\bar{x}$, one for each $x \in X$. We claim $\psi(X)$ is satisfiable if and only if $\mathbb{S}_\varphi$ is σ-realizable.

Suppose that $\psi(X)$ is satisfiable. We have to show that $\mathbb{S}_\varphi$ is (i) incomparable, (ii) tight, and (iii) conflict-sensitive (cf. Theorem 5.1).

The set $\mathbb{S}_\varphi$ can be characterized by $\mathbb{S}_\varphi = \{M \cup \overline{X \setminus M} \mid M \text{ is model of } \psi(X)\}$, that is each model contains either x or $\bar{x}$ for each $x \in X$. This implies (i) that all the sets in $\mathbb{S}_\varphi$ are pairwise incomparable.

For (ii) notice that $x, \bar{x}$ never appear together in a model of φ and thus $(x, \bar{x}) \notin \mathit{Pairs}_{\mathbb{S}_\varphi}$. Hence whenever we have an $S \in \mathbb{S}_\varphi$ and an $x \in X$ ($\bar{x} \in \bar{X}$) such that $S \cup \{x\} \notin \mathbb{S}_\varphi$ ($S \cup \{\bar{x}\} \notin \mathbb{S}_\varphi$) we have that $\bar{x} \in S$ ($x \in S$) and $(x, \bar{x}) \notin \mathit{Pairs}_{\mathbb{S}_\varphi}$. Hence, $\mathbb{S}_\varphi$ is tight. (iii) Consider $M, M' \in \mathbb{S}_\varphi, M \neq M'$. Then w.l.o.g there is an $x \in M$ such that $x \notin M'$ but $\bar{x} \in M'$. By the above observation $(x, \bar{x}) \notin \mathit{Pairs}_{\mathbb{S}_\varphi}$ and thus $\mathbb{S}_\varphi$ is conflict-sensitive.

We obtain that $\mathbb{S}_\varphi$ is σ-realizable for $\sigma \in \{pref, stage, sem\}$.

Conversely suppose $\psi(X)$ is unsatisfiable. Then $\varphi(X, \bar{X})$ is also unsatisfiable, i.e. $\mathbb{S}_\varphi = \emptyset$, and thus $\mathbb{S}_\varphi$ is not σ-realizable for $\sigma \in \{pref, stage, sem\}$.

We deduce that deciding if $\mathbb{S}_\varphi$ is σ-realizable is NP-hard. $\square$

Lemma A.9 *The σ-realizability problem in which instances are propositional formulas, $\varphi(X)$, has AND_2 for $\sigma \in \{pref, stage, sem\}$.*

Proof. Consider two instances $\psi(X), \psi'(Y)$ of the σ-realizability and w.l.o.g. assume that $X \cap Y = \emptyset$ (otherwise rename variables). Now consider the formula $\varphi(X, Y)$ defined as follows.

$$\varphi(X, Y) \;=\; \left(\psi(X) \wedge \bigwedge_{y \in Y} \neg y\right) \vee \left(\psi'(Y) \wedge \bigwedge_{x \in X} \neg x\right)$$

Then we have that $\mathbb{S}_\varphi = \mathbb{S}_\psi \cup \mathbb{S}_{\psi'}$, where each set $S \in \mathbb{S}_\varphi$ either is as subset of X or an subset of Y.

We have to show that $\mathbb{S}_\varphi$ is incomparable (tight, conflict-sensitive) iff both $\mathbb{S}_\psi$ and $\mathbb{S}_{\psi'}$ are incomparable (tight, conflict-sensitive).

- Incomparable: Assume $\mathbb{S}_\varphi$ is comparable, i.e. there are $A, B \in \mathbb{S}_\varphi$ with $A \subset B$. Either both A, B are subsets of X and thus contained in $\mathbb{S}_\psi$ or both are subsets of Y and thus contained in $\mathbb{S}_{\psi'}$. In the former case $\mathbb{S}_\psi$ is comparable in the latter $\mathbb{S}_{\psi'}$ is comparable. For the converse consider that one of $\mathbb{S}_\psi$, $\mathbb{S}_{\psi'}$ is comparable. W.l.o.g. assume there are $A, B \in \mathbb{S}_\psi$ such that $A \subset B$. Then also $A, B \in \mathbb{S}_\varphi$ and thus $\mathbb{S}_\varphi$ is comparable.

- Tight: Assume $\mathbb{S}_\varphi$ is not tight. That is there an $S \in \mathbb{S}_\varphi$ and an $x \in X \cup Y$ such that $S \cup \{x\} \notin \mathbb{S}_\varphi$ and $(x, s) \in$ ***Pairs***$_{\mathbb{S}_\varphi}$ for each $s \in S$. W.lo.g. let us assume that $S \subset X$, then also $x \in X$ as by construction $(x, y) \notin$ ***Pairs***$_{\mathbb{S}_\varphi}$ for $x \in X, y \in Y$. But then also $S \cup \{x\} \notin \mathbb{S}_\psi$ and $(x, s) \in$ ***Pairs***$_{\mathbb{S}_\psi}$ for each $s \in S$ and thus $\mathbb{S}_\psi$ is not tight. For the converse consider that one of $\mathbb{S}_\psi$, $\mathbb{S}_{\psi'}$ is not tight. W.l.o.g. assume there are $S \in \mathbb{S}_\psi$ and $x \in X$ such that $S \cup \{x\} \notin \mathbb{S}_\psi$ and $(x, s) \in$ ***Pairs***$_{\mathbb{S}_\psi}$. But then also $S \cup \{x\} \notin \mathbb{S}_\varphi$ and $(x, s) \in$ ***Pairs***$_{\mathbb{S}_\varphi}$ for each $s \in S$ and thus $\mathbb{S}_\varphi$ is not tight.

- Conflict-sensitive: Assume $\mathbb{S}_\varphi$ is not conflict-sensitive. That is there are $A, B \in \mathbb{S}_\varphi$ such that $A \cup B \notin \mathbb{S}_\varphi$ and $(a, b) \in$ ***Pairs***$_{\mathbb{S}_\varphi}$ for all $a \in A$, $b \in B$. As by construction $(x, y) \notin$ ***Pairs***$_{\mathbb{S}_\varphi}$ for $x \in X, y \in Y$ we have that either $A, B \subseteq X$ or $A, B \subseteq Y$. W.l.o.g. we assume the former to hold. Then we have that $A, B \in \mathbb{S}_\psi$, $A \cup B \notin \mathbb{S}_\psi$, and $(a, b) \in$ ***Pairs***$_{\mathbb{S}_\psi}$ for all $a \in A$, $b \in B$. Hence, $\mathbb{S}_\psi$ is not conflict-sensitive. For the converse consider that one of $\mathbb{S}_\psi$, $\mathbb{S}_{\psi'}$ is not conflict-sensitive. W.l.o.g. assume there are $A, B \in \mathbb{S}_\psi$ such that $A \cup B \notin \mathbb{S}_\psi$ and $(a, b) \in$ ***Pairs***$_{\mathbb{S}_\psi}$ for all $a \in A$, $b \in B$. Then also $A, B \in \mathbb{S}_\varphi$, $A \cup B \notin \mathbb{S}_\varphi$ and $(a, b) \in$ ***Pairs***$_{\mathbb{S}_\varphi}$ for all $a \in A$, $b \in B$ and thus $\mathbb{S}_\varphi$ is not conflict-sensitive.

Consequently, we have $\varphi(X, Y)$ is σ-realizable iff both $\psi(X)$ and $\psi'(Y)$ are σ-realizable and therefore the σ-realizability problem has AND_2. $\square$

Finally, by the above lemmas and Theorem A.6, we have that the σ-realizability problem in which instances are propositional formulas is DP-hard for $\sigma \in \{$*pref*, *stage*, *sem*$\}$.