Exercise 1 (5 credits) Recall the \( \Sigma_2^p \)-hardness proof of the Abduction Solvability problem by reduction from \( \text{QSAT}_2 \): Let an arbitrary instance of the \( \text{QSAT}_2 \) problem be given by the formula \( \varphi = (\exists X)(\forall Y)\psi(X, Y) \) with \( X = \{x_1, \ldots, x_k\} \) and \( Y = \{y_1, \ldots, y_l\} \). Moreover, let \( X' = \{x'_1, \ldots, x'_k\} \), \( R = \{r_1, \ldots, r_k\} \), and \( t \) be fresh variables. Then we define an instance of Solvability as \( P = (V, H, M, T) \) with

\[
V = X \cup Y \cup X' \cup R \cup \{t\} \\
H = X \cup X' \\
M = R \cup \{t\} \\
T = \{\psi(X, Y) \rightarrow t\} \cup \{\neg x_i \lor \neg x'_i, x_i \rightarrow r_i, x'_i \rightarrow r_i \mid 1 \leq i \leq k\}
\]

Give a rigorous correctness proof of this problem reduction, i.e., \( \varphi \equiv \text{true} \iff \text{Sol}(P) \neq \emptyset \).

**Hint.** As usual, prove both directions of the equivalence separately. It is convenient to use the notation from the lecture: For \( A \subseteq X \), let \( A' \) denote the set \( \{x' \mid x \in A\} \).

For the “\( \Rightarrow \)”-direction, you start off with a partial assignment \( I \) on \( X \). Let \( I^{-1}(\text{true}) = A \). Then it can be shown that \( S = A \cup (X \setminus A)' \) is a solution of \( P \). In order to show that \( S \) is indeed a solution, you must prove carefully the two conditions that (1) \( T \cup S \) is satisfiable and (2) \( T \cup S \models M \).

For the “\( \Leftarrow \)”-direction, first show that a solution \( S \) of \( P \) contains exactly one of \( \{x_i, x'_i\} \). This is due to the clauses \( \{\neg x_i \lor \neg x'_i, x_i \rightarrow r_i, x'_i \rightarrow r_i \mid 1 \leq i \leq k\} \) in \( T \). Hence, \( S \) must be of the form \( S = A \cup (X \setminus A)' \) for some \( A \subseteq X \). It remains to show that for the assignment \( I \) on \( X \) with \( I^{-1}(\text{true}) = A \), every extension \( J \) of \( I \) to the variables \( Y \) satisfies the formula \( \psi(X, Y) \).

Exercise 2 (5 credits) Recall the \( \Sigma_2^p \)-hardness proof of the Abduction Relevance problem by reduction from the Solvability problem: Let an arbitrary instance of the Solvability problem be given by the PAP \( P = (V, H, M, T) \). W.l.o.g., let \( T \) consist of a single formula...
\( \varphi \) and let \( h, h', m' \) be fresh variables. Then we define an instance of the Relevance (resp. the Necessity) problem with the following PAP \( \mathcal{P}' = (V', H', M', T') \):

\[
\begin{align*}
V' &= V \cup \{ h, h', m' \} \\
H' &= H \cup \{ h, h' \} \\
M' &= M \cup \{ m' \} \\
T' &= \{ \neg h \lor \varphi \} \cup \{ h' \lor m \mid m \in M \} \cup \{ \neg h \lor \neg h', h \rightarrow m', h' \rightarrow m' \}
\end{align*}
\]

This reduction fulfills the following equivalences:

\[ \mathcal{P} \text{ has at least one solution iff } h \text{ is relevant in } \mathcal{P}' \text{ iff } h' \text{ is not necessary in } \mathcal{P}'. \]

Give a rigorous proof of these equivalences.

**Hints.**

- Show both directions of the first equivalence separately:

  For the “\( \Rightarrow \)”-direction, you start off with a solution \( S \) of \( \mathcal{P} \) and construct a solution \( S' \) of \( \mathcal{P}' \) with \( h \in S' \). Prove carefully that \( S' \) is indeed a solution of \( \mathcal{P}' \), i.e. (1) \( T' \cup S' \) is satisfiable and (2) \( T' \cup S' \models M' \).

  For the “\( \Leftarrow \)”-direction, you start off with a solution \( S' \) of \( \mathcal{P}' \), s.t. \( h \in S' \) and construct a solution \( S \) of \( \mathcal{P} \). Prove carefully that \( S \) is indeed a solution of \( \mathcal{P} \), i.e. (1) \( T \cup S \) is satisfiable and (2) \( T \cup S \models M \).

- The second equivalence follows easily from the clauses \( \{ \neg h \lor \neg h', h \rightarrow m', h' \rightarrow m' \} \) in \( T' \), i.e., every solution of \( \mathcal{P}' \) contains exactly one of \( \{ h, h' \} \).