Journal of Computer and System Sciences 76 (2010) 634–649

Contents lists available at ScienceDirect

Journal of Computer and System Sciences

www.elsevier.com/locate/jcss

Counting complexity of propositional abduction ☆

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ARTICLE INFO

Article history:
Received 16 January 2009
Received in revised form 4 November 2009
Available online 21 December 2009

Keywords:
Computational complexity
Counting complexity
Propositional abduction
Horn
Definite Horn
Dual Horn
Bijunctive formulas

ABSTRACT

Abduction is an important method of non-monotonic reasoning with many applications in artificial intelligence and related topics. In this paper, we concentrate on propositional abduction, where the background knowledge is given by a propositional formula. Decision problems of great interest are the existence and the relevance problems. The complexity of these decision problems has been systematically studied while the counting complexity of propositional abduction has remained obscure. The goal of this work is to provide a comprehensive analysis of the counting complexity of propositional abduction in various settings.

1. Introduction

Abduction is a method of non-monotonic reasoning which has taken a fundamental importance in artificial intelligence and related topics. It is widely used to produce explanations for observed symptoms and manifestations, therefore it has an important application field in diagnosis – notably in the medical domain (see [24]). Other important applications of abduction can be found in planning, database updates, data-mining and many more areas (see e.g. [16,17,23]).

Logic-based abduction can be formally described as follows. Given a logical theory T formalizing an application, a set M of manifestations, and a set H of hypotheses, find an explanation $\mathcal{S}$ for M, i.e., a suitable set $\mathcal{S} \subseteq H$ such that $T \cup \mathcal{S}$ is consistent and logically entails M. In this paper we consider *propositional abduction problems*, where the theory T is represented by a propositional formula over a Boolean algebra $\mathbb{B} = (\{0,1\}; \vee, \wedge, \neg, \rightarrow, \equiv)$ or a Boolean field $\mathbb{Z}_2 = (\{0,1\}; +, \wedge)$, and the sets of hypotheses H together with the manifestations M consist of variables V. A *system diagnosis problem* can be represented by a propositional abduction problem $\mathcal{P} = \langle V, H, M, T\rangle$ as follows. The theory T is the system description. The hypotheses $H \subseteq V$ describe the possibly faulty system components. The manifestations $M \subseteq V$ are the observed symptoms, describing the malfunction of the system. The solutions $\mathcal{S}$ of $\mathcal{P}$ are the possible explanations of the malfunction.

Example 1. Consider the following football knowledge base.

$$T = \left\{ \begin{array}{c} \mathit{weak\_defense} \wedge \mathit{weak\_attack} \rightarrow \mathit{match\_lost}, \\ \mathit{match\_lost} \rightarrow \mathit{manager\_sad} \wedge \mathit{press\_angry}, \\ \mathit{star\_injured} \rightarrow \mathit{manager\_sad} \wedge \mathit{press\_sad} \end{array} \right\}$$

☆ This paper is an extended version of results which appeared as [13] and [14].
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[1] The author was supported by the Austrian Science Fund (FWF), project P20704-N18.

0022-0000/$ – see front matter
doi:10.1016/j.jcss.2009.12.001

Moreover, let the set of observed manifestations and the set of hypotheses be

$$M = \{manager\_sad\}$$

$$H = \{star\_injured, weak\_defense, weak\_attack\}$$

This propositional abduction problem has five abductive explanations (= "solutions").

$$\mathcal{S}_1 = \{star\_injured\}$$

$$\mathcal{S}_2 = \{weak\_defense, weak\_attack\}$$

$$\mathcal{S}_3 = \{star\_injured, weak\_attack\}$$

$$\mathcal{S}_4 = \{star\_injured, weak\_defense\}$$

$$\mathcal{S}_5 = \{star\_injured, weak\_defense, weak\_attack\}$$

Obviously, in the above example, not all solutions are equally intuitive. Indeed, for many applications, one is not interested in *all* solutions of a given propositional abduction problem $\mathcal{P}$ but only in *all acceptable* solutions of $\mathcal{P}$. *Acceptable* in this context means *minimal* with respect to some preorder $\preccurlyeq$ on the powerset 2^H. Two natural preorders are set-inclusion $\subseteq$ and smaller cardinality denoted as $\leqslant$. If we have a weight function on the hypotheses then we may define the acceptable solutions as the weight-minimal ones. This preorder (i.e., smaller weight) is denoted as $\sqsubseteq$. Finally, if indeed all solutions are acceptable, then the corresponding preorder is the syntactic equality $=$.

Note that the various notions of minimality arising from the above mentioned preorders $\subseteq$, $\leqslant$, and $\sqsubseteq$ are very natural requirements. The intuition behind $\subseteq$-minimality is essentially that of redundancy elimination from the solutions, i.e.: we want to eliminate all those hypotheses from an explanation which are not needed to explain the observed symptoms. The intuition behind $\leqslant$ and $\sqsubseteq$ is usually twofold, namely to find explanations with highest probability and/or with minimal repair requirements. More precisely, if the failure of any component in a system is independent of the failure of the other components and all components have equal failure probability, then explanations with minimum cardinality are the ones with highest probability. Likewise, if the repair of each individual component is assumed to cause essentially the same cost, then the solutions with minimal cardinality are precisely the ones with minimal cost of repair. If we have numeric values available for the repair cost or for the robustness of each component (e.g., based on data such as the empirically collected mean time to failure and component age), then weight-minimal abduction seeks for the cheapest repair respectively for the most likely explanation. In Example 1, only the solutions $\mathcal{S}_1$ and $\mathcal{S}_2$ are subset-minimal and only $\mathcal{S}_1$ is cardinality-minimal. Moreover, suppose that we have a weight function w on the hypotheses with $f(weak\_defense) = 10$, $f(weak\_attack) = 20$, $f(star\_injured) = 50$. These weights could, for instance, express the cost of repair (in millions of € to engage a new player in order to reinforce the defense or the attack, or to engage a new star, respectively). Then $\mathcal{S}_2$ is the only weight-minimal solution.

All three criteria $\subseteq$, $\leqslant$, and $\sqsubseteq$ can be further refined by a hierarchical organization of the hypotheses according to some *priorities* (cf. [9]). The resulting preorder is denoted by $\subseteq_P$, $\leqslant_P$, and $\sqsubseteq_P$, respectively. Priorities are particularly useful if different sets of components can be ranked according to some criterion that is not well-suited for numeric values (like, e.g., a qualitative rather than a quantitative robustness measure of components, the accessibility of components, or how critical the failure of a certain component would be). Then this ranking can be expressed by priorities on the hypotheses. For instance, suppose that for some reason we know that (for a specific team) *star_injured* is much less likely to occur than *weak_defense* and *weak_attack*. This judgment can be formalized by assigning lower priority to the former. Then $\mathcal{S}_2$ is the only minimal solution with respect to the preorders $\subseteq_P$ and $\leqslant_P$. Actually, in this simple example, $\mathcal{S}_2$ is also the only $\sqsubseteq_P$-minimal solution independently of the particular weight function.

The usually observed algorithmic problem in logic-based abduction is the existence problem, i.e. deciding whether at least one solution $\mathcal{S}$ exists for a given abduction problem $\mathcal{P}$. Another well-studied decision problem is the so-called relevance problem, i.e. given a propositional abduction problem $\mathcal{P}$ and a hypothesis $h \in H$, is h part of at least one acceptable solution? However, this approach is not always satisfactory. Especially in database applications, in diagnosis, and in data-mining there exist situations where we need to know *all* acceptable solutions of the abduction problem or at least an important part of them. Consequently, the enumeration problem (i.e., the computation of all acceptable solutions) has received much interest (see e.g. [6,7]). Another natural question is concerned with the total number of solutions to the considered problem. The latter problem refers to the *counting complexity* of abduction. Clearly, the counting complexity provides a lower bound for the complexity of the enumeration problem. Moreover, counting the number of abductive explanations can be useful for probabilistic abduction problems (see e.g. [25]). Indeed, in order to compute the probability of failure of a given component in a diagnosis problem (under the assumption that all preferred explanations are equiprobable), we need to count the number of preferred explanations as well as the number of preferred explanations that contain a given hypothesis.

The study of counting complexity has been initiated by Valiant [28,29] and is now a well-established part of complexity theory, where the best known class is #P. Many counting variants of decision problems have been proved #P-complete. Higher counting complexity classes do exist, but they are not commonly known. A counting equivalent of the polynomial

Table 1
Counting complexity of propositional abduction.

#-ABDUCTION	$=$	$\subseteq$	$\subseteq_P$	$\leqslant$	$\leqslant_P$, $\sqsubseteq$, $\sqsubseteq_P$
General case	$\#\cdot\mathrm{coNP}$	$\#\cdot\mathrm{coNP}$	$\#\cdot\Pi_2\mathrm{P}$	$\#\cdot\mathrm{Opt}_2\mathrm{P}[\log n]$	$\#\cdot\mathrm{Opt}_2\mathrm{P}$
Horn	#P	#P	$\#\cdot\mathrm{coNP}$	$\#\cdot\mathrm{OptP}[\log n]$	$\#\cdot\mathrm{OptP}$
Definite Horn	#P	#P	#P	$\#\cdot\mathrm{OptP}[\log n]$	$\#\cdot\mathrm{OptP}$
Dual Horn	#P	#P	#P	$\#\cdot\mathrm{OptP}[\log n]$	$\#\cdot\mathrm{OptP}$
Bijunctive	#P	#P	$\#\cdot\mathrm{coNP}$	$\#\cdot\mathrm{OptP}[\log n]$	$\#\cdot\mathrm{OptP}$

hierarchy was defined by Hemaspaandra and Vollmer [12], whereas generic complete problems for these counting hierarchy classes were presented in [4]. For our complexity analysis here, the classes #P, $\#\cdot\mathrm{coNP} = \#\cdot\Pi_1\mathrm{P}$, and $\#\cdot\Pi_2\mathrm{P}$ will play an important role (for details, see Section 2.2). More specifically, we shall show that all relevant counting problems for propositional abduction with the preorders $=$, $\subseteq$, and $\subseteq_P$ are either tractable or complete for one of these classes.

In [15], we enlarged the approach of Hemaspaandra and Vollmer to classes of optimization problems, obtaining this way a new hierarchy of classes $\#\cdot\mathrm{Opt}_k\mathrm{P}[\log n]$ and $\#\cdot\mathrm{Opt}_k\mathrm{P}$ for arbitrary $k \in \mathbb{N}$ (again, see Section 2.2, for details). These classes are sandwiched between the previously known counting classes $\#\cdot\Pi_k\mathrm{P}$, i.e., for each $k \in \mathbb{N}$ we have

$$\#\cdot\Pi_k\mathrm{P} \subseteq \#\cdot\mathrm{Opt}_{k+1}\mathrm{P}[\log n] \subseteq \#\cdot\mathrm{Opt}_{k+1}\mathrm{P} \subseteq \#\cdot\Pi_{k+1}\mathrm{P}.$$

It was shown in [15] that these inclusions are proper unless the polynomial hierarchy collapses to the k-th level. The most important special case is $k = 1$, where we write $\#\cdot\mathrm{OptP}[\log n]$ and $\#\cdot\mathrm{OptP}$ as a short-hand for $\#\cdot\mathrm{Opt}_1\mathrm{P}[\log n]$ and $\#\cdot\mathrm{Opt}_1\mathrm{P}$. On the first two levels, we thus have the inclusions $\#\mathrm{P} \subseteq \#\cdot\mathrm{OptP}[\log n] \subseteq \#\cdot\mathrm{OptP} \subseteq \#\cdot\mathrm{coNP} \subseteq \#\cdot\mathrm{Opt}_2\mathrm{P}[\log n] \subseteq \#\cdot\mathrm{Opt}_2\mathrm{P} \subseteq \#\cdot\Pi_2\mathrm{P}$. We shall show that these new counting complexity classes are precisely the ones needed to pinpoint the exact counting complexity of propositional abduction with the preorders $\leqslant$, $\sqsubseteq$, $\leqslant_P$ and $\sqsubseteq_P$.

1.1. Results

The goal of this work is to provide a comprehensive analysis of the counting complexity of propositional abduction in various settings. An overview of our results is given in Table 1. The columns of this table correspond to the seven preorders on 2^H considered here for defining the notion of *acceptable* solutions, namely equality $=$, subset-minimality $\subseteq$, subset-minimality with priorities $\subseteq_P$, cardinality-minimality $\leqslant$, cardinality-minimality with priorities $\leqslant_P$, weight-minimality $\sqsubseteq$, and weight-minimality with priorities $\sqsubseteq_P$. All entries in Table 1 refer to completeness results.

Apart from the general case where the theory T is an arbitrary propositional formula, we also consider the subclasses of Horn, definite Horn, dual Horn, and bijunctive theories T. These classes enjoy several favorable properties. Among other properties, they are closed under conjunction and existential quantification, i.e., a conjunction of two formulas from C belongs to the class C and a formula from C with an existentially quantified variable is logically equivalent to another formula from C. Moreover, they represent the most studied formulas in logic, complexity, and artificial intelligence. This is mainly due to Schaefer's famous result that the satisfiability problem for them (as well as for affine formulas) is polynomial as opposed to the NP-completeness of the general case (see [26]). A counting complexity analysis for abduction with affine theories is more subtle: affine theories are conjunctions of linear equations over the Boolean ring $\mathbb{Z}_2$, hence they cannot be expressed as a conjunction of clauses, and they require the application of methods from linear algebra. They will therefore be the subject of a standalone upcoming work.

1.2. Related work

The complexity of logic-based propositional abduction, formulated as a decision problem asking for the existence of a solution, has been intensively investigated in the literature. It was a common folklore to believe that abduction is intractable in general. A first result[2] on intractability of propositional abduction was published by Bylander et al. in [1], where the authors also identified several tractable cases. The computational complexity effort concerning abduction was pursued by Selman and Levesque [27], proving that abduction with Horn clauses is NP-complete. Eshghi presented in [8] a tractable subclass of abduction problems. Eiter and Gottlob [5] were the first to prove the $\Sigma_2\mathrm{P}$-completeness result for the general case, as well as a plethora of other complexity results for several special cases and minimality criteria. In [3] del Val generalized and enlarged the analysis of tractable cases performed by Eshghi. Zanuttini presented in [30] yet another collection of new polynomial-time classes for abduction. Nordh and Zanuttini presented in [21] a classification of propositional abduction based on algebraic properties. Finally, Creignou and Zanuttini published in [2] a complete classification of the complexity of propositional abduction. See also the excellent survey by Nordh and Zanuttini [22] on complexity results for propositional abduction. However, all the aforementioned results, apart from a #P-completeness result briefly mentioned in [1] for *independent* abduction problems (which correspond in our notation to $\subseteq$-minimal bijunctive definite Horn abduction), concern

[2] Bylander et al. published first their results in two conferences in 1987 and 1989 before writing the final version for a scientific journal.

only the decision problem, sometimes with slight differences in the definition of the abduction problem, whereas almost no general complexity analysis is known so far for the corresponding counting problem, with the already mentioned exception.

1.3. Structure of the paper

The paper is organized as follows. After recalling some basic definitions and results in Section 2, we analyze the counting complexity of propositional abduction for general theories (Section 3), for Horn, definite Horn, dual Horn and bijunctive theories (Section 4). We conclude with Section 5.

2. Preliminaries

2.1. Propositional abduction

A *propositional abduction problem* (PAP) $\mathcal{P}$ consists of a tuple $\langle V, H, M, T\rangle$, where V is a finite set of *variables*, $H \subseteq V$ is the set of *hypotheses*, $M \subseteq V$ is the set of *manifestations*, and T is a consistent *theory* in the form of a propositional formula. A set $\mathcal{S} \subseteq H$ is a *solution* (also called *explanation*) to $\mathcal{P}$ if $T \cup \mathcal{S}$ is consistent and $T \cup \mathcal{S} \models M$ holds.

A *preorder* is a reflexive and transitive binary relation. Below, we define several preorders $\preccurlyeq$ on the powerset 2^H. They will allow us to define the corresponding restrictions of propositional abduction where only $\preccurlyeq$-minimal solutions are considered.

Definition 2. Let a propositional abduction problem $\mathcal{P}$ consist of a tuple $\langle V, H, M, T\rangle$ and let $A, B \subseteq H$. We consider the following preorders $\preccurlyeq$ on the powerset 2^H.

- The *equality* and *subset* preorders $A = B$ and $A \subseteq B$, respectively, are obvious.
- The *cardinality* preorder $A \leqslant B$ holds if the condition $|A| \leqslant |B|$ is satisfied by the cardinalities of the sets A and B.
- Suppose that a weight function w on the hypotheses H is given, i.e., $w : H \to \mathbb{N}$. The *weight* preorder $A \sqsubseteq B$ holds if the condition $\sum_{a\in A} w(a) \leqslant \sum_{b\in B} w(b)$ is satisfied.

Definition 3. Let a propositional abduction problem $\mathcal{P}$ consist of a tuple $\langle V, H, M, T\rangle$, let $A, B \subseteq H$, and let $P = \langle H_1, \dots, H_K\rangle$ be a stratification of the hypotheses $H = H_1 \cup \dots \cup H_K$ into disjoint sets $H_1, \dots, H_K$. The sets $H_1, \dots, H_K$ are referred to as *priorities*. Then we consider the following additional preorders $\preccurlyeq$ on the powerset 2^H:

- The *subset with priorities* preorder $A \subseteq_P B$ holds if $A = B$ or there exists an $i \in \{1, \dots, K\}$ such that $A \cap H_j = B \cap H_j$ for all $j < i$ and $A \cap H_i \subsetneq B \cap H_i$.
- The *cardinality with priorities* preorder $A \leqslant_P B$ holds if $A = B$ or there exists an $i \in \{1, \dots, K\}$ such that $|A \cap H_j| = |B \cap H_j|$ for all $j < i$ and $|A \cap H_i| < |B \cap H_i|$.
- Suppose that a weight function w on the hypotheses H is given, i.e., $w : H \to \mathbb{N}$. The *weight with priorities* preorder $A \sqsubseteq_P B$ holds if $A = B$ or there exists an $i \in \{1, \dots, K\}$ such that $\sum_{a\in A\cap H_j} w(a) = \sum_{b\in B\cap H_j} w(b)$ for all $j < i$ and $\sum_{a\in A\cap H_i} w(a) < \sum_{b\in B\cap H_i} w(b)$.

Definition 4. Let a PAP $\mathcal{P}$ consist of a tuple $\langle V, H, M, T\rangle$ and let $\preccurlyeq \in \{=, \subseteq, \leqslant, \sqsubseteq, \subseteq_P, \leqslant_P, \sqsubseteq_P\}$ be a preorder on 2^H. Moreover, let $\mathcal{S} \subseteq H$ be a solution of $\mathcal{P}$. We say that $\mathcal{S}$ *is* $\preccurlyeq$*-minimal* if there does not exist a solution $\mathcal{S}'$ of $\mathcal{P}$ with $\mathcal{S}' \prec \mathcal{S}$, i.e., $\mathcal{S}' \preccurlyeq \mathcal{S}$ and $\mathcal{S} \neq \mathcal{S}'$.

Let $\preccurlyeq \in \{=, \subseteq, \leqslant, \sqsubseteq, \subseteq_P, \leqslant_P, \sqsubseteq\}$ be one of the preorders defined in Definitions 2 and 3. We study the following family of counting problems, parameterized by $\preccurlyeq$:

Problem: #-$\preccurlyeq$-ABDUCTION
Input: A propositional abduction problem $\mathcal{P} = \langle V, H, M, T\rangle$.
Output: Number of $\preccurlyeq$-minimal solutions (explanations) of $\mathcal{P}$.

The abduction counting problem with the equality preorder is usually denoted by #-ABDUCTION rather than #-=-abduction. Throughout this paper we follow the formalism of Eiter and Gottlob [5], allowing only positive literals in the solutions. In contrast, Creignou and Zanuttini [2] also allow negative literals in the solutions.

Example 5. Recall the PAP $\mathcal{P} = \langle V, H, M, T\rangle$ from Example 1 with

$$V = \{star\_injured, weak\_defense, weak\_attack, manager\_sad, press\_sad, match\_lost\}$$

$$H = \{star\_injured, weak\_defense, weak\_attack\}$$

$$M = \{manager\_sad\}$$
$$T = \left\{ \begin{array}{c} weak\_defense \wedge weak\_attack \rightarrow match\_lost, \\ match\_lost \rightarrow manager\_sad \wedge press\_angry, \\ star\_injured \rightarrow manager\_sad \wedge press\_sad \end{array} \right\}$$

It is convenient to use the abbreviations *SI*, *WD*, and *WA* for the hypotheses in H. For the various preorders $\preccurlyeq$ from Definitions 2 and 3, $\mathcal{P}$ has the following $\preccurlyeq$-minimal solutions:

Preorder $=$. Every solution of a PAP is $=$-minimal. Hence, $\mathcal{P}$ has the following $=$-minimal solutions:

$$\mathcal{S}_1 = \{SI\}, \qquad \mathcal{S}_2 = \{WD, WA\}, \qquad \mathcal{S}_3 = \{SI, WA\}, \qquad \mathcal{S}_4 = \{SI, WD\}, \qquad \mathcal{S}_5 = \{SI, WD, WA\}$$

Preorders $\subseteq$ *and* $\leqslant$. The $\subseteq$-minimal solutions are $\mathcal{S}_1$ and $\mathcal{S}_2$. The only $\leqslant$-minimal solution is $\mathcal{S}_1$.

Preorder $\sqsubseteq$. Suppose that a weight function w on the hypotheses H is given with $w(SI) = 50$, $w(WD) = 10$, and $w(WA) = 20$. Then $\mathcal{S}_2$ is the only $\sqsubseteq$-minimal solution of $\mathcal{P}$.

Preorders $\subseteq_P$, $\leqslant_P$, *and* $\sqsubseteq_P$. We consider the priorities $H_1 = \{SI\}$ and $H_2 = \{WD, WA\}$. As we have already discussed in Section 1, $\mathcal{S}_2$ is the only $\subseteq_P$-minimal and the only $\leqslant_P$-minimal. Moreover, for any weight function w on the hypotheses H, $\mathcal{S}_2$ is also the only $\sqsubseteq_P$-minimal solution. Below, we illustrate that a different choice of priorities may change the situation.

Preorder $\leqslant_P$. We consider the priorities $H'_1 = \{SI, WD\}$ and $H'_2 = \{WA\}$. In this case, $\mathcal{S}_1 = \{SI\}$ is the only $\leqslant_P$-minimal solution of $\mathcal{P}$. This can be easily seen as follows: We have $|\mathcal{S}_1 \cap H'_1| = 1$ and $|\mathcal{S}_1 \cap H'_2| = 0$. The only possibility that $\mathcal{S}_1$ is *not* $\leqslant_P$-minimal is that there exists a solution $\mathcal{S}'$ of $\mathcal{P}$ with $|\mathcal{S}' \cap H'_1| = 0$. However, none of the solutions $\mathcal{S}_2, \dots, \mathcal{S}_5$ fulfills this condition.

Preorder $\subseteq_P$. We consider the priorities $H'_1 = \{SI, WD\}$ and $H'_2 = \{WA\}$. There are two $\subseteq_P$-minimal solutions, namely $\mathcal{S}_1 = \{SI\}$ and $\mathcal{S}_2 = \{WD, WA\}$. The $\subseteq_P$-minimality of $\mathcal{S}_1$ is seen as follows: Since $(\mathcal{S}_1 \cap H'_2) = \emptyset$, the only possibility that $\mathcal{S}_1$ is *not* $\subseteq_P$-minimal is that there exists a solution $\mathcal{S}'$ of $\mathcal{P}$ with $(\mathcal{S}' \cap H'_1) \subset (\mathcal{S}_1 \cap H'_1)$. Since $(\mathcal{S}_1 \cap H'_1)$ is a singleton, this means that $(\mathcal{S}' \cap H'_1) = \emptyset$, i.e., $\mathcal{S}' \subseteq \{WA\}$. Clearly, no such solution exists.

Now let us verify that also $\mathcal{S}_2$ is $\subseteq_P$-minimal. Suppose to the contrary that it is not, i.e., there exists a $\subseteq_P$-smaller solution $\mathcal{S}'$. By the definition of $\subseteq_P$, this means that one of the following conditions holds: either (1) $(\mathcal{S}' \cap H'_1) \subset (\mathcal{S}_2 \cap H'_1)$ or (2) $(\mathcal{S}' \cap H'_1) = (\mathcal{S}_2 \cap H'_1)$ and $(\mathcal{S}' \cap H'_2) \subset (\mathcal{S}_2 \cap H'_2)$. The first possibility can be dismissed as before. Now suppose that condition (2) is fulfilled. The only set $\mathcal{S}'$ with this property is $\mathcal{S}' = \{WD\}$, which is clearly not a solution of $\mathcal{P}$. Hence, $\mathcal{S}_2$ is indeed $\subseteq_P$-minimal.

Preorder $\sqsubseteq_P$. We consider the priorities $H'_1 = \{SI, WD\}$ and $H'_2 = \{WA\}$. Now consider the weight function w' on the hypotheses H with $w'(SI) = 50$, $w'(WD) = 40$, and $w'(WA) = 20$. Then $\mathcal{S}_2$ is the only $\sqsubseteq_P$-minimal solution, even though the total weight of the hypotheses in $\mathcal{S}_2$ is 60 and thus exceeds the total weight of $\mathcal{S}_1$. Indeed, on the first priority level, we have $\sum_{s \in \mathcal{S}_2 \cap H_1} w'(s) = 40$ and there exists no solution $\mathcal{S}'$ with $\sum_{s \in \mathcal{S}' \cap H_1} w'(s) < 40$. Moreover, the only possibility to attain the minimal value $\sum_{s \in \mathcal{S}' \cap H_1} w'(s) = 40$ is if $\mathcal{S}' \cap H'_1 = \{WD\}$. But then, there exists no extension of $\mathcal{S}'$ to H'_2 with a smaller weight than $\mathcal{S}_2$, i.e., for any solution $\mathcal{S}'$ with $\mathcal{S}' \cap H'_1 = \{WD\}$, we clearly have $\sum_{s \in \mathcal{S}' \cap H'_2} w'(s) \geqslant 20 = \sum_{s \in \mathcal{S}_2 \cap H'_2} w'(s)$. Hence, $\mathcal{S}_2$ is indeed the only $\sqsubseteq_P$-minimal solution.

Together with the general case where T can be an arbitrary propositional formula, we consider the special cases where T is Horn, definite Horn, dual Horn, and bijunctive. Due to Schaefer's famous dichotomy result (see [26]), these classes of formulas (as well as the affine formulas not considered here) are the most frequently studied subcases of propositional formulas. A propositional clause C is said to be *Horn*, *definite Horn*, *dual Horn*, or *bijunctive* if it has at most one positive literal, exactly one positive literal, at most one negative literal, or at most two literals, respectively. A theory T is Horn, definite Horn, dual Horn, or bijunctive if it is a conjunction (or, equivalently, a set) of Horn, definite Horn, dual Horn, or bijunctive clauses, respectively.

2.2. Counting complexity

2.2.1. Counting problems and the complexity classes $\#\cdot\mathcal{C}$

The study of *counting problems* was initiated by Valiant in [28,29]. While decision problems ask if at least one solution of a given problem instance exists, counting problems ask for the number of different solutions. The most intensively studied counting complexity class is #P, which denotes the functions counting the number of accepting paths of a non-deterministic polynomial-time Turing machine. In other words, #P captures the counting problems corresponding to decision problems in NP. By allowing the non-deterministic polynomial-time Turing machine access to an oracle in NP, Σ_2P, Σ_3P, …, we can define an infinite hierarchy of counting complexity classes.

Alternatively, a *counting problem* is presented using a *witness* function which for every input x returns a set of *witnesses* for x. A *witness* function is a function $w : \Sigma^* \to \mathcal{P}^{<\omega}(\Gamma^*)$, where Σ and Γ are two alphabets, and $\mathcal{P}^{<\omega}(\Gamma^*)$ is the collection of all finite subsets of Γ^*. Every such witness function gives rise to the following *counting problem*: given a string $x \in \Sigma^*$, find the cardinality $|w(x)|$ of the *witness* set $w(x)$. According to [12], if $\mathcal{C}$ is a complexity class of decision problems, we define $\#\cdot\mathcal{C}$ to be the class of all counting problems whose witness function w satisfies the following conditions:

1. There is a polynomial $p(n)$ such that for every $x \in \Sigma^*$ and every $y \in w(x)$ we have $|y| \leqslant p(|x|)$;
2. The problem "given x and y, is $y \in w(x)$?" is in $\mathcal{C}$.

It is easy to verify that $\#\mathrm{P} = \#\cdot\mathrm{P}$. The counting hierarchy is ordered by linear inclusion [12]. In particular, we have that $\#\mathrm{P} \subseteq \#\cdot\mathrm{coNP} \subseteq \#\cdot\Pi_2\mathrm{P} \subseteq \#\cdot\Pi_3\mathrm{P}$, etc. Note that we can, of course, also consider the classes $\#\cdot\mathrm{NP}$, $\#\cdot\Sigma_2\mathrm{P}$, $\#\cdot\Sigma_3\mathrm{P}$, etc. However, they play no role in this work.

2.2.2. Counting the optimal solutions

In [15], we introduced new counting complexity classes for counting *optimal* solutions. We followed the aforementioned approach, where the complexity class $\mathcal{C}$ was chosen among OptP and OptP[$\log n$], or, more generally, $\mathrm{Opt}_k\mathrm{P}$ and $\mathrm{Opt}_k\mathrm{P}[\log n]$ for arbitrary $k \in \mathbb{N}$, respectively. These classes were previously defined by Krentel [19,20]. A large collection of completeness results for these classes is given in [11]. As Krentel observed, the classes OptP[$\log n$] and OptP, which are closely related to $\mathrm{FP}^{\mathrm{NP}[\log n]}$ and $\mathrm{FP}^{\mathrm{NP}}$, contain problems of computing optimal solutions with a logarithmic and polynomial number of calls to an NP-oracle, respectively.

The application of the counting operator to the aforementioned optimization classes allowed us to define in [15] the counting complexity classes $\#\cdot\mathrm{OptP}$, $\#\cdot\mathrm{OptP}[\log n]$ and, more generally, $\#\cdot\mathrm{Opt}_k\mathrm{P}$, $\#\cdot\mathrm{Opt}_k\mathrm{P}[\log n]$ for each $k \in \mathbb{N}$. To formally introduce these classes, we need some supplementary notions.

A *non-deterministic transducer M* is a non-deterministic polynomial-time bounded Turing machine, which writes a binary number on the output at the end of every accepting path. If M is equipped with an oracle from the complexity class $\mathcal{C}$, then it is called a *non-deterministic transducer with* $\mathcal{C}$*-oracle*. A $\Sigma_k\mathrm{P}$*-transducer* M is a non-deterministic transducer with a $\Sigma_{k-1}\mathrm{P}$ oracle. We identify non-deterministic transducers without oracle and $\Sigma_1\mathrm{P}$-transducers. For $x \in \Sigma^*$, we write $\mathrm{opt}_M(x)$ to denote the *optimal* value, which can be either the *maximum* or the *minimum*, on any accepting path of the computation of M on x. If no accepting path exists then $\mathrm{opt}_M(x)$ is undefined.

We say that a counting problem $\#\cdot A : \Sigma^* \to \mathbb{N}$ is in the class $\#\cdot\mathrm{Opt}_k\mathrm{P}$ for some $k \in \mathbb{N}$, if there is a $\Sigma_k\mathrm{P}$-transducer M, such that $\#\cdot A(x)$ is the number of accepting paths of the computation of M on x yielding the optimum value $\mathrm{opt}_M(x)$. If no accepting path exists then $\#\cdot A(x) = 0$. If the length of the binary number written by M is bounded by $O(\log|x|)$, then $\#\cdot A$ is in the class $\#\cdot\mathrm{Opt}_k\mathrm{P}[\log n]$. For $k = 1$, we write $\#\cdot\mathrm{OptP}[\log n]$ and $\#\cdot\mathrm{OptP}$ as a short-hand for $\#\cdot\mathrm{Opt}_1\mathrm{P}[\log n]$ and $\#\cdot\mathrm{Opt}_1\mathrm{P}$, respectively. It was shown in [15] that these new classes $\#\cdot\mathrm{Opt}_k\mathrm{P}[\log n]$ and $\#\cdot\mathrm{Opt}_k\mathrm{P}$ indeed form an infinite hierarchy and do not coincide with already known counting complexity classes unless the polynomial hierarchy collapses. Finally, these new counting classes were shown to be sandwiched between the classes $\#\cdot\Pi_k\mathrm{P}$, i.e., we obtained the inclusions $\#\mathrm{P} \subseteq \#\cdot\mathrm{OptP}[\log n] \subseteq \#\cdot\mathrm{OptP} \subseteq \#\cdot\mathrm{coNP} \subseteq \#\cdot\mathrm{Opt}_2\mathrm{P}[\log n] \subseteq \#\cdot\mathrm{Opt}_2\mathrm{P} \subseteq \#\cdot\Pi_2\mathrm{P}$, etc.

2.2.3. Reductions

Completeness of counting problems in #P is usually proved by means of Turing reductions. However, these reductions preserve neither the counting classes $\#\cdot\Pi_k\mathrm{P}$, nor $\#\cdot\mathrm{Opt}_k\mathrm{P}$. It is therefore better to use *subtractive reductions* [4] which preserve the aforementioned counting classes. We write $\#\cdot R$ to denote the following counting problem: given a string $x \in \Sigma^*$, find the cardinality $|R(x)|$ of the witness set $R(x)$ associated with x. The counting problem $\#\cdot A$ reduces to $\#\cdot B$ via a *strong subtractive reduction* if there exist two polynomial-time computable functions f and g such that for each $x \in \Sigma^*$ we have

$$B\bigl(f(x)\bigr) \subseteq B\bigl(g(x)\bigr) \quad \text{and} \quad \bigl|A(x)\bigr| = \bigl|B\bigl(g(x)\bigr)\bigr| - \bigl|B\bigl(f(x)\bigr)\bigr|$$

A strong subtractive reduction with $B(f(x)) = \emptyset$ is called *parsimonious*. *Subtractive reductions* are the transitive closure of strong subtractive reductions.

2.2.4. Complete problems

The prototypical $\#\cdot\Pi_k\mathrm{P}$-complete problem for $k \in \mathbb{N}$ is $\#\Pi_k\mathrm{SAT}$ [4], defined as follows. Given a formula

$$\psi(X) = \forall Y_1\, \exists Y_2 \cdots \mathsf{Q}_k Y_k\, \varphi(X, Y_1, \ldots, Y_k)$$

where φ is a Boolean formula and $X, Y_1, \ldots, Y_k$ are sets of propositional variables, count the number of truth assignments to the variables in X that satisfy ψ. We obtain the prototypical $\#\cdot\mathrm{Opt}_{k+1}\mathrm{P}[\log n]$-complete counting problem $\#\textsc{min-card-}\Pi_k\textsc{sat}$ and the prototypical $\#\cdot\mathrm{Opt}_{k+1}\mathrm{P}$-complete problem $\#\textsc{min-weight-}\Pi_k\textsc{sat}$ [15] by asking for the number of cardinality-minimal and weight-minimal models of $\varphi(X)$, respectively. In the latter case, there exists a weight function $w : X \to \mathbb{N}$ assigning positive values to each variable $x \in X$. As usual, the counting problems $\#\textsc{min-card-}\Pi_0\textsc{sat}$ and $\#\textsc{min-weight-}\Pi_0\textsc{sat}$ are just denoted by #min-card-sat and #min-weight-sat, being respectively $\#\cdot\mathrm{OptP}[\log n]$- and $\#\cdot\mathrm{OptP}$-complete.

3. General propositional theories

The decidability problem of propositional abduction was shown to be Σ_2P-complete in [5]. The hardness part was proved via a reduction from QSAT$_2$. A modification of this reduction yields the following counting complexity result.

Theorem 6. *The counting problems #-ABDUCTION and #-$\subseteq$-ABDUCTION are #·coNP-complete via parsimonious reductions.*

Proof. The #·coNP-membership is clear by the fact that it is in Δ_2P to test whether a subset $\mathcal{S} \subseteq H$ is a solution (respectively a subset-minimal solution) of a given propositional abduction problem (see [5, Proposition 2.1.5]). The #·coNP-hardness is shown via the following parsimonious reduction from #Π_1SAT. Let an instance of the #Π_1SAT problem be given by a formula

$$\psi(X) = \forall Y\, \varphi(X, Y)$$

with the variable sets $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_l\}$. Moreover, let $x'_1, \ldots, x'_k$, $r_1, \ldots, r_k$, t denote fresh, pairwise distinct variables. Let $X' = \{x'_1, \ldots, x'_k\}$ and $R = \{r_1, \ldots, r_k\}$. We define the propositional abduction problem $\mathcal{P} = \langle V, H, M, T\rangle$ as follows:

$$V = X \cup X' \cup Y \cup R \cup \{t\}$$
$$H = X \cup X'$$
$$M = R \cup \{t\}$$
$$T = \bigl\{\neg x_i \vee \neg x'_i, x_i \to r_i, x'_i \to r_i \mid 1 \leqslant i \leqslant k\bigr\} \cup \bigl\{\varphi(X, Y) \to t\bigr\}$$

Obviously, this reduction is feasible in polynomial time. We now show that the reduction is indeed parsimonious.

The manifestations R together with the formulas $x_i \to r_i$, $x'_i \to r_i$ in T enforce that in every solution $\mathcal{S}$ of the propositional abduction problem, we have to select at least one of x_i and x'_i. The additional formula $\neg x_i \vee \neg x'_i$ enforces that we have to select at most one of x_i and x'_i. By these two conditions, the value of x'_i is fully determined by x_i, namely x'_i is the opposite of x_i.

Moreover, it is easy to check that there is a one-to-one relationship between the solutions $\mathcal{S} \subseteq X$ of $\mathcal{P}$ and the models of $\forall Y\, \varphi(X, Y)$. Hence, this reduction is indeed parsimonious. The complementarity of X and X' enforces each solution of be incomparable with the others and, therefore, to be subset-minimal. □

According to the above theorem, #-ABDUCTION and #-$\subseteq$-ABDUCTION have the same counting complexity. Intuitively, this is due to the following equivalence (cf. [5]): $\mathcal{S}$ is a $\subseteq$-minimal solution of the propositional abduction problem $\mathcal{P}$ if and only if $\mathcal{S}$ is a solution of $\mathcal{P}$ and for every $h \in \mathcal{S}$, $\mathcal{S} \setminus \{h\}$ is not a solution. Hence, taking the $\subseteq$-minimality into account makes things only polynomially harder. In contrast, as soon as there are at least 2 priority levels, the following effect may occur. Suppose that $\mathcal{S}$ is a solution of the propositional abduction problem and that $\mathcal{S} \setminus \{h\}$ is a solution for no $h \in \mathcal{S}$. Then it might well happen that, for some $h \in \mathcal{S}$, some set of the form $\mathcal{S}' = (\mathcal{S} \setminus \{h\}) \cup X$ is a solution, where all hypotheses in X have higher priority than h. Checking if such a set $\mathcal{S}'$ (and, in particular, if such a set X) exists comes down to yet another non-deterministic guess. Formally, we thus get the following complexity result.

Theorem 7. *The counting problem #-$\subseteq_P$-ABDUCTION is #·Π_2P-complete via subtractive reductions.*

Proof. The $\subseteq_P$-minimal solutions of a propositional abduction problem can be computed by a non-deterministic polynomial-time Turing machine with Π_2P-oracle as follows: The Turing machine non-deterministically generates all subsets $\mathcal{S} \subseteq H$ and

(i) checks by an oracle call whether $\mathcal{S}$ is a solution of the propositional abduction problem and
(ii) if so, checks by another oracle call whether $\mathcal{S}$ is $\subseteq_P$-minimal.

The latter test – which is the most expensive part – can be done by a Π_2P-oracle. Indeed, the problem of testing that $\mathcal{S}$ is *not* $\subseteq_P$-minimal can be done by the following Σ_2P-algorithm: guess a subset $\mathcal{S}' \subseteq H$ such that $\mathcal{S}'$ is $\subseteq_P$-smaller than $\mathcal{S}$ and check that $\mathcal{S}'$ is a solution of the propositional abduction problem. Hence, the #-$\subseteq_P$-ABDUCTION problem is in #·Π_2P.

The #·Π_2P-hardness is shown by the following (strong) subtractive reduction from #Π_2SAT. Let an instance of the #Π_2SAT problem be given by a formula

$$\psi(X) = \forall Y\, \exists Z\, \varphi(X, Y, Z)$$

with the variables $X = \{x_1, \ldots, x_k\}$, $Y = \{y_1, \ldots, y_l\}$, and $Z = \{z_1, \ldots, z_m\}$. Moreover, let $x'_1, \ldots, x'_k$, $p_1, \ldots, p_k$, $y'_1, \ldots, y'_l$, $q_1, \ldots, q_l, r, t$ be new, pairwise distinct variables distributed among the sets $X' = \{x'_1, \ldots, x'_k\}$, $P = \{p_1, \ldots, p_k\}$, $Y' = \{y'_1, \ldots, y'_l\}$, and $Q = \{q_1, \ldots, q_l\}$. Then we define two propositional abduction problems $\mathcal{P}_1$ and $\mathcal{P}_2$ as follows:

$$V = X \cup X' \cup Y \cup Y' \cup Z \cup P \cup Q \cup \{r, t\}$$
$$H = X \cup X' \cup Y \cup Y' \cup \{r\} \quad \text{with priorities } H_1 = H \setminus Y' \quad \text{and} \quad H_2 = Y'$$
$$M = P \cup Q \cup \{t\}$$
$$T_1 = \{\neg x_i \vee \neg x_i', x_i \rightarrow p_i, x_i' \rightarrow p_i \mid 1 \leqslant i \leqslant k\} \cup \{\neg y_i \vee \neg y_i', y_i \rightarrow q_i, y_i' \rightarrow q_i \mid 1 \leqslant i \leqslant l\} \cup \{\neg\varphi(X, Y, Z) \rightarrow t\}$$
$$T_2 = T_1 \cup \{r \wedge y_1 \wedge \cdots \wedge y_l \rightarrow t\}$$

Finally we set $\mathcal{P}_1 = \langle V, H, M, T_1\rangle$ and $\mathcal{P}_2 = \langle V, H, M, T_2\rangle$.

Obviously, this reduction is feasible in polynomial time. Now let $A(\psi)$ denote the set of all satisfying assignments of a $\#\Pi_2$SAT-formula ψ and let $B(\mathcal{P})$ denote the set of $\subseteq_P$-minimal solutions of a propositional abduction problem $\mathcal{P}$. We claim that the above definition of the propositional abduction problems $\mathcal{P}_1$ and $\mathcal{P}_2$ is indeed a (strong) subtractive reduction, i.e. that

$$B(\mathcal{P}_1) \subseteq B(\mathcal{P}_2) \quad \text{and} \quad |A(\psi)| = |B(\mathcal{P}_2)| - |B(\mathcal{P}_1)|$$

In order to prove this claim, we describe what the $\subseteq_P$-minimal solutions of the propositional abduction problems $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively look like. To facilitate the discussion, we introduce the following notation. We denote solutions as bit-vectors (x, x', y, r, y'), where x, x' are themselves vectors of arity k and y, y' are vectors of arity l. The representation of a subset of H by such a bit-vector is obvious. Moreover, let $D(x, x')$ and $D(y, y')$ be a short-hand for $\bigwedge_{i=1}^{k} (x_i \equiv \neg x_i')$ respectively $\bigwedge_{i=1}^{l} (y_i \equiv \neg y_i')$.

Finally, we write $\varphi(x, Y, Z)$ to denote that the variables X in $\varphi(X, Y, Z)$ are replaced by 0 and 1 according to the vector x. Analogously, we write $\varphi(x, y, Z)$ if also all occurrences of Y are replaced according to y.

The $\subseteq_P$-minimal solutions of $\mathcal{P}_1$ correspond to vectors of type (i) below while the $\subseteq_P$-minimal solutions of $\mathcal{P}_2$ are either of type (i) or (ii). The vectors of type (i) and (ii) are defined as follows:

(i) All vectors $(x, x', y, 0, y')$, such that $D(x, x') \wedge D(y, y')$ is valid and y is minimal such that $\forall Z\, \neg\varphi(x, y, Z)$ is valid, too;
(ii) All vectors $(x, x', (1, \ldots, 1), 1, (0, \ldots, 0))$, such that the formulas $D(x, x')$ and $\forall Y\, \exists Z\, \varphi(x, Y, Z)$ are valid.

The idea of (i) is similar to the proof of Theorem 6. Moreover, we have $r = 0$ because of minimality, since the value of r is unconstrained. Finally, y is minimal following the structure of priorities, but there is no minimality on x, since the variables X and X' have the same priority and any two tuples (x_1, x_1'), (x_2, x_2') are incomparable to each other by construction.

The idea of (ii) can be described as follows. It is clear that any vector $(x, x', (1, \ldots, 1), 1, (0, \ldots, 0))$ is a (representation of a) solution of the propositional abduction problem $\mathcal{P}_2$. The only question remaining is whether it is indeed $\subseteq_P$-minimal. Note that for this particular x, by which x' is fully determined due to the validity of $D(x, x')$, we actually deduce t via the formula $r \wedge y_1 \wedge \cdots \wedge y_l \rightarrow t$ in T_2. By the priority structure and incomparability of different tuples (x, x'), this vector is $\subseteq_P$-minimal if and only if we cannot deduce t by a $\subseteq_P$-smaller vector $(x, x', y, 0, y')$ via the formula $\neg\varphi(X, Y, Z) \rightarrow t$. In other words, $(x, x', (1, \ldots, 1), 1, (0, \ldots, 0))$ is $\subseteq_P$-minimal if and only if there does not exist a vector y such that $\forall Z\, \neg\varphi(x, y, Z)$ is valid. The variable r ensures that for the case when y is the all-1 vector, even if $\forall Z\, \neg\varphi(x, \mathbf{1}, Z)$ is valid, we have a $\subseteq_P$-smaller vector of type (i), since that one has $r = 0$. This is in turn equivalent to stating that $\neg\exists Y\, \forall Z\, \neg\varphi(x, Y, Z)$ is valid or, equivalently, $\forall Y\, \exists Z\, \varphi(x, Y, Z)$ is valid.

But then, this is also equivalent to saying that x is a satisfying assignment of the $\#\Pi_2$SAT-formula ψ. Note that the vectors of type (i) and of type (ii) are disjoint. Hence, the $\subseteq_P$-minimal solutions of $\mathcal{P}_2$ minus the $\subseteq_P$-minimal solutions of $\mathcal{P}_1$ corresponds to the vectors of type (ii) above. Their cardinality is indeed identical to the number of satisfying assignments x of the $\#\Pi_2$SAT-formula ψ. $\square$

Theorem 8. *The counting problem #-$\leqslant$-ABDUCTION is* $\#\cdot\mathrm{Opt}_2\mathrm{P}[\log n]$*-complete and the counting problem #-$\sqsubseteq$-ABDUCTION is* $\#\cdot\mathrm{Opt}_2\mathrm{P}$*-complete.*

Proof. In order to prove the membership, we show that these problems can be solved by an appropriate Σ_2P-transducer M, i.e., M works in non-deterministic polynomial time with access to an NP-oracle and, in the case of #-$\leqslant$-ABDUCTION, the output of M is logarithmically bounded. We give a high-level description of M: It takes an arbitrary propositional abduction problem $\mathcal{P} = \langle V, H, M, T\rangle$ as input and non-deterministically enumerates all subsets $\mathcal{S} \subseteq H$, such that every computation path of M corresponds to exactly one $\mathcal{S} \subseteq H$. By two calls to an NP-oracle, M checks on every path whether $T \cup \mathcal{S}$ is consistent (i.e., satisfiable) and if $T \cup \mathcal{S} \models M$ holds. If both oracle calls answer “yes”, then $\mathcal{S}$ is a solution of $\mathcal{P}$ and the computation path is accepting. The output written by M on each path is the cardinality of the corresponding set $\mathcal{S}$ (respectively the sum of the weights of the elements in $\mathcal{S}$) for the #-$\leqslant$-ABDUCTION problem (respectively for the #-$\sqsubseteq$-ABDUCTION problem). Finally, we define the optimal value of M to be the minimum. Obviously, the accepting paths of M outputting the optimal value correspond one-to-one to the cardinality-minimal (respectively weight-minimal) solutions of the propositional abduction problem $\mathcal{P}$.

The hardness of #-$\leqslant$-ABDUCTION (respectively of #-$\sqsubseteq$-ABDUCTION) is shown by reduction from #MIN-CARD-Π_1SAT (respectively from #MIN-WEIGHT-Π_1SAT). Let an arbitrary instance of #MIN-CARD-Π_1SAT (respectively of #MIN-WEIGHT-Π_1SAT) be given by the quantified Boolean formula $\varphi(X) = \forall Y\, \psi(X, Y)$ with $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_l\}$. In the case of #MIN-WEIGHT-Π_1SAT, we additionally have a weight function w defined on the variables in X. Let $X' = \{x'_1, \ldots, x'_k\}$, $X'' = \{x''_1, \ldots, x''_k\}$, $Q = \{q_1, \ldots, q_k\}$, $R = \{r_1, \ldots, r_k\}$, and t be fresh variables. Then we define the propositional abduction problem $\mathcal{P} = \langle V, H, M, T\rangle$ as follows:

$$V = X \cup X' \cup X'' \cup Y \cup Q \cup R \cup \{t\}$$

$$H = X \cup X' \cup X''$$

$$M = Q \cup R \cup \{t\}$$

$$T = \big\{\psi(X, Y) \to t\big\} \cup \big\{\neg x_i \vee \neg x'_i, x_i \to q_i, x'_i \to q_i \mid i = 1, \ldots, k\big\} \cup \big\{\neg x'_i \vee \neg x''_i, x'_i \to r_i, x''_i \to r_i \mid i = 1, \ldots, k\big\}$$

In the case of #-$\sqsubseteq$-ABDUCTION, we leave the weights of the variables in X unchanged. For the remaining hypotheses, we set $w(x_i) = w(x'_i) = w(x''_i)$ for every $i \in \{1, \ldots, k\}$.

For each i, the clauses $\neg x_i \vee \neg x'_i$, $x_i \to q_i$, $x'_i \to q_i$ in T ensure that every solution $\mathcal{S}$ of $\mathcal{P}$ contains exactly one of $\{x_i, x'_i\}$. Similarly, the clauses $\neg x'_i \vee \neg x''_i$, $x'_i \to r_i$, $x''_i \to r_i$ ensure that every solution contains exactly one of $\{x'_i, x''_i\}$. In total, for every $i \in \{1, \ldots, k\}$, every solution contains either $\{x_i, x''_i\}$ or $\{x'_i\}$. The intuition of the set X' is the same as in the proof of Theorem 6, namely to provide a means for forcing a hypothesis $x_i \in X$ to false (by including x'_i in the solution). As a consequence, the solutions of $\mathcal{P}$ are incomparable on $X \cup X'$, since every solution contains exactly k out of these $2k$ hypotheses. The intuition of X'' is to add a "copy" x''_i of each $x_i \in \mathcal{S}$ to the solution $\mathcal{S}$ in order to make the cardinalities of $\mathcal{S} \cap X$ for various solutions $\mathcal{S}$ comparable again. Hence, a solution $\mathcal{S}$ is (cardinality- respectively weight-)minimal if and only if $\mathcal{S} \cap X''$ is minimal, which is in turn the case if and only if $\mathcal{S} \cap X$ is minimal.

For a subset of variables $A \subseteq X$, let A' and A'' be defined as $A' = \{x' \mid x \in A\}$ and $A'' = \{x'' \mid x \in A\}$. Then, for every subset $A \subseteq X$, the following equivalence holds. The assignment I on X with $I^{-1}(1) = A$ is a model of $\varphi(X)$ if and only if $A \cup (X \smallsetminus A)' \cup \{\neg x_i \vee \neg x'_i \mid i = 1, \ldots, k\} \cup \{\psi(X, Y) \to t\} \models \{t\}$. Thus, for every $A \subseteq X$, we have the following equivalences. The assignment I on X with $I^{-1}(1) = A$ is a model of $\varphi(X)$ if and only if $A \cup (X \smallsetminus A)' \cup A''$ is a solution of $\mathcal{P}$. Moreover, the previous assignment I is cardinality-minimal (respectively weight-minimal) if and only if $A \cup (X \smallsetminus A)' \cup A''$ is a cardinality-minimal (respectively a weight-minimal) solution of $\mathcal{P}$. This accomplishes a parsimonious reduction to #-$\leqslant$-ABDUCTION (respectively #-$\sqsubseteq$-ABDUCTION). □

The counting problem #-$\leqslant_P$-ABDUCTION with no restriction on the number of priorities requires some preparatory work. For this purpose, we first consider the appropriate version of #SAT.

Problem: #MIN-LEX–Π_kSAT
Input: A quantified Boolean formula $\varphi(X) = \forall Y_1 \exists Y_2 \cdots Q\, Y_k\, \psi(X, Y_1, \ldots, Y_k)$ and a subset of variables $X' = \{x_1, \ldots, x_\ell\} \subseteq X$, such that $Q = \forall$ (respectively $Q = \exists$) and $\psi(X, Y_1, \ldots, Y_k)$ is in DNF (respectively in CNF) if k is odd (respectively k is even).
Output: Number of satisfying assignments $I : X \to \{0, 1\}$ of the formula $\varphi(X)$, such that the vector $(I(x_1), \ldots, I(x_\ell))$ is lexicographically minimal.

As usual, #MIN-LEX–Π_0SAT represents the aforementioned problem for unquantified formulas, therefore we denote it as #MIN-LEX-SAT.

Theorem 9. *The counting problem* #MIN-LEX–Π_kSAT *is* #·Opt$_{k+1}$P-*complete. In particular, the problem* #MIN-LEX-SAT *is* #·OptP-*complete.*

Proof. We only give the proof for #MIN-LEX-SAT, since the generalization to higher levels of the hierarchy is obvious.

In order to prove the membership, we show that #MIN-LEX-SAT can be solved by an appropriate NP-transducer M. We give a high-level description of M: It takes as input an arbitrary propositional formula φ with variables in X plus a subset $X' = \{x_1, \ldots, x_\ell\} \subseteq X$ of distinguished variables. M non-deterministically enumerates all possible truth assignments $I : X \to \{0, 1\}$, such that every computation path of M corresponds to exactly one assignment I. On each path, M checks in polynomial time if I is a model of φ. If this is the case, then the computation path is accepting. The output written by M on each path is the binary string $(I(x_1), \ldots, I(x_\ell))$. Finally, we define the optimal value of M to be the minimum. Obviously, the accepting paths of M outputting the optimal value correspond one-to-one to the satisfying assignments I of φ, such that $(I(x_1), \ldots, I(x_\ell))$ is lexicographically minimal.

For the hardness proof, let L be an arbitrary *minimum* problem in #·OptP. We show that there exists a parsimonious reduction from L to #MIN-LEX-SAT. Since L is in #·OptP, there exists an NP-transducer M for L. On input w, the transducer M produces an output of length smaller or equal to $p(|w|)$ on every branch for some polynomial p. Without loss of generality, we may assume that M actually produces an output of length exactly equal to $p(|w|)$. Now let w be an arbitrary instance of L and let $N = p(|w|)$ denote the length of the output on every computation path. Analogously to Cook's theorem (see [10]), there exists a propositional formula φ with variables X, such that there is a one-to-one correspondence between the satisfying truth assignments of φ and the successful computations of M on w. Moreover, X and φ can be extended in such a way that the output on each successful computation path is encoded by the variables $X' = \{x_1, \dots, x_N\}$, i.e., for every successful computation path π, the truth values $(I(x_1), \dots, I(x_N))$ of the corresponding model I of φ represent exactly the output on the path π. But then there is indeed a one-to-one correspondence between the computation paths of M on w, such that M outputs the minimum on these paths and the satisfying assignments of the (extended) formula φ, such that the truth values on $(x_1, \dots, x_N)$ are lexicographically minimal. $\square$

We also need the usual restriction of the previous problem to three literals per clause.

Problem: #MIN-LEX-3SAT
Input: A propositional formula φ in conjunctive normal form over the variables X with at most three literals per clause and a subset $X' = \{x_1, \dots, x_\ell\} \subseteq X$.
Output: Number of satisfying assignments $I : X \to \{0, 1\}$ of the formula φ, such that the Boolean vector $(I(x_1), \dots, I(x_\ell))$ is lexicographically minimal.

Since there exists a parsimonious reduction from #SAT to #3SAT (see [18]), the same reduction implies the following consequence of Theorem 9.

Corollary 10. *The counting problem* #MIN-LEX-3SAT *is* #·OptP*-complete.*

Theorem 11. *The counting problems* #-$\leqslant_P$-ABDUCTION *without restriction on the number of priorities and* #-$\sqsubseteq_P$-ABDUCTION *with or without restriction on the number of priorities are* #·Opt$_2$P*-complete. The problem* #-$\leqslant_P$-ABDUCTION *is* #·Opt$_2$P[$\log n$]*-complete if the number of priorities is bounded by a constant.*

Proof. For the membership proof, we slightly modify the Σ_2P-transducer M from the membership proof of Theorem 8. Again, M non-deterministically enumerates all subsets $\mathcal{S} \subseteq H$, such that every computation path of M corresponds to exactly one $\mathcal{S} \subseteq H$. By two calls to an NP-oracle, M checks on every path whether $T \cup \mathcal{S}$ is consistent (i.e., satisfiable) and whether $T \cup \mathcal{S} \models M$ holds. If both oracle calls answer "yes", then $\mathcal{S}$ is a solution of $\mathcal{P}$ and the computation path is accepting. Only the output written by M on each path has to be modified with respect to the proof of Theorem 8: Suppose that the input propositional abduction problem $\mathcal{P}$ has K priorities $H_1, \dots, H_K$. Then M computes on every computation path the vector $(c_1, \dots, c_K)$, where c_i is the cardinality (respectively the total weight) of $\mathcal{S} \cap H_i$ for every i. Without loss of generality we may assume for every i that, on all paths, the binary representation of the numbers c_i has identical length (by adding appropriately many leading zeros). Then M simply outputs this vector $(c_1, \dots, c_K)$, considered as a single number in binary. Finally, we again define the optimal value of M as the minimum. Obviously, the accepting paths of M outputting the optimal value correspond one-to-one to the $\leqslant_P$-minimal (respectively $\sqsubseteq_P$-minimal) solutions of the propositional abduction problem $\mathcal{P}$. If we consider $\leqslant_P$-minimality, the length of each c_i is bounded by $\log|H|$ bits, since $c_i \leqslant |H|$ holds. For $\sqsubseteq_P$-minimality, c_i is bounded by $|b_w| \cdot \log|H|$ bits, where $|b_w|$ is the number of bits needed to represent the biggest weight of the hypotheses. Hence, we need $O(K \log|H|)$ respectively $O(K|b_w| \cdot \log|H|)$ bits to represent the vector $(c_1, \dots, c_K)$. If there are no restrictions on the number K of priorities or if we consider weight-minimality, then the output of M has polynomial length. In contrast, for $\leqslant_P$-minimality with a constant number K of priorities, this upper bound becomes $O(\log|H|)$.

For the hardness part, only the #·Opt$_2$P-hardness of #-$\leqslant_P$-ABDUCTION without restriction on the number of priorities has to be shown. The remaining cases follow from the corresponding hardness result without priorities in Theorem 8. We reduce the #MIN-LEX–Π_1SAT problem to #-$\leqslant_P$-ABDUCTION. Let an arbitrary instance of #MIN-LEX–Π_1SAT be given by the quantified Boolean formula $\varphi(X) = \forall Y\, \psi(X, Y)$ with $X = \{x_1, \dots, x_n\}$ and the subset $X' = \{x_1, \dots, x_\ell\} \subseteq X$. Let t, $Q = \{q_1, \dots, q_n\}$, $R = \{r_1, \dots, r_\ell\}$, $Z = \{z_1, \dots, z_n\}$, and $Z' = \{z'_1, \dots, z'_\ell\}$ be fresh variables. Then we define the propositional abduction problem $\mathcal{P} = \langle V, H, M, T\rangle$ as follows:

$$V = X \cup Y \cup Z \cup Z' \cup Q \cup R \cup \{t\}$$
$$H = X \cup Z \cup Z' \quad \text{with } H_1 = \{x_1\}, \dots, H_\ell = \{x_\ell\}, \quad \text{and} \quad H_{\ell+1} = \big(X \smallsetminus X'\big) \cup Z \cup Z'$$
$$M = Q \cup R \cup \{t\}$$
$$T = \big\{\psi(X, Y) \to t\big\} \cup \{\neg x_i \vee \neg z_i, x_i \to q_i, z_i \to q_i \mid 1 \leqslant i \leqslant n\} \cup \big\{\neg z_i \vee \neg z'_i, z_i \to r_i, z'_i \to r_i \mid 1 \leqslant i \leqslant \ell\big\}$$

The idea of the variables in Q, R, Z, and Z' is similar to the variables Q, R, X', and X'' in the proof of Theorem 8. They ensure that every solution $\mathcal{S}$ of $\mathcal{P}$ contains exactly n variables out of the $2n$ variables in $H_{\ell+1}$. This can be seen as follows. By the clauses $\neg x_i \vee \neg z_i, x_i \to q_i, z_i \to q_i$ with $i \in \{1, \dots, n\}$, every solution contains exactly one of $\{x_i, z_i\}$. Of course, the variables x_i with $i \in \{1, \dots, \ell\}$ are not in $H_{\ell+1}$. However, the clauses $\neg z_i \vee \neg z'_i, z_i \to r_i, z'_i \to r_i$ with $i \in \{1, \dots, \ell\}$ ensure that every solution contains exactly one of $\{z_i, z'_i\}$. In other words, for every $i \in \{1, \ldots, \ell\}$, every solution contains either $\{x_i, z'_i\}$ or $\{z_i\}$.

There is a one-to-one correspondence between the models of $\varphi(X)$ which are lexicographically minimal on X' and the $\leqslant_P$-minimal solutions of $\mathcal{P}$. Indeed, let I be a model of $\varphi(X)$ which is lexicographically minimal on X'. Then I can be extended to exactly one $\leqslant_P$-minimal solution $\mathcal{S}$ of $\mathcal{P}$, namely $\mathcal{S} = I^{-1}(1) \cup \{z_i \mid 1 \leqslant i \leqslant n \text{ and } I(x_i) = 0\} \cup \{z'_i \mid 1 \leqslant i \leqslant \ell \text{ and } I(x_i) = 1\}$.

Conversely, let $\mathcal{S}$ be a $\leqslant_P$-minimal solution of $\mathcal{P}$. Then we obtain a lexicographically minimal model I of $\varphi(X)$ simply by restricting $\mathcal{S}$ to X, i.e. $I(x) = 1$ for all $x \in \mathcal{S} \cap X$ and $I(x) = 0$ otherwise. □

4. Horn, dual Horn, and bijunctive theories

In this section, we consider the special case where the theory T is a set of (arbitrary or definite) Horn, dual Horn, or bijunctive clauses. If no minimality criterion is applied to the solutions then we get the following result.

Theorem 12. *The counting problem #-ABDUCTION for Horn, definite Horn, dual Horn, or bijunctive theories is #P-complete.*

Proof. The #P-membership is easily seen by the fact that it can be checked in polynomial time whether some subset $\mathcal{S} \subseteq H$ is a solution, since the satisfiability and also the unsatisfiability of a set of (dual) Horn or bijunctive clauses can be checked in polynomial time.

For the #P-hardness, we reduce the #POSITIVE-2SAT problem (which is known to be #P-complete by [29]) to it and show that this reduction is parsimonious. Let an arbitrary instance of #POSITIVE-2SAT be given as a 2CNF-formula

$$\psi = (p_1 \vee q_1) \wedge \cdots \wedge (p_n \vee q_n)$$

where the p_i's and q_i's are propositional variables from the set $X = \{x_1, \ldots, x_k\}$. Moreover, let $g_1, \ldots, g_n$ denote fresh, pairwise distinct variables and let $G = \{g_1, \ldots, g_n\}$. Then we define the propositional abduction problem $\mathcal{P} = \langle V, H, M, T\rangle$ as follows:

$$V = X \cup G$$

$$H = X$$

$$M = G$$

$$T = \{p_i \rightarrow g_i \mid 1 \leqslant i \leqslant n\} \cup \{q_i \rightarrow g_i \mid 1 \leqslant i \leqslant n\}$$

Obviously, this reduction is feasible in polynomial time. Moreover, it is easy to check that there is a one-to-one relationship between the solutions $\mathcal{S} \subseteq X$ of $\mathcal{P}$ and the models of ψ. Note that the clauses in T are at the same time definite Horn, bijunctive, and dual Horn. □

Analogously to the case of general theories, the counting complexity remains unchanged when we restrict our attention to subset-minimal solutions.

Theorem 13. *The counting problem #-⊆-ABDUCTION for Horn, definite Horn, dual Horn, or bijunctive theories is #P-complete.*

Proof. The #P-membership holds analogously to the case of abduction without subset-minimality. This is due to the following property (see [5], Proposition 2.1.5). $\mathcal{S} \subseteq H$ is a subset-minimal solution of $\mathcal{P}$ if and only if $\mathcal{S}$ is a solution and for all $h \in \mathcal{S}$, the set $\mathcal{S} \setminus \{h\}$ is not a solution of $\mathcal{P}$.

For the #P-hardness, we modify the reduction from the #POSITIVE-2SAT problem in Theorem 12. Let ψ, X, and G be defined as before. Moreover, let $X' = \{x'_1, \ldots, x'_k\}$ and $R = \{r_1, \ldots, r_k\}$ be fresh, pairwise distinct variables. Then we define $\mathcal{P} = \langle V, H, M, T\rangle$ as follows:

$$V = X \cup X' \cup G \cup R$$

$$H = X \cup X'$$

$$M = R \cup G$$

$$T = \{p_i \rightarrow g_i,\ q_i \rightarrow g_i \mid 1 \leqslant i \leqslant n\} \cup \big\{x_i \rightarrow r_i, x'_i \rightarrow r_i \mid 1 \leqslant i \leqslant k\big\}$$

The idea of the variables X' and the additional manifestations G is similar to the proof of Theorem 6, with the following slight change. Whenever a subset $\mathcal{S} \subseteq H$ with $x_i, x'_i \in \mathcal{S}$ is a solution of $\mathcal{P}$, then $\mathcal{S} \setminus \{x'_i\}$ is also a solution since x'_i is useless as soon as x_i is present (note that the only use of x'_i is to derive r_i in the absence of x_i). Therefore in a subset-minimal solution of the propositional abduction problem $\mathcal{P}$, we will never select both x_i and x'_i even without the formula $\neg x_i \vee \neg x'_i$. The formulas in T are indeed definite Horn, dual Horn, and bijunctive. □

The result of Theorem 13, although using a different vocabulary, has been proved in [1] by a reduction from #MINIMAL VERTEX COVERS. The latter was proved #P-complete by Valiant [29].

Below we consider propositional abduction problems with $\subseteq_P$-minimality. It turns out that for definite Horn and dual Horn clauses, the priorities leave the counting complexity unchanged. In all other cases, the counting complexity increases.

Theorem 14. *The counting problem #-$\subseteq_P$-ABDUCTION for definite Horn and for dual Horn theories is #P-complete.*

Proof. The #P-hardness is clear, since it holds even without priorities. The #P-membership for definite Horn clauses is proved as follows. Let $\mathcal{P} = \langle V, H, M, T\rangle$ where T consists only of Horn clauses. According to [5, Theorem 5.3.3], for any $\mathcal{S} \subseteq H$, we can check in polynomial time whether $\mathcal{S}$ is a $\subseteq_P$-minimal solution. The #P-membership for definite Horn clauses is thus proved.

Now suppose that T is dual Horn. Let $N = \{h \mid h \in H$ and $T \models \neg h\}$. Clearly, for T dual Horn, N can be computed in polynomial time. Then for every solution $\mathcal{S}$ of $\mathcal{P}$, we have $\mathcal{S} \subseteq H \setminus N$, since otherwise $T \cup \mathcal{S}$ would be inconsistent. Moreover, for any $\mathcal{S}'$ with $\mathcal{S} \subseteq \mathcal{S}' \subseteq H \setminus N$, the set $\mathcal{S}'$ is also a solution of $\mathcal{P}$, since (by the special form of dual Horn) $\mathcal{S}' \cup T$ is also consistent and (by the monotonicity of $\models$) $\mathcal{S}' \cup T$ also implies M.

So let $H_1, \dots, H_k$ denote the priorities of H. Then $\mathcal{S}$ is a $\subseteq_P$-minimal solution of $\mathcal{P}$ if and only if $\mathcal{S}$ is a solution of $\mathcal{P}$ and for all $i \in \{1, \dots, k\}$ and for all $x \in (\mathcal{S} \cap H_i)$ the set

$$\mathcal{S}' = \big(\mathcal{S} \setminus \{x\}\big) \cup (H_{i+1} \setminus N) \cup \dots \cup (H_k \setminus N)$$

is *not* a solution of $\mathcal{P}$, because any solution $\subseteq_P$-smaller than $\mathcal{S}$ would be a subset of such an $\mathcal{S}'$. The latter test is clearly feasible in polynomial time in the dual Horn case. Moreover, there are only polynomially many such tests required. □

Recall from our remark preceding Theorem 7 that the effect of at least 2 priority levels is as follows. In order to check that some solution $\mathcal{S}$ is *not* $\subseteq_P$-minimal, we have to test that there exists some solution of the form $\mathcal{S}' = (\mathcal{S} \setminus \{h\}) \cup X$, where all hypotheses in X have higher priority than h. In general, the difficulty of determining if such a set X exists is the following one. If we choose X too small, then $\mathcal{S}'$ might not entail the manifestations M. If we choose X too big, then $\mathcal{S}' \cup T$ might be inconsistent. The intuition underlying Theorem 14 is that the problem of choosing X too big disappears for definite Horn and dual Horn clauses. For definite Horn, the only candidate X that has to be checked is $X = H_{i+1} \cup \dots \cup H_K$. For dual Horn, the only candidate X is $X = (H_{i+1} \cup \dots \cup H_K) \setminus N$, where N contains all hypotheses $h \in H$ with $T \models \neg h$.

Theorem 15. *The counting problem #-$\subseteq_P$-ABDUCTION for Horn or bijunctive theories is #·coNP-complete via subtractive reductions.*

Proof. The #·coNP-membership is clear. Given a set of variables $\mathcal{S}$, we have to (i) check whether $\mathcal{S}$ is a solution of the propositional abduction problem and (ii) if so, check whether $\mathcal{S}$ is $\subseteq_P$-minimal. The latter test, which dominates the overall complexity, can be done by a coNP-oracle. The #·coNP-hardness is shown by a (strong) subtractive reduction from #Π_1SAT. Let an instance of the #Π_1SAT problem be given by a formula

$$\psi(X) = \forall Y\, \varphi(X, Y)$$

with the variables $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_l\}$. Without loss of generality, we may assume that $\varphi(X, Y)$ is in 3DNF, i.e., it is of the form $C_1 \vee \dots \vee C_n$ where each C_i is of the form $C_i = l_{i1} \wedge l_{i2} \wedge l_{i3}$ and the l_{ij}'s are propositional literals over $X \cup Y$.

Let $x'_1, \dots, x'_k$, $p_1, \dots, p_k$, $y'_1, \dots, y'_l$, $q_1, \dots, q_l$, $g_1, \dots, g_n$, t denote fresh, pairwise distinct variables. Let $X' = \{x'_1, \dots, x'_k\}$, $Y' = \{y'_1, \dots, y'_l\}$, $P = \{p_1, \dots, p_k\}$, $Q = \{q_1, \dots, q_l\}$ and $G = \{g_1, \dots, g_n\}$. Then we define two propositional abduction problems $\mathcal{P}_1$ and $\mathcal{P}_2$ as follows.

$$V = X \cup X' \cup Y \cup Y' \cup P \cup Q \cup G \cup \{r\}$$

$$H = X \cup X' \cup Y \cup Y' \cup \{r\} \quad \text{with priorities } H_1 = H \setminus Y' \quad \text{and} \quad H_2 = Y'$$

$$M = P \cup Q \cup G$$

$$T_1 = \big\{\neg x_i \vee \neg x'_i,\ x_i \to p_i,\ x'_i \to p_i \mid 1 \leqslant i \leqslant k\big\} \cup \big\{\neg y_i \vee \neg y'_i,\ y_i \to q_i,\ y'_i \to q_i \mid 1 \leqslant i \leqslant l\big\}$$
$$\cup \{z_{ij} \to g_i \mid 1 \leqslant i \leqslant n \text{ and } 1 \leqslant j \leqslant 3\}$$

where z_{ij} is either of the form x_k, x'_k, y_l, or y'_l depending on whether the literal l_{ij} in C_i is of the form $\neg x_k$, x_k, $\neg y_l$, or y_l, respectively. In other words, the variable z_{ij} encodes the negation of l_{ij}. The second theory is defined as

$$T_2 = T_1 \cup \{r \wedge y_1 \wedge \dots \wedge y_l \to g_i \mid 1 \leqslant i \leqslant n\}$$

Finally, we set $\mathcal{P}_1 = \langle V, H, M, T_1\rangle$ and $\mathcal{P}_2 = \langle V, H, M, T_2\rangle$.

Obviously, this reduction is feasible in polynomial time. Now let $A(\psi)$ denote the set of all satisfying assignments of a $\#\Pi_1$SAT-formula ψ and let $B(\mathcal{P})$ denote the set of $\subseteq_P$-minimal solutions of a propositional abduction problem $\mathcal{P}$. We claim that $\mathcal{P}_1$ and $\mathcal{P}_2$ have the following property.

$$B(\mathcal{P}_1) \subseteq B(\mathcal{P}_2) \quad \text{and} \quad \left|A(\psi)\right| = \left|B(\mathcal{P}_2)\right| - \left|B(\mathcal{P}_1)\right|$$

In order to prove this claim, we describe what the $\subseteq_P$-minimal solutions of the propositional abduction problems $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively look like. Analogously to the proof of Theorem 7, we denote subsets of H by bit-vectors of the form (x, x', y, r, y'), where x, x', y, y' are themselves vectors of the obvious arity. The formula $D(x, x')$ (respectively $D(y, y')$) is used as a short-hand for the condition that x' (respectively y') encodes the bitwise opposite of x (respectively y). Finally we write $\varphi(x, Y)$ to denote that the variables X in $\varphi(X, Y)$ are replaced by 0 and 1 according to the vector x. Analogously we write $\varphi(x, y)$ if also all occurrences of Y are replaced according to y.

The $\subseteq_P$-minimal solutions of $\mathcal{P}_1$ correspond to vectors of type (1) below. The $\subseteq_P$-minimal solutions of $\mathcal{P}_2$ correspond to vectors of either type (1) or type (2). The vectors of type (1) and (2) are defined as follows:

(1) All vectors $(x, x', y, 0, y')$, such that $D(x, x') \wedge D(y, y')$ is valid, and y is minimal such that $\varphi(x, y)$ is false;
(2) All vectors $(x, x', (1, \dots, 1), 1, (0, \dots, 0))$, such that the formulas $D(x, x')$ and $\forall Y\, \varphi(x, Y)$ are valid.

For the idea of (1), recall that each z_{ij} in the formula $z_{ij} \to g_i$ encodes the dual of l_{ij}. Hence, all g_i's are implied if $\varphi(x, y)$ is false. Similarly to the proof of Theorem 7, the idea of (2) is as follows. Any vector of the form $(x, x', (1, \dots, 1), 1, (0, \dots, 0))$ is a (representation of a) solution of the propositional abduction problem $\mathcal{P}_2$ since it allows us to deduce the g_i's via the formulas $r \wedge y_1 \wedge \dots \wedge y_k \to g_i$ in T_2. Moreover, $(x, x', (1, \dots, 1), 1, (0, \dots, 0))$ is $\subseteq_P$-minimal if and only if we cannot deduce all g_i's by a $\subseteq_P$-smaller vector (x, x', y, r, y') via the rules $z_{ij} \to g_i$. The latter condition holds if there is no vector y such that $\varphi(x, y)$ is false or, equivalently, if $\forall Y\, \varphi(x, Y)$ is valid (see the proof of Theorem 7). This is in turn equivalent to stating that x is a satisfying assignment of the $\#\Pi_1$SAT-formula ψ.

Thus, the $\subseteq_P$-minimal solutions of $\mathcal{P}_2$ minus the $\subseteq_P$-minimal solutions of $\mathcal{P}_1$ corresponds to the vectors fulfilling condition (2) above. Their cardinality is identical to the number of satisfying assignments x of the $\#\Pi_1$SAT-formula ψ.

The case of Horn clauses is thus proved. It remains to show how the above subtractive reduction can be modified to settle the case of bijunctive clauses. Actually, each clause $r \wedge y_1 \wedge \dots \wedge y_l \to g_i$ in T_2 may be replaced by the set of clauses $\{r \to g_i, r \to y_1, \dots, r \to y_l\}$. It is easy to show that this does not change the set of $\subseteq_P$-minimal solutions of $\mathcal{P}_1$ and $\mathcal{P}_2$. Then the resulting theories T_1 and T_2 indeed consist only of bijunctive clauses. □

We need some additional counting problems to be able to consider the counting problems for propositional abduction with the cardinality and weight preorders. Recall the following counting problem introduced in [15].

Problem: #MIN-CARD-VERTEX-COVER (RESPECTIVELY #MIN-WEIGHT-VERTEX-COVER)
Input: Graph $G = (V, E)$ (plus a weight function $w : V \to \mathbb{N}$ in the case of #MIN-WEIGHT-VERTEX-COVER).
Output: Number of vertex covers of G with minimal cardinality (respectively with minimal weight), i.e., cardinality-minimal (respectively weight-minimal) subsets $C \subseteq V$ such that $(u, v) \in E$ implies $u \in C$ or $v \in C$.

We proved in [15] that #MIN-CARD-VERTEX-COVER is #·OptP[log n]-complete while #MIN-WEIGHT-VERTEX-COVER is #·OptP-complete. We will use these results for proving the lower bounds in the following theorem.

Theorem 16. *The counting problem #-$\leqslant$-ABDUCTION is #·OptP[log n]-complete and the counting problem #-$\sqsubseteq$-ABDUCTION is #·OptP-complete for Horn, definite Horn, dual Horn, or bijunctive theories.*

Proof. For the membership part, we construct a transducer M exactly as in the proof of Theorem 8. The only difference is that we can now check in *deterministic polynomial time* whether $T \cup \mathcal{S}$ is consistent (i.e., satisfiable) and whether $T \cup \mathcal{S} \models M$ holds. Hence, we end up with the desired NP-transducer (rather than a Σ_2P-transducer) since we no longer need an NP-oracle.

Hardness is shown by a reduction from the counting problem #MIN-CARD-VERTEX-COVER (respectively #MIN-WEIGHT-VERTEX-COVER). Let an arbitrary instance of #MIN-CARD-VERTEX-COVER be given by the graph $G = (V, E)$ with $V = \{v_1, \dots, v_n\}$ and $E = \{e_1, \dots, e_m\}$. By slight abuse of notation, we consider the elements in V and E also as propositional variables and we set $X = \{v_1, \dots, v_n\}$ and $R = \{e_1, \dots, e_m\}$. In the case of #MIN-WEIGHT-VERTEX-COVER, we additionally have a weight function w defined on the variables in X. Then we define the propositional abduction problem $\mathcal{P} = \langle W, H, M, T\rangle$ as follows.

$$W = X \cup R$$
$$H = X$$
$$M = R$$
$$T = \{v_i \to e_j \mid v_i \in e_j, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$$

The resulting theory contains only clauses which are, at the same time, Horn, definite Horn, dual Horn, and bijunctive. Obviously, for every subset $X' \subseteq X = V$ the following equivalence holds: X' is a solution of $\mathcal{P}$ if and only if X' is a vertex cover of G. But then there exists also a one-to-one correspondence between the cardinality-minimal (respectively weight-minimal) solutions of $\mathcal{P}$ and the cardinality-minimal (respectively weight-minimal) vertex covers of G. □

Again, #-$\leqslant_P$-ABDUCTION with no restriction on the number of priorities requires some preparatory work. For this purpose, we first consider an appropriate variant of counting the vertex covers of a graph.

Problem: #MIN-LEX-VERTEX-COVER
Input: Graph $G = (V, E)$ and a subset $V' = \{v_1, \ldots, v_\ell\} \subseteq V$.
Output: Number of vertex covers C of G, such that $(\chi(v_1), \ldots, \chi(v_\ell))$ is lexicographically minimal, where χ is the characteristic function of the vertex cover C.

Theorem 17. *The counting problem* #MIN-LEX-VERTEX-COVER *is* #·OptP-*complete.*

Proof. In order to prove the membership, we show that #MIN-LEX-VERTEX-COVER can be solved by the following NP-transducer M. It takes as input an arbitrary graph $G = (V, E)$ with distinguished vertices $V' = \{v_1, \ldots, v_\ell\}$. M nondeterministically enumerates all subsets $C \subseteq V$, such that every computation path of M corresponds to exactly one such subset C. If C is a vertex cover of G, then the computation path is accepting. The output written by M on each path is the binary vector $(\chi_C(v_1), \ldots, \chi_C(v_\ell))$. Obviously, the accepting paths of M outputting the minimal value correspond one-to-one to the vertex covers C of G, such that $(\chi_C(v_1), \ldots, \chi_C(v_\ell))$ is lexicographically minimal.

The hardness proof is by a parsimonious reduction from #MIN-LEX-3SAT. In fact, this is the same reduction as in the standard NP-completeness proof of VERTEX COVER by reduction from 3SAT to VERTEX COVER, see e.g. [10]. Let $\varphi(x_1, \ldots, x_k)$ be a propositional formula in CNF with three literals per clause. We construct the graph $G = (V, E)$ as follows. For each variable x_i we construct an edge $e_i = (x_i, x_i')$. For each clause $c_i = l_i^1 \vee l_i^2 \vee l_i^3$ we construct three edges (l_i^1, l_i^2), (l_i^2, l_i^3), (l_i^3, l_i^1) forming a triangle t_i. Finally, we connect each positive literal z in the triangle t_i to its counterpart z in an edge $e_j = (z, z')$, as well as each negative literal $\neg z$ in the triangle t_i to its counterpart z'. The set of distinguished variables X' from #MIN-LEX-3SAT becomes the set of distinguished vertices V' in #MIN-LEX-VERTEX-COVER. □

Theorem 18. *The counting problem* #-$\leqslant_P$-ABDUCTION *without restriction on the number of priorities and the problem* #-$\sqsubseteq_P$-ABDUCTION *with or without restriction on the number of priorities are* #·OptP-*complete for Horn, definite Horn, dual Horn, or bijunctive theories. The problem* #-$\leqslant_P$-ABDUCTION *for Horn, definite Horn, dual Horn, or bijunctive theories is* #·OptP[$\log n$]-*complete if the number of priorities is restricted by a constant.*

Proof. For the membership part, we construct a transducer M exactly as in the proof of Theorem 11. The only difference is that we get an NP-transducer (rather than a Σ_2P-transducer) since we no longer need an NP-oracle for checking whether $T \cup \mathcal{S}$ is consistent (i.e., satisfiable) and whether $T \cup \mathcal{S} \models M$ holds.

For the hardness part, only the #·OptP-hardness of #-$\leqslant_P$-ABDUCTION without restriction on the number of priorities has to be shown. The remaining cases follow from the corresponding hardness result without priorities in Theorem 16. Let an arbitrary instance of #MIN-LEX-VERTEX-COVER be given by the graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$ and let $V' = \{v_1, \ldots, v_\ell\}$ with $\ell \leqslant n$. As in the proof of Theorem 16, we consider the elements in V and E also as propositional variables and set $X = \{v_1, \ldots, v_n\}$ and $R = \{e_1, \ldots, e_m\}$. In addition, let $Q = \{q_{\ell+1}, \ldots, q_n\}$, and $Z = \{z_{\ell+1}, \ldots, z_n\}$ be fresh variables. Then we define the propositional abduction problem $\mathcal{P} = \langle V, H, M, T \rangle$ as follows:

$$V = X \cup R \cup Q \cup Z$$

$$M = R \cup Q$$

$$H = X \cup Z \quad \text{with } H_1 = \{v_1\}, \ldots, H_\ell = \{v_\ell\}, \quad \text{and} \quad H_{\ell+1} = \big(X \smallsetminus V'\big) \cup Z$$

$$T = \{v_i \to e_j \mid v_i \in e_j, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\} \cup \{v_i \to q_i, z_i \to q_i \mid \ell + 1 \leqslant i \leqslant n\}$$

The resulting theory contains only clauses which are, at the same time, Horn, definite Horn, dual Horn, and bijunctive. The variables Q and Z realize the familiar idea that in every $\leqslant_P$-minimal solution $\mathcal{S}$ of $\mathcal{P}$, for every $i \in \{\ell + 1, \ldots, n\}$, exactly one of v_i and z_i is contained in $\mathcal{S}$. It can then be easily shown that there is a one-to-one correspondence between the lexicographically minimal vertex covers of G and the $\leqslant_P$-minimal solutions of $\mathcal{P}$. □

5. Concluding remarks

Eiter and Gottlob proved in [5] a plethora of complexity results for propositional abduction. Their results were extended to a trichotomy of propositional abduction problems without minimality-criterion by Creignou and Zanuttini [2]. A thorough study of the computational complexity of the abduction problem has been presented by Nordh and Zanuttini in [22]. The use of complexity results is usually twofold. Theoretically, they give us a better understanding of the nature of the considered problem class. Practically, they give us a hint as to which subclass of the problem we should aim at, provided that the application in mind admits such a restriction. In this sense, the counting complexity results shown here are important in complementing the already known decision complexity results. Note that our results indeed reveal differences between the counting complexity behavior of propositional abduction problems and the decision complexity. For instance, definite Horn abduction has been shown to be tractable [2,22]. In contrast, by our Theorem 12, the corresponding counting problem is #P-complete. This is one more example of the often observed "easy to decide, hard to count" phenomenon. In this case, the gap between the complexity for existence and for counting is due to the fact that definite Horn abduction is degenerate for existence of a solution.

From a complexity theoretic point of view, there is another interesting aspect to the counting complexity results shown here. The class #P has been studied intensively and many completeness results for this class can be found in the literature. In contrast, for the higher counting complexity classes $\#\cdot\Pi_k\mathrm{P}$, $\#\cdot\mathrm{Opt}_k\mathrm{P}[\log n]$, and $\#\cdot\mathrm{Opt}_k\mathrm{P}$ (with $k \geqslant 1$), very few problems had been shown to be complete. In fact, to the best of our knowledge, our $\#\cdot\Pi_2\mathrm{P}$-completeness result in Theorem 7 is the first one apart from $\#\Pi_2\mathrm{SAT}$. Our results on the counting complexity of propositional abduction thus also lead to a better understanding of these counting complexity classes.

In this article, we have considered the complexity of determining the number of all $\preccurlyeq$-minimal explanations of a propositional abduction problem, where $\preccurlyeq \in \{=, \subseteq, \subseteq_P, \leqslant, \leqslant_P, \sqsubseteq, \sqsubseteq_P\}$. We were able to prove precise complexity classifications for counting problems of abduction with all these preorders $\preccurlyeq$ in the case of general theories as well as Horn, dual Horn, definite Horn, and bijunctive theories. A thorough complexity analysis for the subclass of affine theories has to be postponed to a standalone upcoming work since these theories do not follow the usual clausal presentation, they require the application of different methods, mainly issued from linear algebra, and we still have gaps in the counting complexity classification of affine abduction for several preorders.

For future work, we also plan to extend the complexity analysis of many more families of decision problems in the artificial intelligence domain to counting problems, like, e.g., counting the number of minimal models of a theory for closed-world reasoning in various settings.

Acknowledgment

We are very grateful to the anonymous referee for the valuable comments on a previous version of this article.

References

[1] T. Bylander, D. Allemang, M.C. Tanner, J.R. Josephson, The computational complexity of abduction, Artificial Intelligence 49 (1–3) (1991) 25–60.
[2] N. Creignou, B. Zanuttini, A complete classification of the complexity of propositional abduction, SIAM J. Comput. 36 (1) (2006) 207–229.
[3] A. del Val, On some tractable classes in deduction and abduction, Artificial Intelligence 116 (1–2) (2000) 297–313.
[4] A. Durand, M. Hermann, P.G. Kolaitis, Subtractive reductions and complete problems for counting complexity classes, Theoret. Comput. Sci. 340 (3) (2005) 496–513.
[5] T. Eiter, G. Gottlob, The complexity of logic-based abduction, J. Assoc. Comput. Machinery 42 (1) (1995) 3–42.
[6] T. Eiter, K. Makino, On computing all abductive explanations, in: Proceedings 18th National Conference on Artificial Intelligence (AAAI 2002), Edmonton (Alberta, Canada), AAAI Press, 2002, pp. 62–67.
[7] T. Eiter, K. Makino, Generating all abductive explanations for queries on propositional Horn theories, in: M. Baaz, J.A. Makowsky (Eds.), Proceedings 12th Annual Conference of the European Association for Computer Science Logic (CSL'03), Vienna (Austria), in: Lecture Notes in Comput. Sci., vol. 2803, Springer-Verlag, August 2003, pp. 197–211.
[8] K. Eshghi, A tractable class of abduction problems, in: R. Bajcsy (Ed.), Proceedings 13th International Joint Conference on Artificial Intelligence (IJCAI'93), Chambéry (France), Morgan Kaufmann, 1993, pp. 3–8.
[9] R. Fagin, J.D. Ullman, M.Y. Vardi, On the semantics of updates in databases, in: Proceedings 2nd ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems (PODS'83), Atlanta (Georgia, USA), Association for Computing Machinery, March 1983, pp. 352–365.
[10] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W.H. Freeman and Co., 1979.
[11] W.I. Gasarch, M.W. Krentel, K.J. Rappoport, OptP as the normal behavior of NP-complete problems, Math. Systems Theory 28 (6) (1995) 487–514.
[12] L.A. Hemaspaandra, H. Vollmer, The satanic notations: Counting classes beyond #P and other definitional adventures, SIGACT News, Complex. Theory Column 8 26 (1) (March 1995) 2–13.
[13] M. Hermann, R. Pichler, Counting complexity of propositional abduction, in: M.M. Veloso (Ed.), Proceedings 20th International Joint Conference on Artificial Intelligence (IJCAI 2007), Hyderabad (India), AAAI Press, January 2007, pp. 417–422.
[14] M. Hermann, R. Pichler, Counting complexity of minimal cardinality and minimal weight abduction, in: S. Hölldobler, C. Lutz, H. Wansing (Eds.), Proceedings 11th European Conference on Logics in Artificial Intelligence (JELIA 2008), Dresden (Germany), in: Lecture Notes in Artificial Intelligence, vol. 5293, Springer-Verlag, 2008, pp. 206–218.
[15] M. Hermann, R. Pichler, Complexity of counting the optimal solutions, Theoret. Comput. Sci. 410 (38–40) (2009) 3814–3825.
[16] A. Herzig, J. Lang, P. Marquis, T. Polacsek, Updates, actions, and planning, in: B. Nebel (Ed.), Proceedings 17th International Joint Conference on Artificial Intelligence (IJCAI 2001), Seattle (Washington, USA), Morgan Kaufmann, August 2001, pp. 119–124.
[17] A.C. Kakas, P. Mancarella, Database updates through abduction, in: D. McLeod, R. Sacks-Davis, H.-J. Schek (Eds.), Proceedings 16th International Conference on Very Large Data Bases (VLDB 1990), Brisbane (Queensland, Australia), Morgan Kaufmann, August 1990, pp. 650–661.
[18] D.C. Kozen, The Design and Analysis of Algorithms, Counting Problems and #P, Springer-Verlag, 1992, pp. 138–143, Chapter 26.
[19] M.W. Krentel, The complexity of optimization problems, J. Comput. System Sci. 36 (3) (1988) 490–509.
[20] M.W. Krentel, Generalizations of OptP to the polynomial hierarchy, Theoret. Comput. Sci. 97 (2) (1992) 183–198.
[21] G. Nordh, B. Zanuttini, Propositional abduction is almost always hard, in: L.P. Kaelbling, A. Saffiotti (Eds.), Proceedings 19th International Joint Conference on Artificial Intelligence (IJCAI 2005), Edinburgh (Scotland, UK), Professional Book Center, 2005, pp. 534–539.
[22] G. Nordh, B. Zanuttini, What makes propositional abduction tractable, Artificial Intelligence 172 (10) (2008) 1245–1284.

[23] I. Papatheodorou, A.C. Kakas, M.J. Sergot, Inference of gene relations from microarray data by abduction, in: Ch. Baral, G. Greco, N. Leone, G. Terracina (Eds.), Proceedings 8th International Conference on Logic Programming and Nonmonotonic Reasoning (LPNMR 2005), Diamante (Italy), in: Lecture Notes in Comput. Sci., vol. 3662, Springer-Verlag, September 2005, pp. 389–393.
[24] Y. Peng, J.A. Reggia, Abductive Inference Models for Diagnostic Problem Solving, Springer-Verlag, 1990.
[25] D. Poole, Probabilistic Horn abduction and Bayesian networks, Artificial Intelligence 64 (1) (1993) 81–129.
[26] T.J. Schaefer, The complexity of satisfiability problems, in: Proceedings 10th Symposium on Theory of Computing (STOC'78), San Diego (California, USA), 1978, pp. 216–226.
[27] B. Selman, H.J. Levesque, Abductive and default reasoning: A computational core, in: Proceedings 8th National Conference on Artificial Intelligence, Boston (Massachusetts, USA), MIT Press, 1990, pp. 343–348.
[28] L.G. Valiant, The complexity of computing the permanent, Theoret. Comput. Sci. 8 (2) (1979) 189–201.
[29] L.G. Valiant, The complexity of enumeration and reliability problems, SIAM J. Comput. 8 (3) (1979) 410–421.
[30] B. Zanuttini, New polynomial classes for logic-based abduction, J. Artificial Intelligence Res. 19 (2003) 1–10.