COMPLEXITY THEORY

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Part 1:

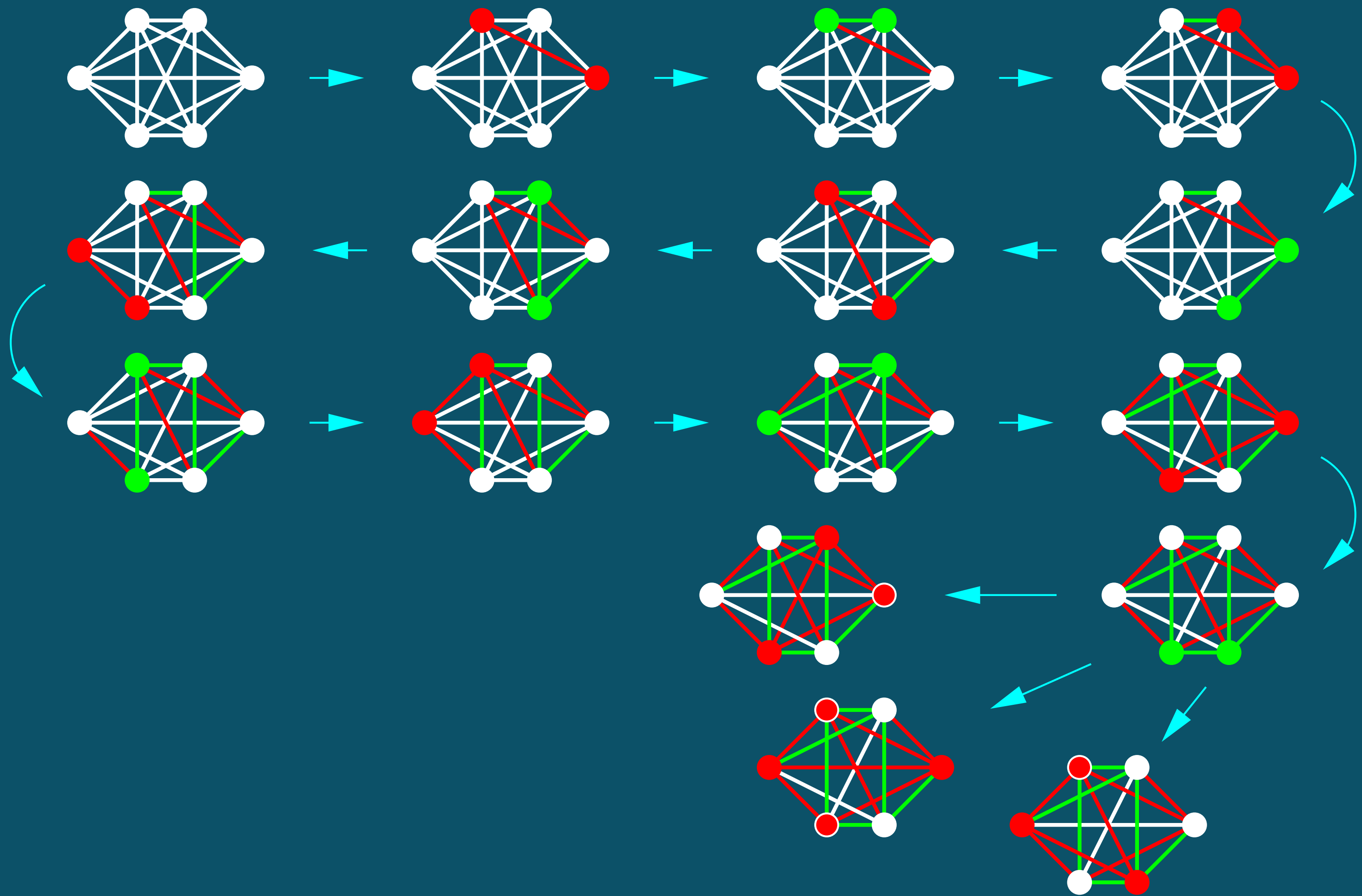
A panorama of complexity theory via graph Ramsey games

Given two graphs G and A, two players, Red and Green, alternate in coloring the edges of G in their respective color. Aim is to avoid (achieve) to build a monochromatic subgraph isomorphic to A. How difficult are these games?

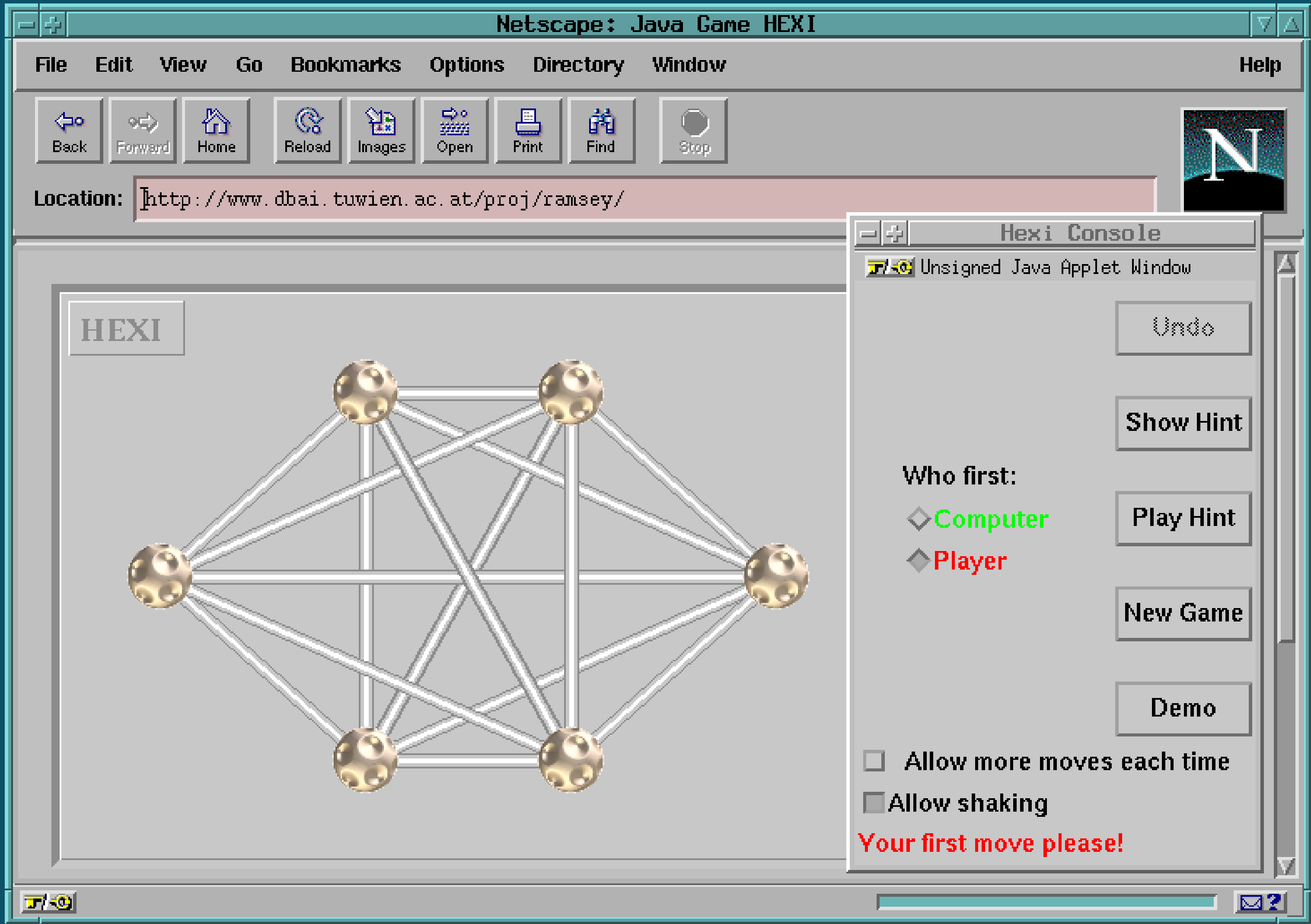
Overview

- In medias res: Let's play . . .
- Complexity of the graph Ramsey games
- Ultra-strongly solving Sim and Sim$^+$
- About the unlikeliness of solving Sim$_4$ etc.
- Tractable cases
- Provably intractable cases
- The complexity of games: another view
- Open problems and conjectures
- Further remarks

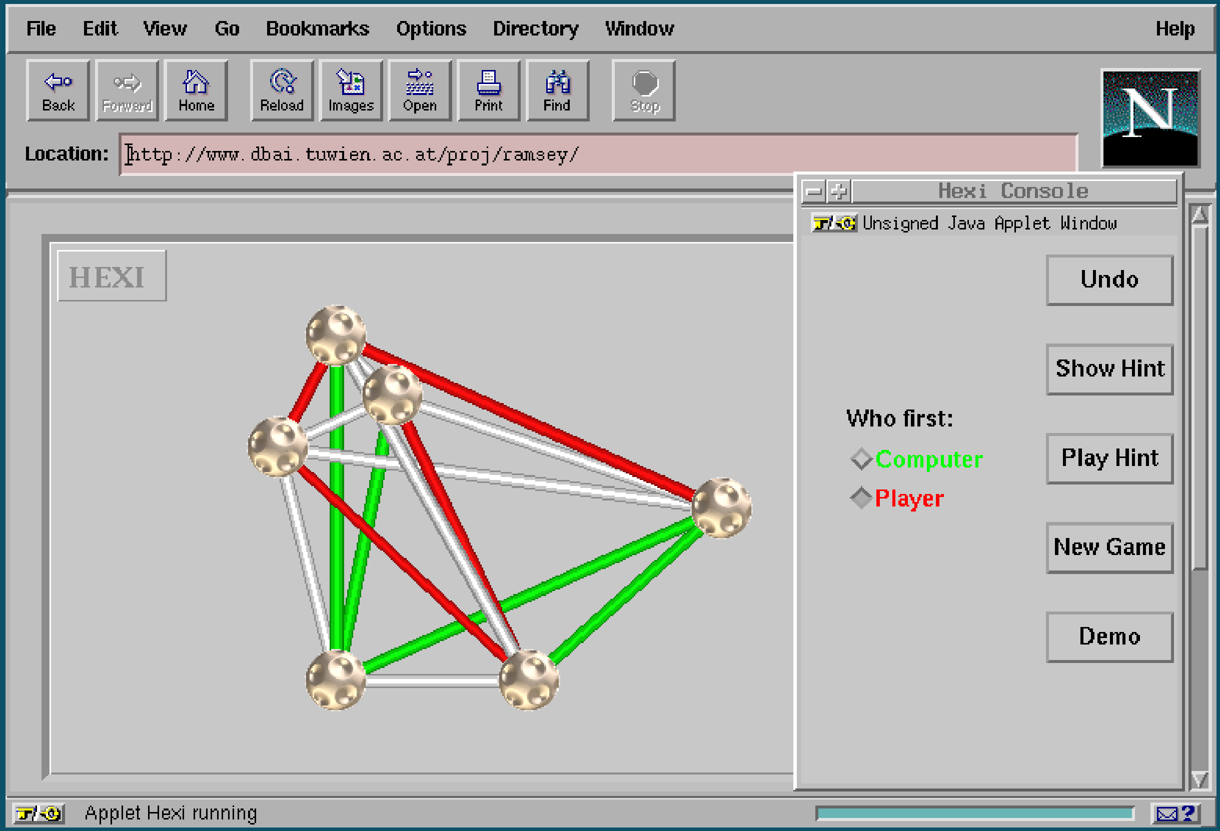
Sim: $G = K_{\text{Ramsey}(3,3)} = K_6,\ A = K_3$ on $G_{\text{Avoid-Ramsey}}$



Considering that a hands-on session with an interactive system often is worth more than a thousand images:



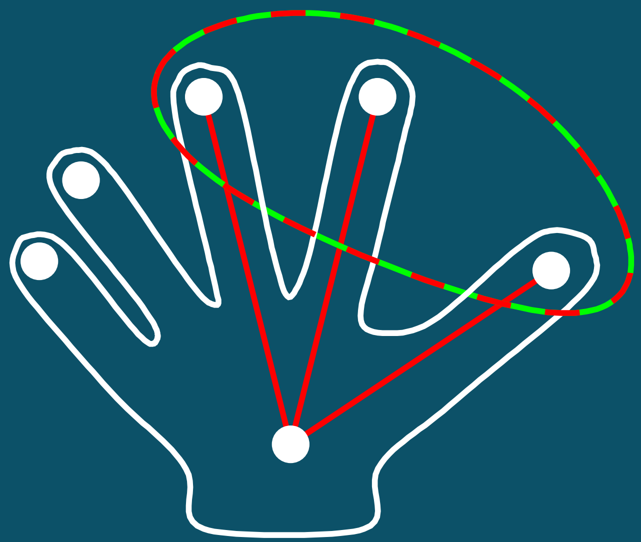
... with random permutations between moves:

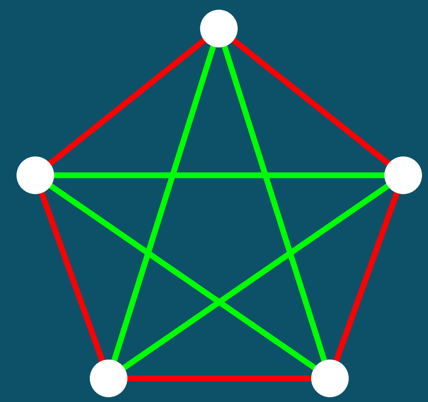


This Java applet plays Sim and a variant, Sim$^+$ (players color one *or more* edges per move). In case you win, you will be allowed to leave your name in our hall-of-fame!

Sim and Sim$^+$ can never end in a tie:

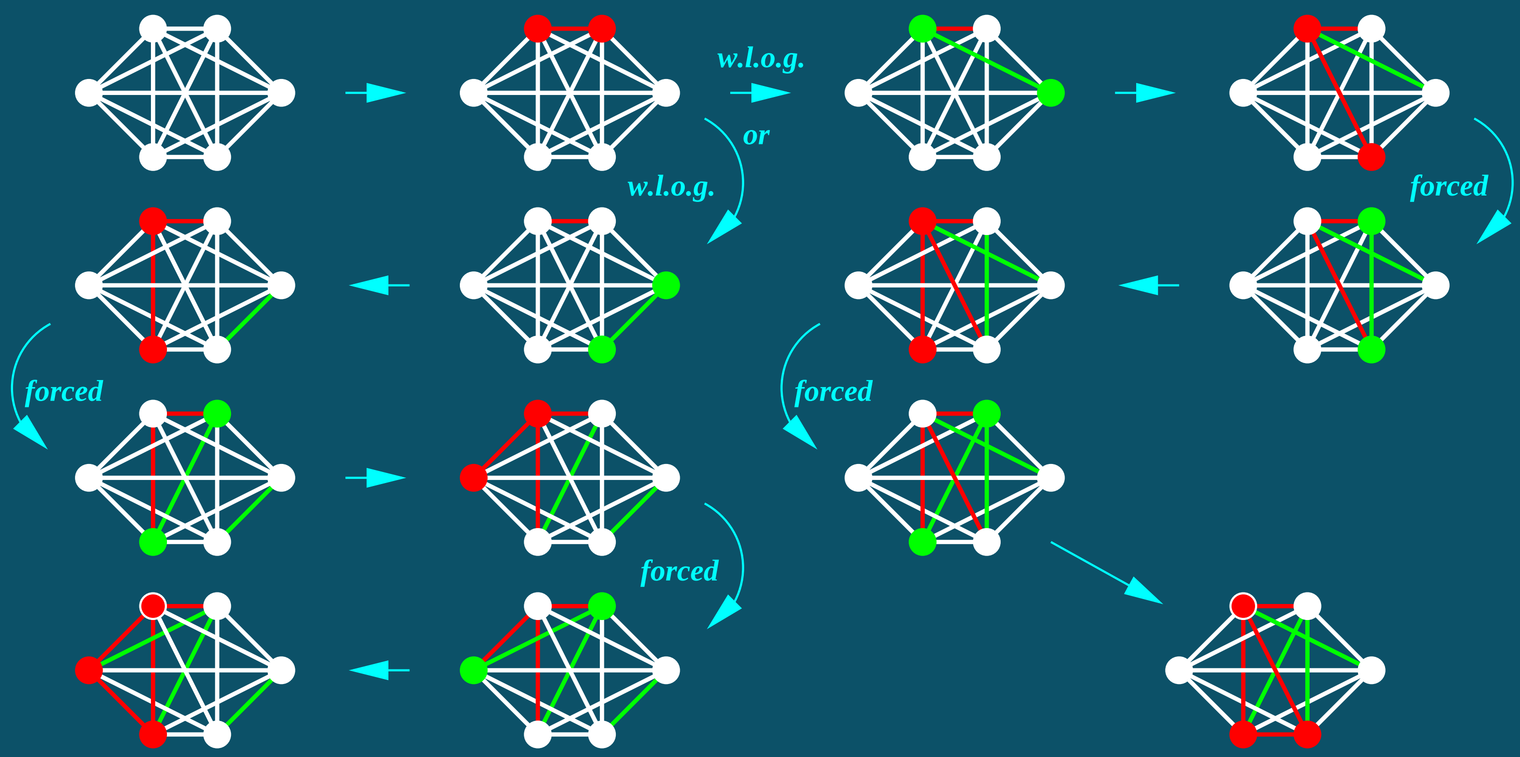
Ramsey(3,3)=6





(visual proof by courtesy of Ranan Banerji)

A winning strategy for the $G_{\text{Achieve-Ramsey}}$ game Sim$_a$:



No simple winning strategies are known for Sim and Sim$^+$.

$\Rightarrow$ **Natural question: How “difficult” is a game?**

Translation to complexity theory:

How does the function bounding the computational resources that are needed in the worst case to determine a winning strategy for the first player grow in relation to the size of the game description?

Typical results: Generalizations of well-known games such as Chess, Checkers, and Go to boards of size $n \times n$ have been classified as polynomial space and exponential time complete (Fraenkel & Lichtenstein 1981, Fraenkel & al. 1978, Lichtenstein & Sipser 1980).

Note: $\textbf{P} \subseteq \textbf{NP} \subseteq \textbf{PSPACE} \subseteq \textbf{EXPTIME}$ and
$\textbf{P} \subset \textbf{EXPTIME}$

How to generalize Sim to game boards of arbitrary size?

$\Rightarrow$ **Graph Ramsey theory**

Definition 1 $G \to A$:
We say that a graph G arrows *a graph A if every edge-coloring of G with colors red and green contains a monochromatic subgraph isomorphic to A. G is called a* Ramsey graph *of A.*

Theorem 1 (Chvátal & Harary 1972, Deuber 1975, Erdős & al. 1975, Rödl 1973) *Every graph has Ramsey graphs.*

Theorem 2 (Burr 1976) *Deciding $G \not\to A$ when G and A are part of the input is* **NP***-complete.*

Theorem 3 (M. Schaefer 1999) *Deciding $G \to A$ when G and A are part of the input is π_2^P-complete.*

Generalizing Sim to graph Ramsey theory leads to:

Definition 2 $G_{\text{Avoid-Ramsey}}(G, A, E^r, E^g)$: *Given two graphs $G = (V, E)$ and A and two non-intersecting sets $E^r \cup E^g \subseteq E$ that contain edges initially colored in red and green, respectively. Two players, Red and Green, take turns in selecting at each move one so-far uncolored edge from E and color it in red for player Red respectively in green for player Green. However, both players are forbidden to choose an edge such that A becomes isomorphic to a subgraph of the red or the green part of G. It is Red's turn. The first player who cannot move loses.*

Similar definitions of $G_{\text{Avoid'-Ramsey}}$ (a misère variant) and $G_{\text{Avoid-Ramsey}^+}$ (one or more edges colored per move).

▷

Definition 3 $G_{\text{Achieve-Ramsey}}(G, A, E^r, E^g)$*:*
Achievement variant: the first player who builds a monochromatic subgraph isomorphic to A wins.

Definition 4 *A simple strategy-stealing argument tells us that with optimal play on an uncolored board,* $G_{\text{Achieve-Ramsey}}$ *must be either a first-player win or a draw, so it is only fair to count a draw as a second-player win. Let us call this variant* $G_{\text{Achieve'-Ramsey}}$*.*

Definition 5 *Following the terminology of (Beck & Csirmaz 1982), let us call the variant of* $G_{\text{Achieve-Ramsey}}$ *where all the second player does is to try to prevent the first player to build* A*, without winning by building it himself, the “weak” graph Ramsey achievement game* $G_{\text{Achieve''-Ramsey}}$*.*

Main complexity results (Slany 1999)

Theorem 4

$G_{\textit{Avoid-Ramsey}}$ *and* $G_{\textit{Avoid'-Ramsey}}$ *are* **PSPACE***-complete.*

Theorem 5 $G_{\textit{Avoid-Ramsey}^+}$ *is* **PSPACE***-complete.*

And, surprisingly,

Theorem 6

$G_{\textit{Achieve''-Ramsey}}$ *and* $G_{\textit{Achieve'-Ramsey}}$ *are* **PSPACE***-complete.*

Theorem 7 $G_{\textit{Achieve-Ramsey}}$ *is* **PSPACE***-complete.*

Significance: These games thus are as difficult as other well-known difficult games such as Go, and at least as difficult as any **NP**-complete problem.

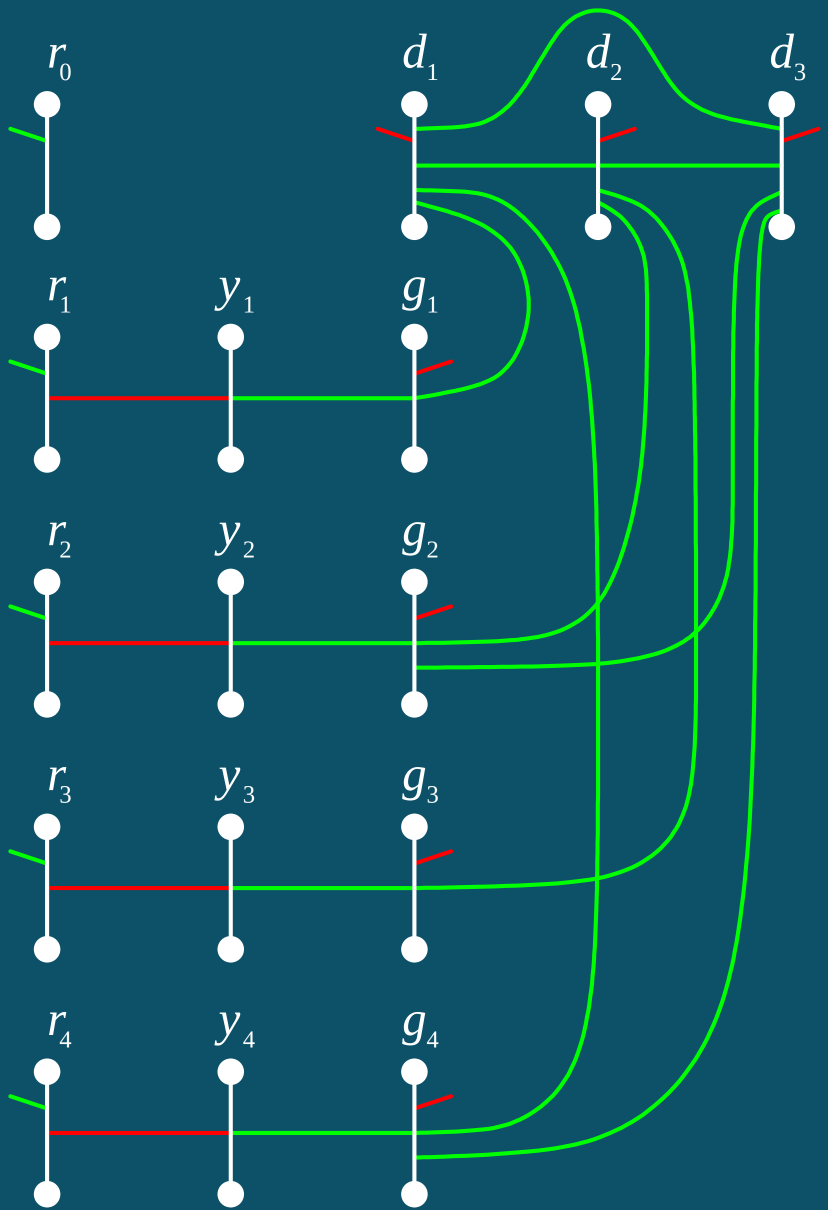
Proof sketch of Theorem 4

- Membership in **PSPACE**: easy.
- Hardness: via a **LOGSPACE** reduction from the **PSPACE**-complete game $G_{\text{Achieve-POS-CNF}}$ (T. Schaefer 1978):

Definition 6 $G_{\text{Achieve-POS-CNF}}(F)$: *We are given a positive CNF formula F. A move consists of choosing some variable of F which has not yet been chosen. Player I starts the game. The game ends after all variables of F have been chosen. Player I wins iff F is* `true` *when all variables chosen by player I are set to* `true` *and all variables chosen by player II are set to* `false`*.*

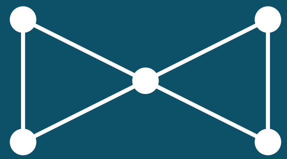
Ex.: On $F = (x_1 \vee x_4) \wedge (x_2 \vee x_3) \wedge (x_2 \vee x_4)$ player I wins. F is reduced to the following $G_{\text{Avoid-Ramsey}}$ game . . .

▷

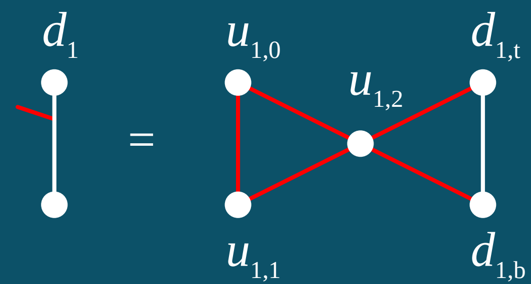


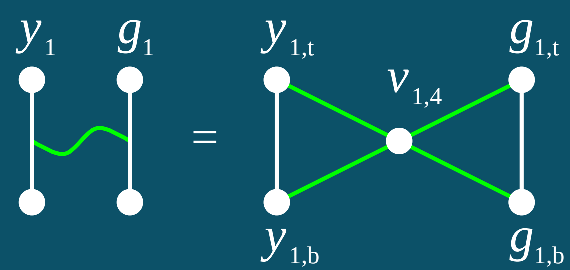
$$\mathbf{F} = (\mathbf{x_1} \vee \mathbf{x_4}) \wedge (\mathbf{x_2} \vee \mathbf{x_3}) \wedge (\mathbf{x_2} \vee \mathbf{x_4})$$

$A:$



Abbreviations:





Proof sketch of Theorem 5

- A careful analysis of the proof of Theorem 4 reveals that we can reuse the reduction of that proof to show the **PSPACE**-completeness of $G_{\text{Avoid-Ramsey}^+}$.
- Indeed, all arguments go through even when both players are allowed to color more than one edge per move.
- The difficulty here lies in the analysis of the cases when the opponent plays non-optimally.

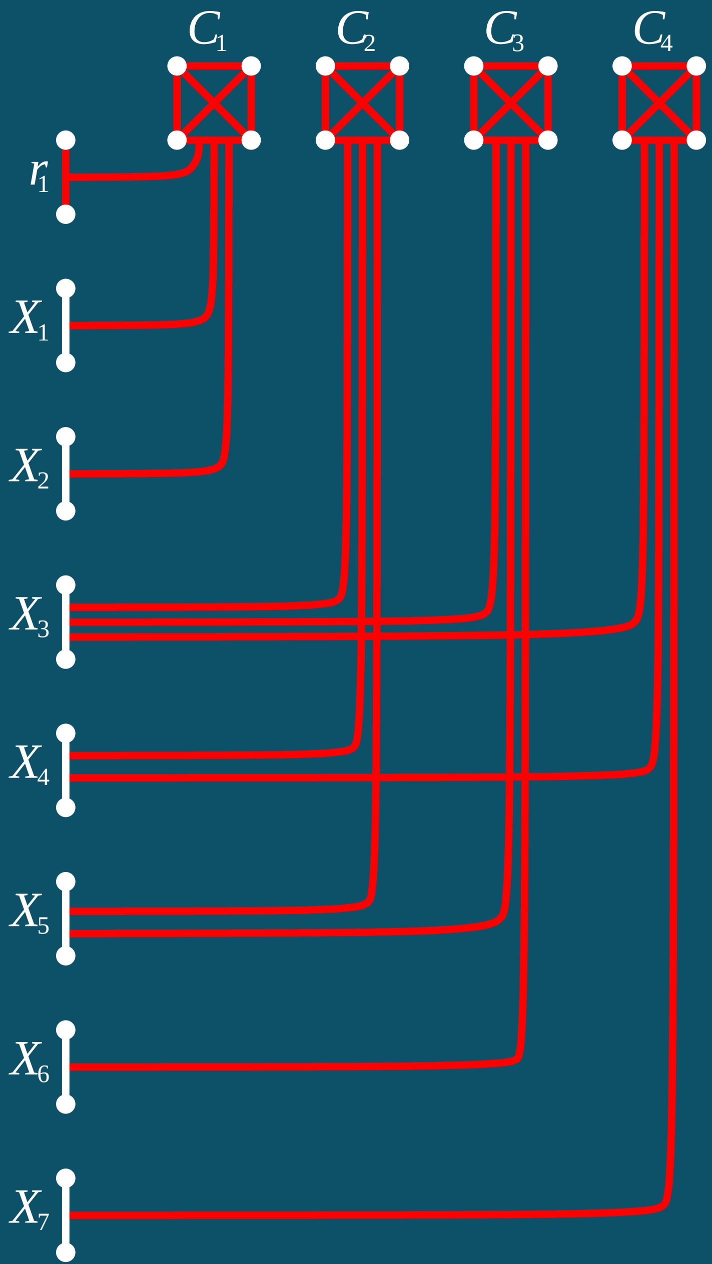
Proof sketch of Theorem 6

- Membership in **PSPACE**: easy.
- Hardness: via a **LOGSPACE** reduction from the **PSPACE**-complete game $G_{\text{Achieve-POS-DNF}}$ (T. Schaefer 1978):

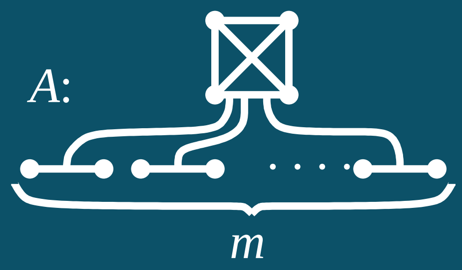
Definition 7 $G_{\text{Achieve-POS-DNF}}(F)$: *We are given a positive DNF formula F. A move consists of choosing some variable of F which has not yet been chosen. Player I starts the game. The game ends after all variables of F have been chosen. Player I wins iff F is* `true` *when all variables chosen by player I are set to* `true` *and all variables chosen by player II are set to* `false`*.*

Ex.: On $F = (x_1 \wedge x_2) \vee (x_3 \wedge x_4 \wedge x_5) \vee (x_3 \wedge x_5 \wedge x_6) \vee (x_3 \wedge x_4 \wedge x_7)$ player II wins. The $G_{\text{Achieve"-Ramsey}}$ game . . .

▷

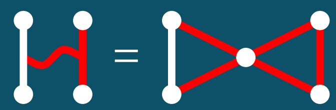


$$\mathbf{F} = (\mathbf{x_1} \wedge \mathbf{x_2}) \vee (\mathbf{x_3} \wedge \mathbf{x_4} \wedge \mathbf{x_5}) \vee (\mathbf{x_3} \wedge \mathbf{x_5} \wedge \mathbf{x_6}) \vee (\mathbf{x_3} \wedge \mathbf{x_4} \wedge \mathbf{x_7})$$



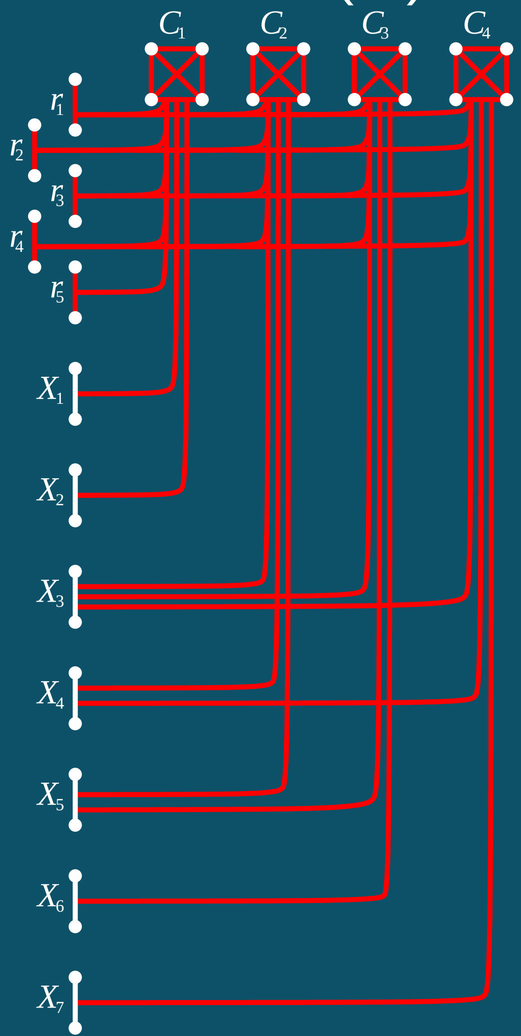
m . . . size of largest clause

Abbreviation:

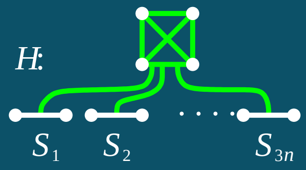


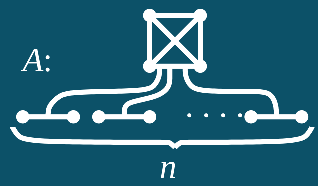
Proof sketch of Theorem 7

Similar to the proof for $G_{\text{Achieve"-Ramsey}}$, but some changes ($A$) and one addition ($H$) in the reduction are necessary:



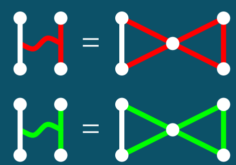
$$\mathbf{F} = (\mathbf{x_1} \wedge \mathbf{x_2}) \vee (\mathbf{x_3} \wedge \mathbf{x_4} \wedge \mathbf{x_5}) \vee (\mathbf{x_3} \wedge \mathbf{x_5} \wedge \mathbf{x_6}) \vee (\mathbf{x_3} \wedge \mathbf{x_4} \wedge \mathbf{x_7})$$





n . . . number of variables

Abbreviations:



Definition 8 (J. Schaeffer & Lake 1996)

A combinatorial game is . . .

- *ultra-weakly solved if the game-theoretic value for the initial position has been determined,*
- *weakly solved if it is ultra-weakly solved and if a strategy exists for achieving the game-theoretic value from the opening position, assuming reasonable computing resources,*
- *strongly solved if for all possible positions, a strategy is known for determining the game-theoretic value for both players, assuming reasonable computing resources, and*
- *ultra-strongly solved if for all positions in a strongly solved game, a strategy is known that improves the chances of achieving more than the game-theoretic value against a fallible opponent.*

Theoretical size of Sim's game tree: $15! \approx 1.3 \times 10^{12}$.
In case of Sim^{+}: $15! \times 2^{15-1} \approx 2.1 \times 10^{16}$.

Practical size of their directed acyclic game graphs:

- Sim: 2,309 non-isomorphic positions
- Sim^{+}: 13,158 non-isomorphic positions

$\Rightarrow$ Strong solutions of Sim and Sim^{+} are easily feasible.

To ultra-strongly solve Sim, we additionally need a strategy for non-winning positions. In our Java applet, we:

- maximize static chance of opponent to make a mistake
- improve this strategy by probabilistically learning the value of moves through playing over the Internet

$\Rightarrow$ **Sim and Sim^{+} are ultra-strongly solved.**

Definition 9 *Sim$_n$:*
$G = K_{\text{Ramsey}(n,n)}$, $A = K_n$ *played on* $G_{\text{Avoid-Ramsey}}$.

Problem: Despite much effort, only Ramsey(4,4) $=$ 18 is known so far (conjecture (McKay 1998) Ramsey(5,5) $\stackrel{?}{=}$ 43 based on 10 cpu-years of computations ...).

Let us consider the game Sim$_4$ played on a game board G having $\binom{18}{2} = 153$ edges, the graph A to avoid being a tetrahedron. Unfortunately, we found that the number of non-isomorphic game positions in Sim$_4$ is around 2×10^{54}.

$\Rightarrow$ There is not much hope to even weakly solve any game Sim$_n$ and even less so any game Sim$_n^+$ for $n > 3$.

Tractable cases:

Theorem 8 (Harary, Slany, Verbitsky 2000)

$$G_{\text{Avoid-Ramsey}}(K_n, (\{a,b,c\}, \{\{a,b\},\{b,c\}\}), \{\}, \{\})$$

for $n \geq 3$ is a win for the second player.

Proof sketch:

There is a relatively simple two-phase winning strategy for the second player. The proof uses a counting argument and several lemmas.

Provably intractable cases:

Because of the known exponential lower bounds for classic symmetric binary Ramsey numbers

$$n2^{n/2}(1/e\sqrt{2}) + o(1)) < \text{Ramsey}(n,n)$$

already computing the size of the game graph of a graph Ramsey game played on $(K_{\text{Ramsey}(n,n)}, K_n, \{\}, \{\})$ given only n for input will require at least doubly exponential time because of the succintness of the input (Graham et al. 1990).

The complexity of games: another view

Problem: **PSPACE**-completeness is a very coarse instrument to measure the difficulty of combinatorial games: no statement about particular instances are possible. For example, how does the complexity of the *real* game Go compare to that of Sim_4 or Sim_5?

$\Rightarrow$ **time-bounded Kolmogorov complexity of combinatorial game instances:**

What is the "size" n of the "smallest program" that, using at most n "time units", wins game G whenever a winning strategy exists and plays "optimally" otherwise?

Good upper and lower bounds are most likely difficult . . .

Open Problem 1 *Consider* $G_{\textit{Avoid-Ramsey}}(K_k, K_n, \{\}, \{\})$ *where* $k = \textit{Ramsey}(n,n)$. *Is it always true that the first player has a winning strategy in this game iff* $\binom{k}{2}$ *is even?*

Open Problem 2 *Consider* $G_{\textit{Avoid-Ramsey}}(G, A, E^r, E^g)$, *where*

$$c \overset{\text{def}}{=} \min_{\substack{(r,g) \in \mathbb{N}^2 \\ r=g \text{ or } r=g+1 \\ r+g \le |E(G)|-|E^r|-|E^g|}} \{r+g \,|\, (G, E^r, E^g)^{(r,g)} \to A\} \,,$$

and where $(G, E^r, E^g)^{(r,g)}$ *denotes an* (r,g) *edge-red-green-coloring of the uncolored edges of the precolored graph* (G, E^r, E^g). *Is it always true that the first player has a winning strategy in this game iff* c *is even?*

Conjecture 1 *Graph Ramsey games played on* $(G, A, \{\}, \{\})$ *are* **PSPACE***-complete.*

Conjecture 2 *Graph Ramsey achievement games played on* (K_n, A, E^r, E^g) *are tractable.*

Conjecture 3 *Graph Ramsey avoidance games played on* (K_k, K_n, E^r, E^g) *where* $k \geq$ *Ramsey*(n,n) *are* **PSPACE***-complete.*

Conjecture 4 *The graph Ramsey avoidance games played on* $(K_{\text{Ramsey}(n,n)}, K_n, \{\}, \{\})$ *are 2-***EXPSPACE***-complete.*

Open Problem 3 *Show that $G_{\text{Achieve-Ramsey}}$ remains* **PSPACE**-*complete even if the achievement graph A is restricted to a meaningful subclass of graphs such as fixed, bipartite or degree-restricted graphs.*

Open Problem 4 *Show that Theorems 4–7 hold even if the game graph G is restricted to a meaningful subclass of graphs such as bipartite or degree-restricted graphs.*

Further remarks

- Sim and Sim$^+$: simple enough to analyze perfectly, yet far from trivial.
- Applications: competitive situations where opposing parties try to achieve or to avoid a certain pattern in the structure of their commitments, e.g., analysis of mobile Internet agent warfare (Thomsen & Thomsen 1998).
- Sim and Sim$^+$ are to be integrated in a role-playing game $\Rightarrow$ "cheats" will be made very difficult.
- Please try out our applet that plays Sim and Sim$^+$ on

  http://www.dbai.tuwien.ac.at/proj/ramsey/

  so that it can continue to become even better.