In the lecture “Formale Methoden der Informatik”, the following Cook reduction from co-2-SAT to the REACHABILITY problem was given:

- The variables of $\varphi$ and their negations form the vertices of $G(\varphi)$.
- There is an arc $(\alpha, \beta)$ iff there is a clause $\bar{\alpha} \lor \beta$ or $\beta \lor \bar{\alpha}$ in $\varphi$, where $\bar{\alpha}$ is the complement of $\alpha$, i.e.: If $\alpha$ is true in some satisfying assignment $I$ of $\varphi$, then $\beta$ must also be true in $I$.
- It can be shown that $\varphi$ is unsatisfiable iff there is a variable $x$ such that there are paths from $x$ to $\neg x$ and from $\neg x$ to $x$ in $G(\varphi)$.

**Notation.** It is convenient to write $x \Rightarrow y$ if $y$ is reachable from $x$ in the graph $G(\varphi)$.

**Exercise 1 (4 credits)** Give a rigorous proof of the “if”-direction of the correctness of the above reduction, i.e.: If there exists a variable $x$, s.t. $x \Rightarrow \neg x$ and $\neg x \Rightarrow x$ in $G(\varphi)$, then $\varphi$ is unsatisfiable.

**Hint.** Carefully distinguish between what is assumed, what is defined and what has to be shown. The proof could thus start as follows:

**Proof.** (indirect) Suppose that there exists a variable $x$, s.t. $x \Rightarrow \neg x$ and $\neg x \Rightarrow x$ in $G(\varphi)$. Moreover, suppose that there exists a model $I$ of $\varphi$. We derive a contradiction by showing that then both $x$ and $\neg x$ are true in $I$.

For the “only if”-direction of the correctness proof of the problem reduction from co-2-SAT to REACHABILITY, we have to show the following implication: If there exists no variable $x$, s.t. $x \Rightarrow \neg x$ and $\neg x \Rightarrow x$ in $G(\varphi)$, then there exists a model $I$ of $\varphi$. To this end, consider the truth assignment $I$ constructed by the following algorithm:

/* Step 1 */
for each literal $x$, s.t. $\bar{x} \Rightarrow x$ do
        $I(x) := \text{true}; /*$ hence, implicitly, $I(\bar{x}) := \text{false}; */$
for each literal $z$ with $x \Rightarrow z$ do $I(z) := \text{true}$ od;

/* Step 2 */
while there exists $x \in \text{Var}(\varphi)$, s.t. $I(x)$ is undefined do
  choose an arbitrary variable $x$, s.t. $I(x)$ is undefined;
  $I(x) := \text{true}$;
  for each literal $z$ with $x \Rightarrow z$ do $I(z) := \text{true}$ od;
od;

To show that an assignment $I$ thus constructed is indeed a model of $\varphi$, it is convenient to proof the following lemmas.

**Lemma 1** Suppose that there exists no variable $x$, s.t. both $x \Rightarrow x$ and $\overline{x} \Rightarrow x$ hold. Then a truth assignment made by the above algorithm is never changed later, i.e.: it cannot happen, that at some stage, $I(z) = \text{true}$ for some variable $z$ and later this value is changed to $I(z) = \text{false}$ or vice versa.

**Lemma 2** Suppose that there exists no variable $x$, s.t. both $x \Rightarrow x$ and $\overline{x} \Rightarrow x$ hold. Then the truth assignment $I$ constructed by the above algorithm is a model of $\varphi$.

**Exercise 2 (4 credits)** Give a rigorous proof of Lemma 1.

**Exercise 3 (2 credits)** Give a rigorous proof of Lemma 2.