Complexity Theory
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3. Logarithmic Space

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3. Logarithmic Space
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Example

Log-space computations

If a Turing machine with input and output is supposed to operate within space bound $O(\log(n))$, then it may never copy a substantial portion of the input onto its worktapes. Access to the input and control of the machine is based on the following log-space functionalities:

- maintaining (a constant number of) counters
- maintaining (a constant number of) “pointers” to the input
- “storing” a constant-size fragment of the input in the state

Example

The language of palindromes (over an arbitrary alphabet $\Sigma$) can be recognized by a 3-string Turing machine $M$ with input within space bound $O(\log(n))$. (There is no output tape needed.)
Sketch of a log-space TM for palindromes

The 3-string Turing machine $M$ with input implements a loop from 1 to $n$ with $n = |x|$. For every $i$, it checks if $x[i] = x[n + 1 - i]$ holds:

1. Tape 2 contains the loop counter $i$. It is initialized to 1 (in binary) and will be subsequently incremented.

2. Tape 3 contains another counter $j$ which, for every $i$, controls the cursor movements required for comparing $x[i]$ and $x[n + 1 - i]$:
   (a) Initialize $j$ to 1.
   (b) Move the cursor from the front end of $x$ to $x[i]$. The integer $j$ counts the cursor movements (i.e.: increment $j$ as long as $j < i$).
   (c) “Store” the symbol $x[i]$ in an appropriate state.
   (d) Reinitialize $j$ to 1.
   (e) Move the cursor from the rear end of $x$ to $x[n + 1 - i]$.
   (f) “Compare” $x[n + 1 - i]$ with $x[i]$ (via the state). If $x[n + 1 - i] = x[i]$ then increment $i$ and repeat Step 2. Otherwise halt with “no”.
## Log-Space Reductions

<table>
<thead>
<tr>
<th>Reductions in the “Formale Methoden” lecture</th>
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<tbody>
<tr>
<td>- 2 kinds of reductions: Turing reductions vs. many-one reductions</td>
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<td>- Limit on the resources needed by a reduction: polynomial time vs. logarithmic space reductions.</td>
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<tr>
<td>- Default for problem reductions in “Formale Methoden” (e.g., in NP-completeness proofs) : polynomial time, many-one reductions.</td>
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</table>
| - In this course: We also want to prove completeness results for classes below NP (in particular, completeness in P or NL).  
  \( \implies \) We need reductions in a complexity class below P. |
| - From now on: log-space, many-one reductions, denoted as \( \leq_L \) |
| - Remark. All polynomial-time reductions encountered so far (in the “Formale Methoden” lecture) also work in log-space! |
Composing Reductions

- In the “Formale Methoden” lecture, we have established the following chain of reductions:
  \[ 3\text{-SAT} \leq_L \text{INDEPENDENT SET} \leq_L \text{VERTEX COVER}. \]

- But do reductions compose, i.e., is \( \leq_L \) transitive?

For instance, does \( 3\text{-SAT} \leq_L \text{VERTEX COVER} \) hold?

**Theorem**

*If* \( R \) *is a reduction from language* \( L_1 \) *to* \( L_2 \) *and* \( R' \) *is a reduction from* \( L_2 \) *to* \( L_3 \), *then the composition* \( R \cdot R' \) *is a reduction from* \( L_1 \) *to* \( L_3 \).*

- As \( R, R' \) are reductions, \( x \in L_1 \) iff \( R(x) \in L_2 \) iff \( R'(R(x)) \in L_3 \).

- It remains to show that \( R'(R(x)) \) can be computed in \( O(\log n) \) space where \( n = |x| \).
Logarithmic space consumption

To construct a machine $M$ for the composition $R \cdot R'$ working in space $O(\log n)$ requires care as the intermediate result computed by $M_R$ cannot be stored (possibly longer than $\log n$).

Solution: simulate $M_{R'}$ on input $R(x)$ by remembering the cursor position $i$ of the input string of $M_{R'}$ which is the output string of $M_R$. Only the index $i$ is stored (in binary) and the symbol currently scanned but not the whole string.
Logarithmic space consumption (continued)

- Initially $i = 1$ and it is easy to simulate the first move of $M_R'$ (scanning $\triangleright$).
- If $M_R'$ moves the cursor on the input tape to the right, then simulate $M_R$ to generate the next output symbol and increment $i$ by one.
- If $M_R'$ moves the cursor on the input tape to the left, then decrement $i$ by one and run $M_R$ on $x$ from the beginning, counting the output symbols and stopping when the $i$-th symbol is output.
- The space required for simulating $M_R$ on $x$ as well as $M_R'$ on $R(x)$ is $O(\log n)$ where $n = |x|$.
- The space needed for bookkeeping the output of $M_R$ on $x$ is $O(\log n)$ as $|R(x)| = O(n^k)$ and we need only indices stored in binary.
Motivation

- The intuition of a complexity class is best understood by looking at “natural” problems which are complete for this class.
- The complexity of a problem is only understood if we manage to show its completeness in a “natural” complexity class.

Theorem

**REACHABILITY** *is NL-complete (w.r.t. log-space reductions), i.e.*

- *It can be decided by an NTM in space* \( O(\log(n)) \).
- *Any problem in NL can be reduced to it in log-space.*
Proof sketch of the NL-membership

Let \((V, E)\) be a graph with vertices \(V = \{1, \ldots, n\}\). Moreover, suppose that we attempt to find a path from vertex 1 to \(n\). We sketch an NTM \(N\) with input tape plus 3 worktapes:

1. On tape 2, store the current node \(i\) in binary; initially take node 1.
2. On tape 3, “guess” an integer \(j \leq n\) (\(\equiv\) next node in the path) and check that \((i, j) \in E\).
3. If \((i, j) \in E\), then continue at step 2 with \(j\) as the new current node. Moreover, if \(j = n\), then halt with “yes”. If \((i, j) \notin E\), then halt with “no”.
4. On tape 4, maintain a counter which checks that we do not construct paths of length \(\geq n\).
Proof sketch of the NL-hardness

Let $P$ be an arbitrary problem in NL, i.e., $P$ is decided by a $k$-string nondeterministic TM $M$ with input tape within space $f(n) = O(\log n)$.

Let $x$ be an arbitrary instance of $P$. From this, we construct the instance $R(x) = (G, u, v)$ of Reachability as follows:

The configuration graph $G(M, x)$ of $M$ has as its nodes all possible configurations of $M$ and there is an edge between two nodes (configurations) $C_1$ and $C_2$ iff $C_1 \xrightarrow{M} C_2$.

Our graph $G$ contains an additional node “success” and there is an edge from any configuration $C$ with state “yes” to the “success” node.

Finally, we set $u = C_0 = (s, \triangleright, x, \triangleright, \epsilon, \ldots, \triangleright, \epsilon)$ (= the initial configuration of $M$ on input $x$) and $v = “success”$. 
Proof sketch of the NL-hardness (continued)

Clearly, this problem reduction is correct, i.e., there exists an accepting computation of $M$ on input $x$ iff there exists a path in graph $G$ from node $u = C_0$ to node $v = \text{“success”}$. It remains to show that $R(x)$ can be computed (by a deterministic TM $N$ with input and output) in log-space.

Sketch of a log-space TM $N$.

- $N$ has 3 worktapes. The first two worktapes are used to store one possible configuration of $M$ at a time.

- We are assuming that $M$ is a $k$-string Turing machine with input. Hence, configurations are represented as $(q, i, w_2, u_2, \ldots, w_k, u_k)$ where $1 \leq i \leq n + 1$ gives the cursor position on the input string. Clearly, position $i$ as well as each string $u_j, w_j$ fits into log-space.
Proof sketch of the NL-hardness (continued)

Operating principles of the log-space TM $N$.

- In a loop, $N$ generates all possible configurations of $M$ on its first worktape and writes each configuration to the output tape. At the end, also “success” is output.
- In a nested loop, $N$ generates all possible configurations of $M$ plus the additional node “success” on each of its first two worktapes.
- For every pair of configurations $C_1$ and $C_2$, $N$ checks (by using the third worktape) if $C_1 \xrightarrow{M} C_2$ holds. If this is the case, then $N$ writes the pair (= edge) $(C_1, C_2)$ to the output.
- $N$ also outputs all edges $(C, “success”) if C corresponds to an accepting configuration.
- Finally, the nodes $u = C_0 = (s, 0, \triangleright, \epsilon, \ldots, \triangleright, \epsilon)$ and $v = “success”$ are output.
Nondeterministic Space vs. Deterministic Time

Motivation

- The NL-hardness proof of \textbf{REACHABILITY} implicitly establishes that $\text{NL} \subseteq \text{P}$ holds.
- This result can be rephrased as follows: a nondeterministic machine $M$ working in space $f(n) = \log n$ can be simulated by a deterministic machine $N$, s.t. the time bound of $N$ is exponential w.r.t. $f(n)$.
- We want to generalize this relationship between space and time complexity to any “reasonable” function $f(n) \geq \log n$.
- The idea of “reasonable” functions is formalized by the definition of proper complexity functions.
Proper Complexity Functions

Definition

A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is a proper complexity function if $f$ is nondecreasing and there is a $k$-string TM $M_f$ with input and output such that on any input $x$,

1. $M_f(x) = \sqcap^f(|x|)$ where $\sqcap$ is a quasi-blank symbol,
2. $M_f$ halts after $O(|x| + f(|x|))$ steps, and
3. $M_f$ uses $O(f(|x|))$ space besides its input.

Remark

For the definition of complexity classes $\text{TIME}(f(n))$, $\text{NTIME}(f(n))$, $\text{SPACE}(f(n))$, $\text{NSPACE}(f(n))$, we have to avoid the situation that a function $f$ cannot be computed within the time or space it allows.
Theorem

For any proper complexity function \( f(n) \geq \log n \), we have

\[
\text{NSPACE}(f(n)) \subseteq \text{TIME}(c^{f(n)})
\]

Proof sketch

- We construct the configuration graph as in the NL-hardness proof of \textsc{Reachability}.

- We get the following bound on the number of possible configurations represented as \((q, i, w_2, u_2, \ldots, w_k, u_k)\) where \(1 \leq i \leq n + 1\):

\[
|K|(n + 1)(|\Sigma|^{f(n)})^{2(k-1)} \leq |K|2n(|\Sigma|^{2(k-1)}f(n) \leq nc_1^{\log n + f(n)} \leq c_1^{2f(n)} \leq c^{f(n)}
\]

- Hence, deciding if \(x\) is a positive instance of problem \(P\) reduces to a reachability problem for a graph with at most \(c^{f(n)}\) nodes.
Savitch’s Theorem

**Theorem**

\[
\text{REACHABILITY} \in \text{SPACE}(\log^2 n).
\]

**Proof sketch**

- Given a graph \( G \) and nodes \( x, y \) and \( i \geq 0 \), define \( PATH(x, y, i) \) as the assertion “there is a path from \( x \) to \( y \) of length at most \( 2^i \)’’.
- If \( G \) has \( n \) nodes then any cycle-free path has length \( \leq n \) and we can solve reachability in \( G \) if we can compute whether \( PATH(x, y, \lceil \log n \rceil) \) holds for any given nodes \( x, y \) of \( G \).
- This can be done using middle-first search.
Proof sketch (continued)

- **Idea of the middle-first search.** Guess a midpoint of the alleged path and recursively check for the existence of a half-length path from the start to the midpoint and from the midpoint to the finish.

- **Implementation of the middle-first search.**

  ```
  function path(x, y, i) /* middle-first search */
  if i = 0 then
    if x = y or there is an edge (x, y) in G then return “yes”;
  else for all nodes z do
    if path(x, z, i − 1) and path(z, y, i − 1) then return “yes”;
  return “no”.
  ```
Proof sketch (continued)

- Proof that \( \text{path}(x, y, i) \) correctly determines \( \text{PATH}(x, y, i) \) (by induction on \( i \)):
  
  \( i = 0 \). Clearly, \( \text{path}(x, y, 0) \) correctly determines \( \text{PATH}(x, y, 0) \).

  \( i > 0 \). \( \text{path}(x, y, i) \) returns “yes” iff there is a node \( z \) with both \( \text{path}(x, z, i - 1) \) and \( \text{path}(z, y, i - 1) \) holding.

  By the induction hypothesis, \( \text{path}(x, z, i - 1) \) and \( \text{path}(z, y, i - 1) \) return “yes” iff there are paths from \( x \) to \( z \) and from \( z \) to \( y \) both at most \( 2^{i-1} \) long.

  This is the case iff there is a path from \( x \) to \( y \) at most \( 2^i \) long.
Proof sketch (continued)

- The algorithm is started with $\text{path}(x, y, \lceil \log n \rceil)$.
- The $O(\log^2 n)$ space bound is achieved by handling recursion using a stack containing a triple $(x, y, i)$ for each active recursive call.
  For each node $z$ put $(x, z, i - 1)$ onto the stack and call $\text{path}(x, z, i - 1)$.
  If this fails, erase $(x, z, i - 1)$ and put $(x, z', i - 1)$ for the next $z'$.
  Otherwise erase $(x, z, i - 1)$ and continue with $(z, y, i - 1)$.
- As there are at most $\log n$ recursive calls active with each taking at most $3 \log n$ space, the $O(\log^2 n)$ space bound is achieved.
Corollary

For any proper complexity function $f(n) \geq \log n$, we have

$$\text{NSPACE}(f(n)) \subseteq \text{SPACE}((f(n))^2)$$

Proof sketch

- To simulate an $f(n)$-space bounded NTM $M$ on input $x$, run the previous algorithm on the configuration graph $G(M, x)$.
- The edges of the graph $G(M, x)$ are determined on the fly by examining the input $x$.
- The configuration graph has at most $c^{f(n)}$ nodes.
- By Savitch’s Theorem, the algorithm needs at most $(\log c^{f(n)})^2 = f(n)^2 \log^2 c = O(f(n)^2)$ space.
Basic Complexity Classes Revisited

**Theorem**

\[
L \subseteq NL \subseteq P \subseteq NP \subseteq \text{PSPACE} \subseteq \text{EXPTIME} \subseteq \text{NEXPTIME} \subseteq \text{EXPSPACE}
\]

**Proof**

In the “Formale Methoden” lecture, we showed all inclusions except for \( NL \subseteq P \) and \( \text{PSPACE} \subseteq \text{EXPTIME} \).

These inclusions follow from the more general relationship

\[
\text{NSPACE}(f(n)) \subseteq \text{TIME}(c^{f(n)})
\]

**Remark**

It is now clear, why we have not mentioned \( \text{NPSPACE} \) and \( \text{NEXPSPACE} \). Indeed, \( \text{PSPACE} = \text{NPSPACE} \) and \( \text{EXPSPACE} = \text{NEXPSPACE} \) hold.
### Immerman-Szelepscényi Theorem

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<tr>
<td><em>Given a graph</em> $G$ <em>and a node</em> $u$, <em>the number of nodes reachable from</em> $u$ <em>in</em> $G$ <em>can be computed by an NTM within logarithmic space.</em></td>
</tr>
<tr>
<td><em>More formally, given a graph</em> $G$, <em>a node</em> $u$, <em>and an integer</em> $m$, <em>deciding if the number of nodes reachable from</em> $u$ <em>is</em> $m$ <em>can be done in</em> $NL$.</td>
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<td><strong>REACHABILITY</strong> <em>is in</em> co-$NL$. <em>Hence, NL = co-NL.</em></td>
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<td><em>If</em> $f(n) \geq \log n$ <em>is a proper complexity function, then</em> $\text{NSPACE}(f(n)) = \text{co-NSPACE}(f(n))$.</td>
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Witnesses for NL-Problems

Characterization of positive instances

Let \( \mathcal{P} \) denote a problem in NL. Every positive instance \( x \) of \( \mathcal{P} \) can be characterized by a witness \( y \), s.t.

- the witness \( y \) may be polynomially big,
- but we can check if \( y \) is a witness by a sequence of local consistency checks – each requiring only logarithmic space.

\( \text{REACHABILITY} \)

\((G, u, v)\) is a positive instance of \( \text{REACHABILITY} \) if there exists a path \( \pi \) from \( u \) to \( v \) s.t.

- the path \( \pi = (z_0, z_1, \ldots, z_{k-1}, z_k) \) with \( u = z_0, v = z_k \), and \( k \leq n \) with \( n = |V| \) may be polynomially big,
- but it suffices to check for every pair of neighbouring nodes \( z_i, z_{i+1} \) that an arc \((z_i, z_{i+1})\) indeed exists in \( E \).
co-NL-Membership of \textsc{REACHABILITY}

\textbf{Proof Idea}

What shall a \textit{witness} look like that there is \textbf{no} path from $u$ to $v$?

\textbf{Idea.} Let $S(k)$ denote the set of nodes reachable from $u$ in $k$ steps (with $0 \leq k \leq n - 1$) and suppose that we know $m = |S(n - 1)|$. Then a \textit{witness} $y$ for not-reachability consists of $m$ paths from $u$ to (pairwise distinct) vertices $v_1, \ldots, v_m \in V$, s.t. $v_i \neq v$ for all $i$.

\textbf{Logspace verification of such witnesses}

\texttt{for } j := 1, 2, \ldots, n \texttt{ do } \{
    \texttt{guess flag; } /* \text{ meaning: flag = true if } v_j \in S(n - 1) */
    \texttt{if flag then } \{
        \texttt{decrement } m;
        \texttt{guess a path } \pi \texttt{ from } u \texttt{ to } v_j \texttt{ of length } \leq n - 1;
        \texttt{check that } v \neq v_j; \} \}
\texttt{check that } m = 0;
Counting the Number of Reachable Nodes in “NL”

Idea of the Algorithm

The strategy is to compute values $|S(1)|, |S(2)|, \ldots, |S(n-1)|$ iteratively and recursively, i.e. $|S(i)|$ is computed from $|S(i-1)|$.

$$|S(0)| := 1;$$

for $k := 1, 2, \ldots, n - 1$ do {

$$\ell := 0;$$

for $j := 1, 2, \ldots, n$ do

guess flag; /* meaning: flag = true if $v_j \in S(k)$ */

if flag then { check that $v_j \in S(k)$; increment $\ell$; }

else check that $v_j \not\in S(k)$;

$$|S(k)| := \ell;$$

}
Idea of the Algorithm (continued)

- Clearly, we can check $v_j \in S(k)$ in NL. It remains to show that we can also check $v_j \not\in S(k)$ in NL in the inner loop of our algorithm.
- When checking $v_j \not\in S(k)$, we already know $m = |S(k-1)|$.
- A witness $y$ for $v_j \not\in S(k)$ consists of $m$ paths from $u$ to vertices $v_1, \ldots, v_m \in V$, s.t. $v_i \neq v_j$ and $(v_i, v_j) \not\in E$.
- Below we give a verification of witness $y$ that requires only logspace.

```plaintext
for $v := 1, 2, \ldots, n$ do {
  guess flag;  /* meaning: flag = true if $v \in S(k - 1)$ */
  if flag then {
    decrement m;
    guess a path $\pi$ from $u$ to $v$ of length $\leq k - 1$;
    check that $v \neq v_j$ and $(v, v_j) \not\in E$;
  }
  check that $m = 0$;
}```
Learning Objectives

- Computational power of log-space
- Reductions in this lecture: log-space, many-one
- Composability of reductions
- NL-completeness of **REACHABILITY** (configuration graph)
- Savitch’s Theorem (middle-first search)
- Basic relationships (inclusions) between complexity classes
- Immerman-Szelepscényi Theorem