The complexity of graph Ramsey achievement and avoidance games

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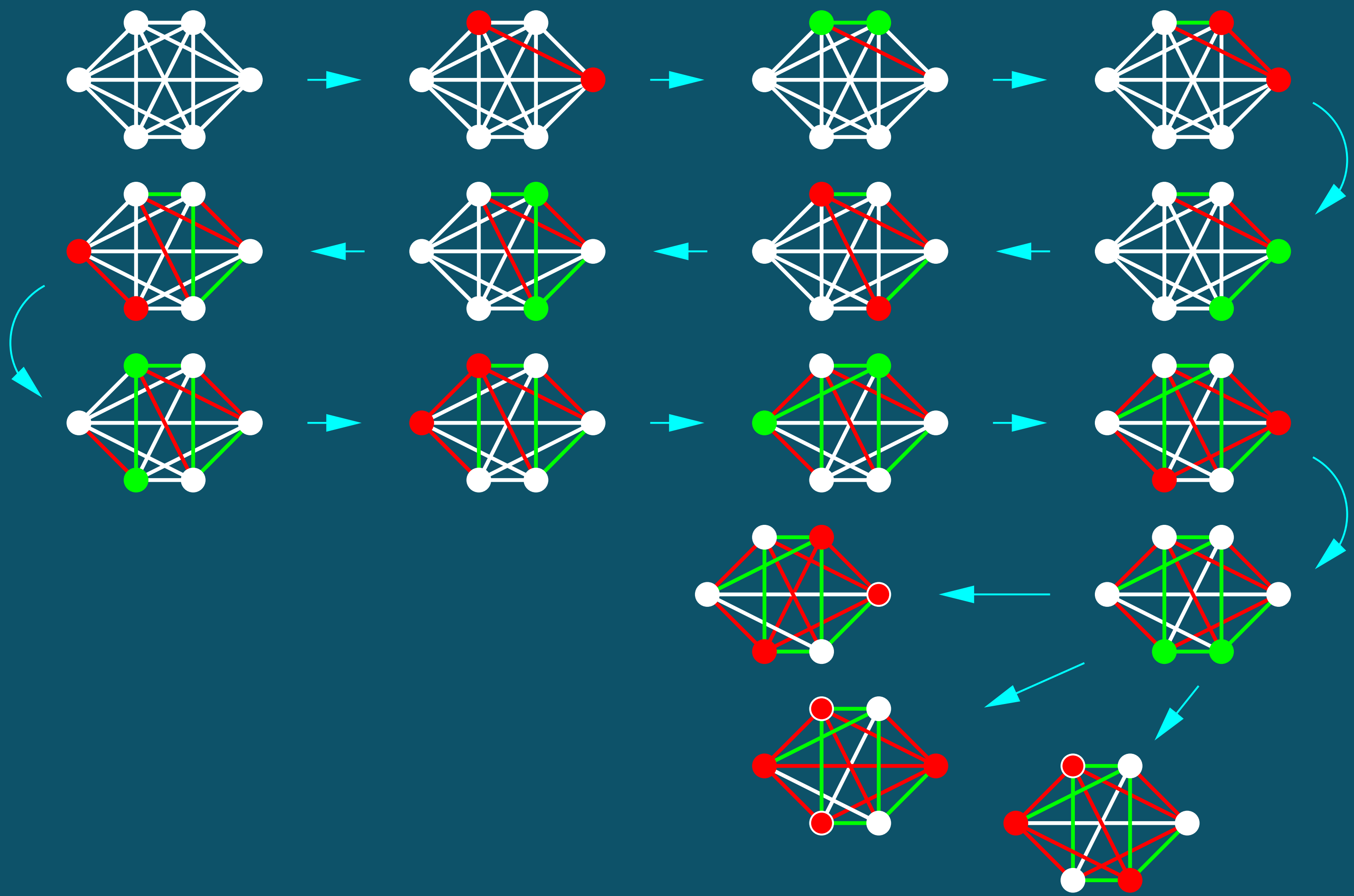
Abstract

Given two graphs G and A, two players, Red and Green, alternate in coloring the edges of G in their respective color. Aim is to avoid (achieve) to build a monochromatic subgraph isomorphic to A. How difficult are these games?

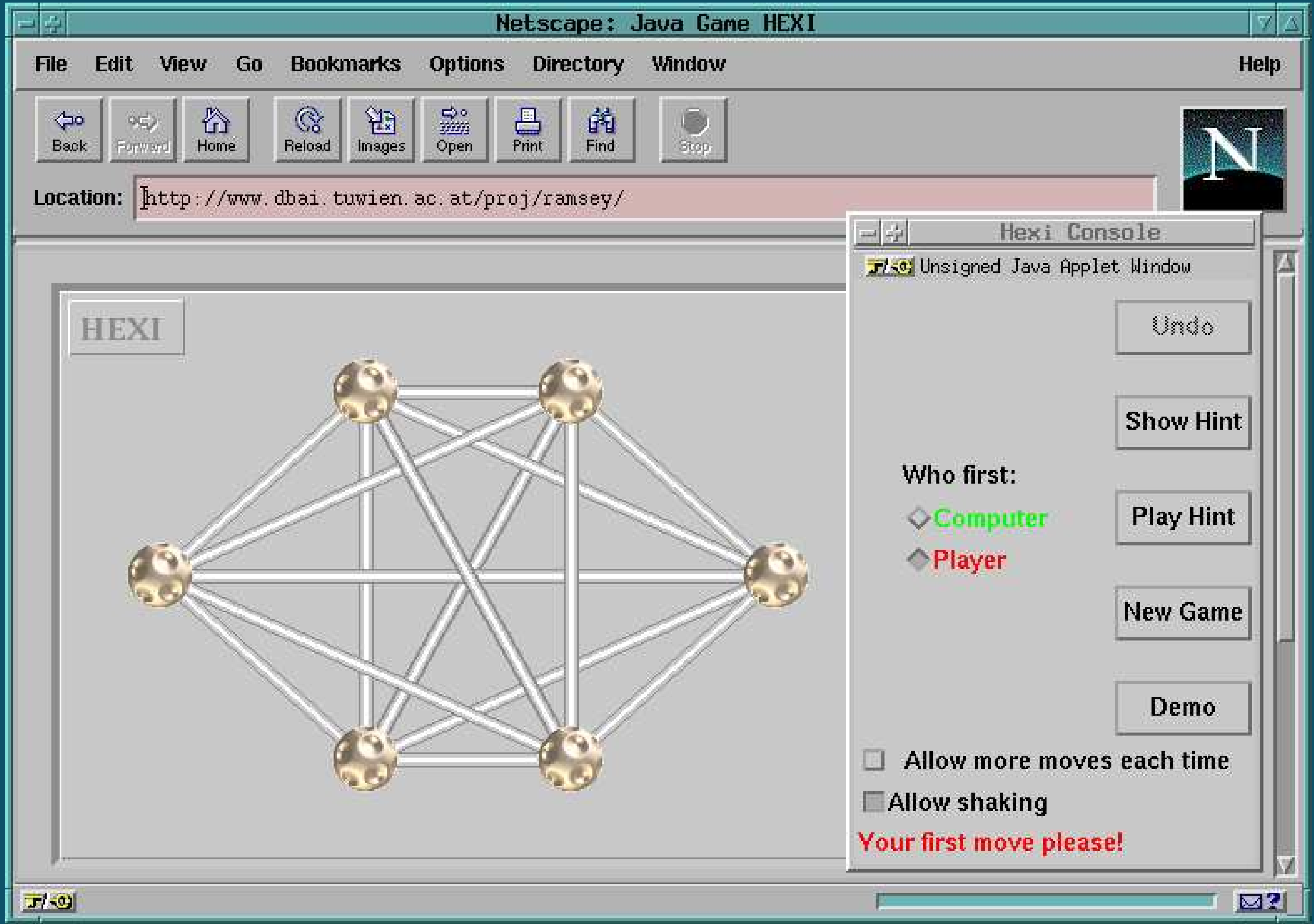
Overview

- In medias res: Let's play ...
- Complexity of the graph Ramsey games
- Ultra-strongly solving Sim and Sim$^+$
- About the unlikeliness of solving Sim$_4$ etc.
- The "difficulty" of combinatorial games revisited
- Some conjectures
- Further remarks

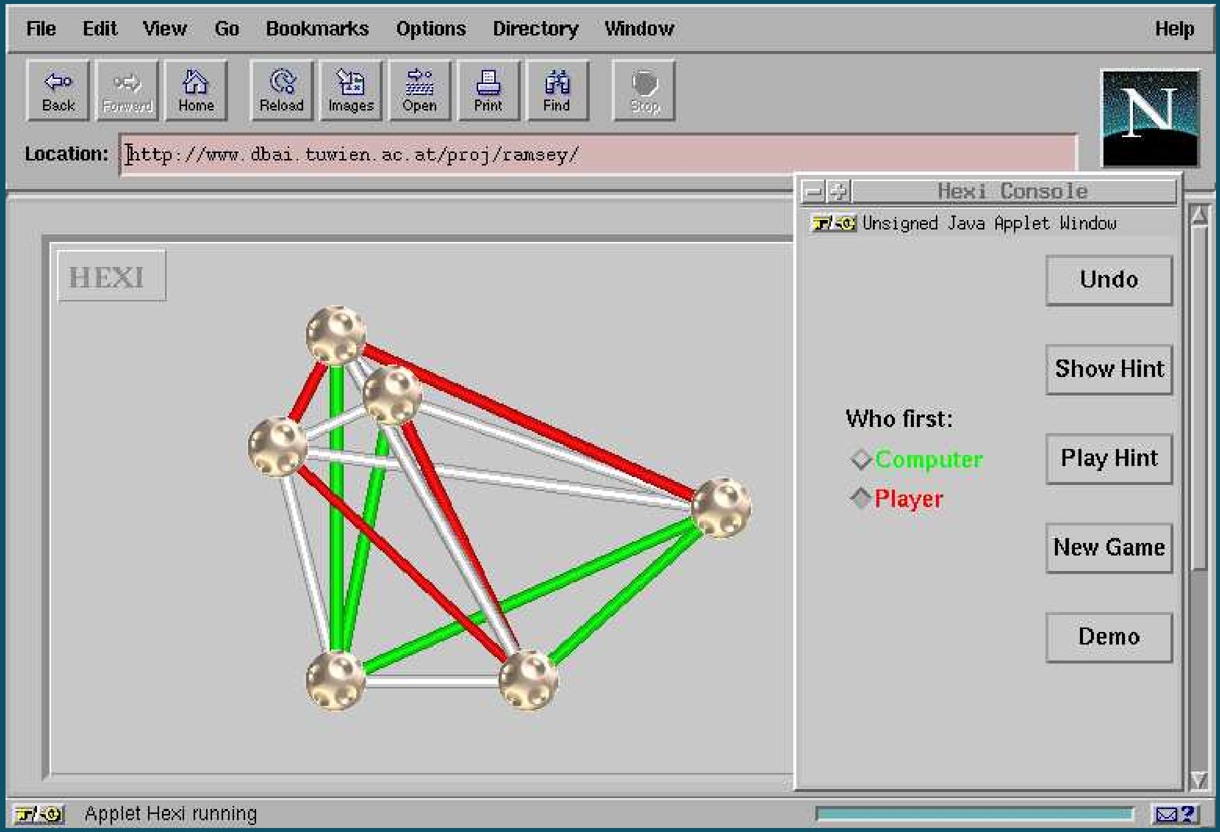
Sim: $G = K_{\text{Ramsey}(3,3)} = K_6,\ A = K_3$ on $G_{\text{Avoid-Ramsey}}$



Considering that a hands-on session with an interactive system often is worth more than a thousand images:



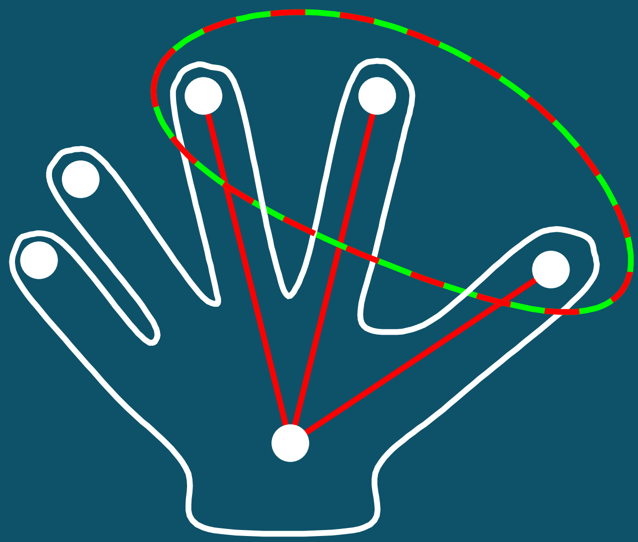
... with random permutations between moves:

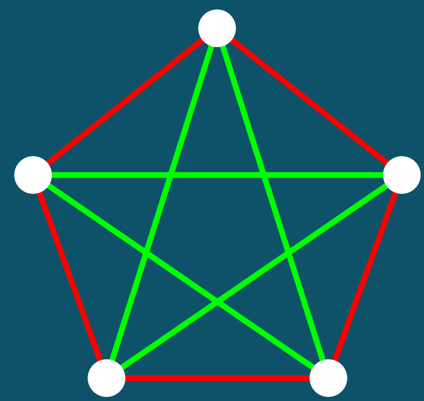


This Java applet plays Sim and a variant, Sim$^+$. In case you win, you will be allowed to leave your name in our hall-of-fame!

Sim and Sim$^+$ can never end in a tie:

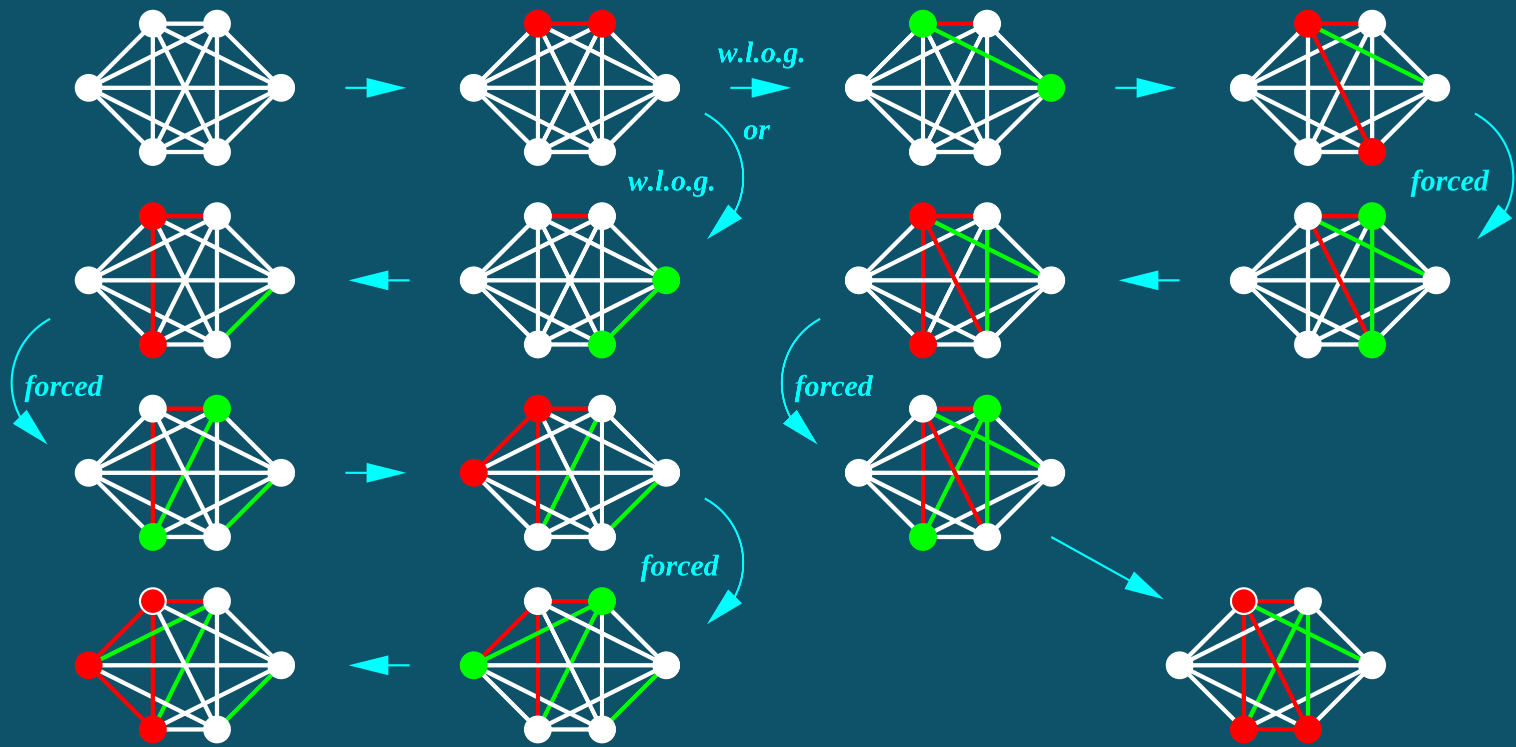
Ramsey(3,3)=6





(visual proof by courtesy of Ranan Banerji)

A winning strategy for the $G_{\text{Achieve-Ramsey}}$ game Sim$_a$:



No simple winning strategies are known for Sim and Sim$^+$.

$\Rightarrow$ **Natural question: How "difficult" is a game?**

Translation to complexity theory:

How does the function bounding the computational resources that are needed in the worst case to determine a winning strategy for the first player grow in relation to the size of the game description?

Typical results: Generalizations of well-known games such as Chess, Checkers, and Go to boards of size $n \times n$ have been classified as polynomial space and exponential time complete [20, 21, 39].

Note: **P** $\subseteq$ **NP** $\subseteq$ **PSPACE** $\subseteq$ **EXPTIME**

How to generalize Sim to game boards of arbitrary size?

$\Rightarrow$ **Graph Ramsey theory**

Definition 1 $G \to A$:
We say that a graph G arrows *a graph A if every edge-coloring of G with colors red and green contains a monochromatic subgraph isomorphic to A. G is called a* Ramsey graph *of A.*

Theorem 1 [10, 17, 52] *Every graph has Ramsey graphs.*

Theorem 2 [54]
Deciding $G \to A$ for given G and A is Π_2^{P}-complete.

Generalizing Sim to graph Ramsey theory leads to:

Definition 2 $G_{\textit{Avoid-Ramsey}}(G, A, E^r, E^g)$*:* *Given two graphs $G = (V, E)$ and A and two non-intersecting sets $E^r \cup E^g \subseteq E$ that contain edges initially colored in red and green, respectively. Two players, Red and Green, take turns in selecting at each move one so-far uncolored edge from E and color it in red for player Red respectively in green for player Green. However, both players are forbidden to choose an edge such that A becomes isomorphic to a subgraph of the red or the green part of G. It is Red's turn. The first player who cannot move loses.*

Similar definitions of $G_{\text{Avoid-Ramsey}^+}$ and $G_{\text{Achieve-Ramsey}}$.

Main complexity results

Theorem 3 $G_{\textit{Avoid-Ramsey}}$ *is* **PSPACE**-*complete.*

Theorem 4 $G_{\textit{Avoid-Ramsey}^{+}}$ *is* **PSPACE**-*complete.*

And, surprisingly,

Theorem 5 $G_{\textit{Achieve-Ramsey}}$ *is* **PSPACE**-*complete.*

Significance: These games thus are as difficult as other well-known difficult games such as Go, and at least as difficult as any **NP**-complete problem.

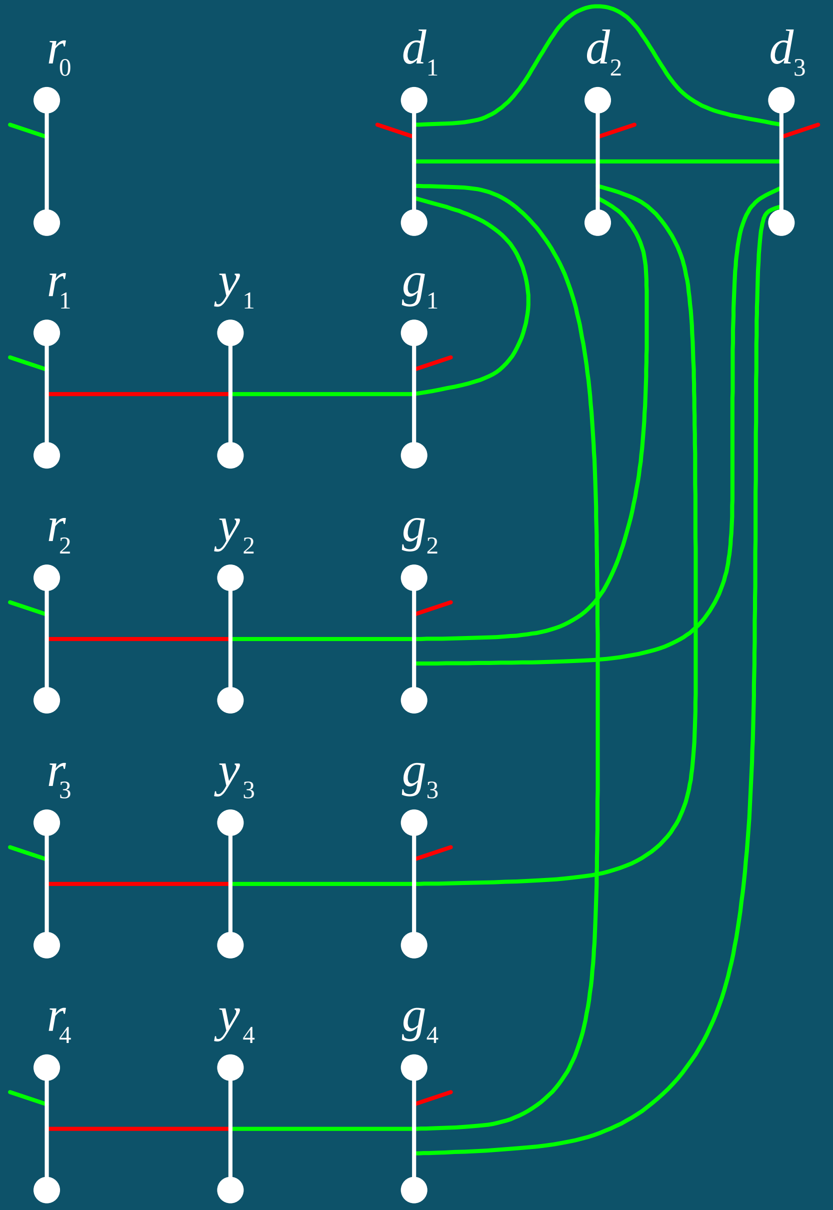
Proof sketch of Theorem 3

- Membership in **PSPACE**: easy.
- Hardness: via a **LOGSPACE** reduction from the **PSPACE**-complete game $G_{\text{Achieve-POS-CNF}}$ [55]:

Definition 3 $G_{\textit{Achieve-POS-CNF}}(F)$*: We are given a positive CNF formula F. A move consists of choosing some variable of F which has not yet been chosen. Player I starts the game. The game ends after all variables of F have been chosen. Player I wins iff F is* `true` *when all variables chosen by player I are set to* `true` *and all variables chosen by player II are set to* `false`.

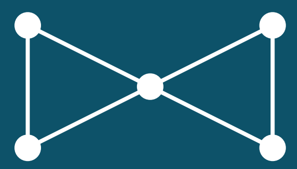
Example: On input $F = (x_1 \vee x_4) \wedge (x_2 \vee x_3) \wedge (x_2 \vee x_4)$ player I has a winning strategy.

F is reduced to the following $G_{\text{Avoid-Ramsey}}$ game . . . ▷

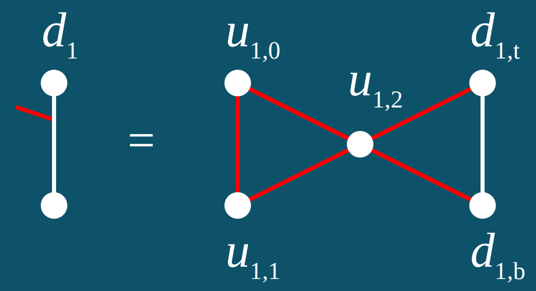


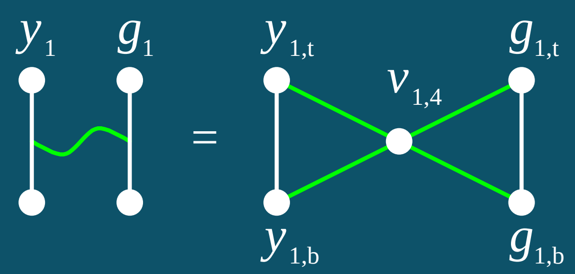
$$\mathbf{F} = (\mathbf{x_1} \vee \mathbf{x_4}) \wedge (\mathbf{x_2} \vee \mathbf{x_3}) \wedge (\mathbf{x_2} \vee \mathbf{x_4})$$

A:



Abbreviations:





Proof sketch of Theorem 4

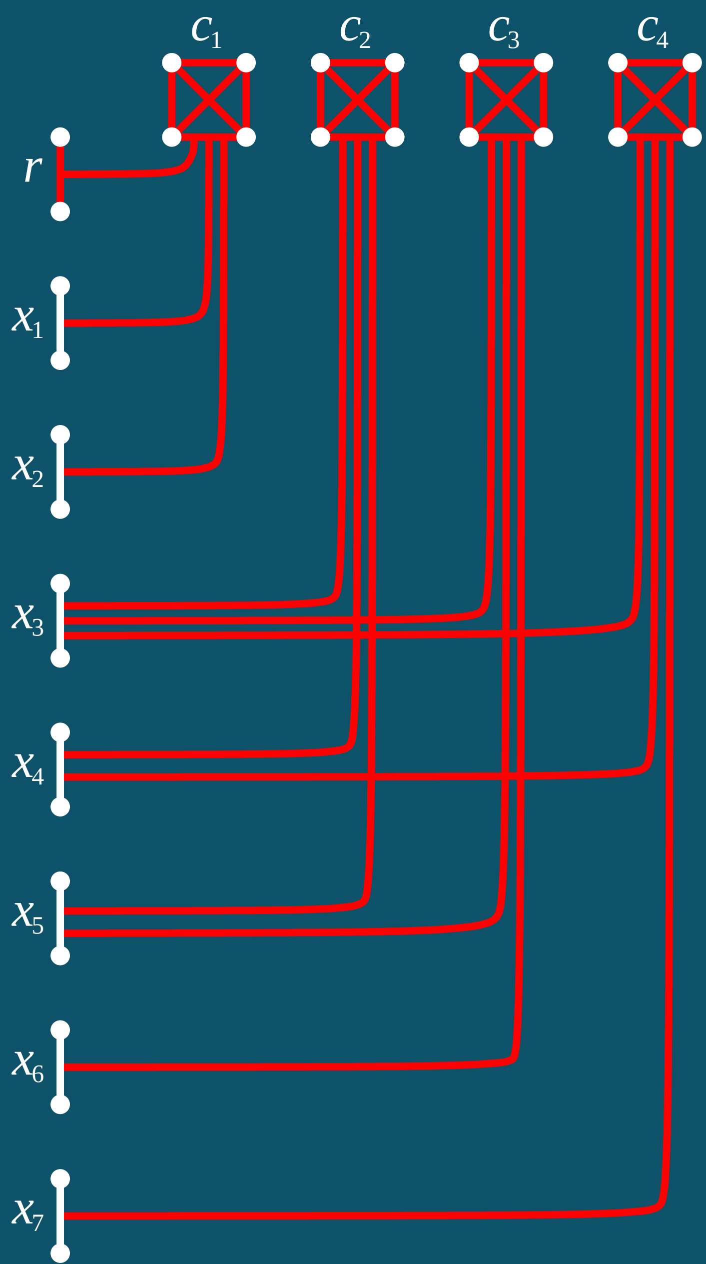
- A careful analysis of the proof of Theorem 3 reveals that we can reuse the reduction of that proof to show the **PSPACE**-completeness of $G_{\text{Avoid-Ramsey}^+}$.
- Indeed, all arguments go through even when both players are allowed to color more than one edge per move.
- The difficulty here lies in the rather involved analysis of the cases when the opponent plays non-optimally.

Proof sketch of Theorem 5

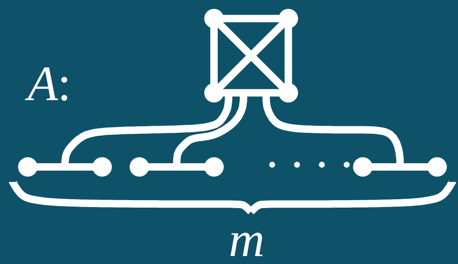
- Membership in **PSPACE**: easy.
- Hardness: via a **LOGSPACE** reduction from the **PSPACE**-complete game $G_{\text{Achieve-POS-DNF}}$ [55]:

Definition 4 $G_{\textit{Achieve-POS-DNF}}(F)$*: We are given a positive DNF formula* F*. A move consists of choosing some variable of* F *which has not yet been chosen. Player I starts the game. The game ends after all variables of* F *have been chosen. Player I wins iff* F *is* `true` *when all variables chosen by player I are set to* `true` *and all variables chosen by player II are set to* `false`*.*

Example: On input $F = (x_1 \wedge x_2) \vee (x_3 \wedge x_4 \wedge x_5) \vee (x_3 \wedge x_5 \wedge x_6) \vee (x_3 \wedge x_4 \wedge x_7)$ player II has a winning strategy. The corresponding $G_{\text{Achieve-Ramsey}}$ game is . . . ▷

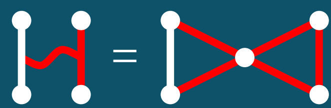


$$\mathbf{F} = (\mathbf{x_1} \wedge \mathbf{x_2}) \vee (\mathbf{x_3} \wedge \mathbf{x_4} \wedge \mathbf{x_5}) \vee (\mathbf{x_3} \wedge \mathbf{x_5} \wedge \mathbf{x_6}) \vee (\mathbf{x_3} \wedge \mathbf{x_4} \wedge \mathbf{x_7})$$



m . . . size of largest clause

Abbreviation:



Definition 5 ([1, 56]) *A combinatorial game is . . .*

- *ultra-weakly solved if the game-theoretic value for the initial position has been determined,*
- *weakly solved if it is ultra-weakly solved and a strategy exists for achieving the game-theoretic value from the opening position, assuming reasonable computing resources,*
- *strongly solved if for all possible positions, a strategy is known for determining the game-theoretic value for both players, assuming reasonable computing resources, and*
- *ultra-strongly solved if for all positions in a strongly solved game, a strategy is known that improves the chances of achieving more than the game-theoretic value against a fallible opponent.*

Theoretical size of Sim's game tree: $15! \approx 1.3 \times 10^{12}$.
In case of Sim$^+$: $15! \times 2^{15-1} \approx 2.1 \times 10^{16}$.

Practical size of their directed acyclic game graphs:

- Sim: 2,309 non-isomorphic positions
- Sim$^+$: 13,158 non-isomorphic positions

$\Rightarrow$ Strong solutions of Sim and Sim$^+$ are easily feasible.

To ultra-strongly solve Sim, we additionally need a strategy for non-winning positions. In our applet, we:

- maximize static chance of opponent to make a mistake
- improve this strategy by learning the value of moves through playing over the Internet

$\Rightarrow$ **Sim and Sim$^+$ are ultra-strongly solved.**

Definition 6 *Sim$_n$:*
$G = K_{\textit{Ramsey}(n,n)}$, $A = K_n$ *played on* $G_{\textit{Avoid-Ramsey}}$.

Problem: Despite much effort, only Ramsey(4,4) = 18 is known so far (conjecture [40] Ramsey(5,5) $\overset{?}{=}$ 43 . . .).

Let us consider the game Sim$_4$ played on a game board G having $\binom{18}{2} = 153$ edges, the graph A to avoid being a tetrahedron. Unfortunately, we found that the number of non-isomorphic game positions in Sim$_4$ is around 10^{54}.

$\Rightarrow$ There is not much hope to even weakly solve any game Sim$_n$ and even less so any game Sim$_n^+$ with $n \geq 4$.

The difficulty of games: another view

Problem: **PSPACE**-completeness is a very coarse instrument to measure the difficulty of combinatorial games: no statement about particular instances are possible.

$\Rightarrow$ **time-bounded Kolmogorov complexity of combinatorial game instances:**

Definition 7 *The time bounded Kolmogorov complexity K^t_G of a game instance x (for example, $x = G_{\textit{Avoid-Ramsey}}(G, A, E^r_0, E^g_0)$) is defined by*

$$K^t_G(x) \overset{\text{def}}{=} \min_p \sup_{y \,\epsilon\, s(x)} \inf_{s(x)} \{l(p) + t(p, y)\},$$

where p is a program of length $l(p)$ that runs on input y (e.g., $y = (E^r_i, E^g_i)$) in $t(p, y)$ steps on a universal Turing machine U. U has the three halting states win, loss, ▷

and draw *such that the halting state of $U(p, y) = x(y)$ is the answer of the decision problem for the first player in situation y of game x. The (not unique) sets $s(x)$ are recursively defined as follows:*

$$s(x) \stackrel{\text{def}}{=} \bigcup_{i=0}^{\infty} F_i,$$
$$F_0 \stackrel{\text{def}}{=} \{y_0\},$$

where y_0 denotes the initial situation of x (in the example, $y_0 = (E_0^r, E_0^g)$), and

$$F_{i+1} \stackrel{\text{def}}{=} \bigcup_{y \in F_i} N(y)$$

▷

$$N(y) \stackrel{\text{def}}{=} \begin{cases} \{\} & \textit{if } G(\{y\}) = \{\} \lor \\ & \quad G(G(\{y\})) = \{\} \\ \\ Y'' \subseteq G(G(\{y\})) & \textit{otherwise, where} \\ & \quad \exists\, y' \in G(\{y\}) \mid x(y') = x(y) \land \\ & \quad \forall\, y'' \in G(\{y\}),\ y'' \in Y'' \end{cases}$$

where

$$G(Y) \stackrel{\text{def}}{=} \{y' \mid \exists\, y \in Y \textit{ such that } y' \textit{ is reachable in one move from } y\}.$$

Conjecture 1

$$K_G^t(\mathrm{Sim}_a) \ll K_G^t(\mathrm{Sim}) < K_G^t(\mathrm{Sim}^+) \ll K_G^t(\mathrm{Sim}^4)$$

Conjecture 2 *Graph Ramsey games played on* $(G, A, \{\}, \{\})$ *are* **PSPACE**-*complete.*

Conjecture 3 *Graph Ramsey achievement games played on* (K_n, A, E^r, E^g) *are tractable.*

Conjecture 4 *Graph Ramsey avoidance games played on* (K_k, K_n, E^r, E^g) *where* $k \geq$ *Ramsey*(n, n) *are* **PSPACE**-*complete.*

Conjecture 5 *The graph Ramsey avoidance games played on* $(K_{\text{Ramsey}(n,n)}, K_n, \{\}, \{\})$ *are 2-***EXPSPACE**-*complete.*

Conjecture 6 *The high-score handling between the server and an unknown remote client can be made tamper-proof for all practical purposes with respect to client-cheating by an appropriate randomized algebraic protocol.*

Further remarks

- The proof of the complexity of $G_{\text{Avoid-Ramsey}^+}$ is, to our best knowledge, the first **PSPACE**-completeness proof for a problem where the number of decisions that can be taken at each alternation is not predetermined.
- Our results can be applied to a number of known abstract and practical problems (detailed in the paper).
- Sim and Sim$^+$ are to be integrated in a role-playing game $\Rightarrow$ "cheats" will be made impossible.
- Please try out our applet that plays Sim and Sim$^+$ on

  http://www.dbai.tuwien.ac.at/proj/ramsey/

  so that it can continue to become even better.

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