Exercise 1 (5 credits) Recall the following characterizations of the complexity classes $\Sigma_i P$ and $\Pi_i P$ for $i \geq 1$.

**Theorem.**

- Let $L$ be a language and $i \geq 1$. Then $L \in \Sigma_i P$ iff there is a polynomially balanced relation $R$ such that the language $\{x \# y \mid (x, y) \in R\}$ is in $\Pi_{i-1} P$ and
  $$L = \{x \mid \text{there exists a } y \text{ with } |y| \leq |x|^k \text{ s.t. } (x, y) \in R\}$$

- Let $L$ be a language and $i \geq 1$. Then $L \in \Pi_i P$ iff there is a polynomially balanced relation $R$ such that the language $\{x \# y \mid (x, y) \in R\}$ is in $\Sigma_{i-1} P$ and
  $$L = \{x \mid \text{for all } y \text{ with } |y| \leq |x|^k, (x, y) \in R\}$$

**Corollary.**

- Let $L$ be a language and $i \geq 1$. Then $L \in \Sigma_i P$ iff there is a polynomially balanced, polynomial-time decidable $(i+1)$-ary relation $R$ such that
  $$L = \{x \mid \exists y_1 \exists y_2 \exists y_3 \cdots Q y_i \text{ such that } (x, y_1, \ldots, y_i) \in R\}$$
  where $Q$ is $\forall$ if $i$ is even and $\exists$ if $i$ is odd.

- Let $L$ be a language and $i \geq 1$. Then $L \in \Pi_i P$ iff there is a polynomially balanced, polynomial-time decidable $(i+1)$-ary relation $R$ such that
  $$L = \{x \mid \forall y_1 \exists y_2 \exists y_3 \cdots Q y_i \text{ such that } (x, y_1, \ldots, y_i) \in R\}$$
  where $Q$ is $\exists$ if $i$ is even and $\forall$ if $i$ is odd.

Give a rigorous proof of this corollary. It suffices to prove the correctness of the characterization of $\Sigma_i P$. The characterization of $\Pi_i P$ follows immediately.
**Hint.** Use the above theorem and proceed by induction on $i$.

**Exercise 2 (5 credits)** Recall the $\Sigma_2^P$-hardness proof of MINIMAL MODEL SAT by reduction from the QSAT$_2$-problem: Let an arbitrary instance of QSAT$_2$ be given by the QBF

$$\psi = (\exists x_1, \ldots, x_k)(\forall y_1, \ldots, y_l)\varphi$$

Now let $\{x'_1, \ldots, x'_k, z\}$ be fresh propositional variables. Then we construct an instance of MINIMAL MODEL SAT by the variable $z$ and the formula

$$\chi = \bigwedge_{i=1}^k (\neg x_i \leftrightarrow x'_i) \land (\neg \varphi \lor (y_1 \land \ldots \land y_l \land z))$$

Recall from the lecture that we have already proved the following implication:

$\psi$ is true (in every interpretation) $\Rightarrow$ $z$ is true in a minimal model of $\chi$.

Give a rigorous proof also of the opposite direction, i.e.:

$z$ is true in a minimal model of $\chi$ $\Rightarrow$ $\psi$ is true (in every interpretation).

**Hint.** Let $J$ be a minimal model of $\chi$ and let $z$ be true in $J$.

- First show that then $J(y_j) = \text{true}$ for every $j$.

- Second, let $I$ be the truth assignment obtained by restricting $J$ to the variables $\{x_1, \ldots, x_k\}$. Show that (by the minimality of $J$) $I$ is indeed a partial assignment on $\{x_1, \ldots, x_k\}$ s.t. for any values assigned to $\{y_1, \ldots, y_l\}$, the formula $\varphi$ is true.