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Graph Ramsey games

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Wolfgang Slany[1]

Abstract. We consider combinatorial avoidance and achievement games based on graph Ramsey theory: The players take turns in coloring still uncolored edges of a graph G, each player being assigned a distinct color, choosing one edge per move. In avoidance games, completing a monochromatic subgraph isomorphic to another graph A leads to immediate defeat or is forbidden and the first player that cannot move loses. In the avoidance$^+$ variants, both players are free to choose more than one edge per move. In achievement games, the first player that completes a monochromatic subgraph isomorphic to A wins. Erdős & Selfridge [16] were the first to identify some tractable subcases of these games, followed by a large number of further studies. We complete these investigations by settling the complexity of all unrestricted cases: We prove that general graph Ramsey avoidance, avoidance$^+$, and achievement games and several variants thereof are **PSPACE**-complete. We ultra-strongly solve some nontrivial instances of graph Ramsey avoidance games that are based on symmetric binary Ramsey numbers and provide strong evidence that all other cases based on symmetric binary Ramsey numbers are effectively intractable.

Keywords: combinatorial games, graph Ramsey theory, Ramsey game, PSPACE-completeness, complexity, edge coloring, winning strategy, achievement game, avoidance game, the game of Sim, Pólya's enumeration formula, probabilistic counting, machine learning, heuristics, Java applet

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1 Introduction and overview

To illustrate the nature of combinatorics, Cameron [7] uses the following simple game: Two players, Red and Green, compete on a game board composed of six vertices and all $\binom{6}{2} = 15$ possible edges between these vertices. The players alternate in coloring at each move one so far uncolored edge using their color, with the restriction that building a complete subgraph with three vertices whose edges all have the same color (a monochromatic triangle) is forbidden. The game ends when one player is forced to give up because there are no legal moves left or when one of the players builds a triangle by mistake.

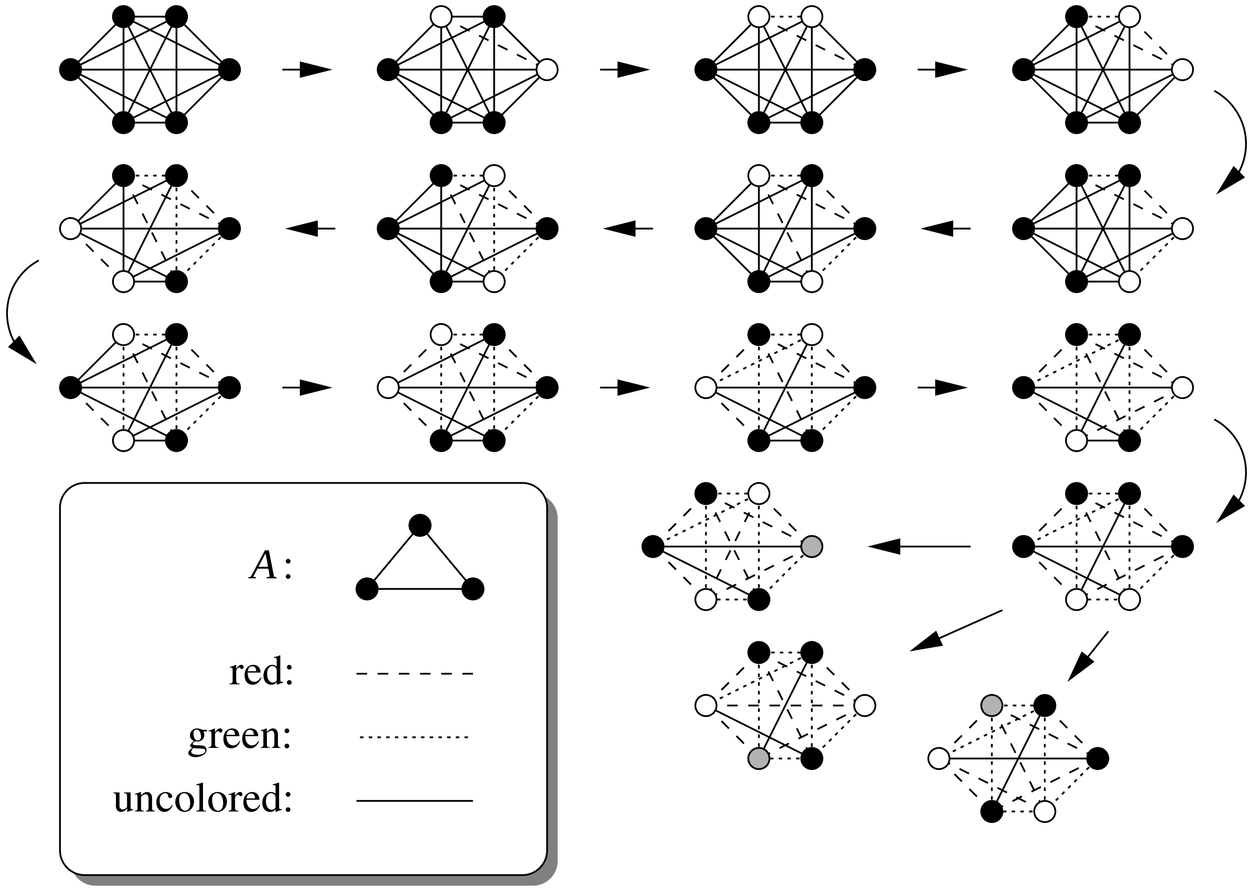

Figure 1: Sample play sequence of Sim. The initial, uncolored game board is shown on the top left corner. Player Red (= dashed lines) starts by coloring some edge, then player Green (= dotted lines) colors another one, etc. Finally, Red is forced to give up since any further coloring would complete a red triangle (= a monochromatic subgraph isomorphic to A).

This game was first described under the name Sim by Simmons [65] in 1969. Since then, it has attracted much interest [2, 3, 4, 5, 7, 9, 10, 11, 12, 15, 16, 20, 25, 26, 27, 28, 34, 38, 39, 40, 44, 45, 46, 48, 51, 55, 57, 61, 62, 63, 64, 66, 69]. Figure 1 shows a typical play sequence[1].

Besides their value as motivational examples for the field of combinatorics, games such as Sim are of practical interest because they can serve as models that simplify the analysis of competitive situations with opposing parties that pursue different interests, or for situations where one is faced with an unforeseeable environment such as Nature. It is easy to see that playing against a perfectly

[1]Considering that a hands-on session with an interactive system often is worth more than a thousand images, you might want to challenge a Java applet at http://www.dbai.tuwien.ac.at/proj/ramsey/ that plays Sim and its avoidance$^+$ variant Sim$^+$, playing perfectly when possible and improving its strategy by playing over the Internet when perfect play is impossible. In case you win, you will be allowed to leave your name in our hall-of-fame!

intelligent opponent with unlimited computational resources is the worst case that can happen. If the problem of winning against such an opponent can be solved, one will also be able to handle all other eventualities that could arise. Finding a winning strategy for a combinatorial game can thus be translated into finding a strategy to cope with many kinds of real world problems such as found in telecommunications, circuit design, scheduling as well as a large number of other problems of industrial relevance [24, 31, 49]. Proving or at least providing strong evidence that finding such a winning strategy is of high complexity helps to explain the great difficulties one often faces in corresponding real world problems [23, 29, 50]. As usual, we mean the complexity of deciding whether the first player has a winning strategy when we speak of the complexity of a game in the rest of this paper.

Another, more psychological reason why humans may be attracted by combinatorial games such as Sim is that they appeal to

> our primal beastly instincts; the desire to corner, torture, or at least dominate our peers. An intellectually refined version of these dark desires, well hidden under the façade of scientific research, is the consuming strive "to beat them all", to be more clever than the most clever, in short — to create the tools to *Math-master* them all in hot *comb*inatorial *comb*at! (Fraenkel [24])

In Section 2, we define the necessary notions from combinatorial games, computational complexity and Ramsey theory, informally introduce graph Ramsey games and discuss previous work that includes some tractable subcases. The exact definitions of all games we study are given in Section 3. Section 4 contains our main complexity results on the previously defined games. Section 5 contains the detailed proofs for all our complexity results. These results imply that the unrestricted graph Ramsey games are at least as hard as a large number of well-known games (e.g., Go [42]) and problems of industrial relevance (e.g., decision-making under uncertainty such as stochastic scheduling [49]) generally recognized as very difficult. Section 6 contains complexity results on some further variants of graph Ramsey games. In Section 7, we turn to concrete game instances and present our implemented winning strategy for Sim. We sketch the heuristics our program uses when perfect play is not possible, present a winning strategy for the avoidance$^+$ variant Sim$^+$ of Sim, and provide strong evidence that graph Ramsey avoidance games based on symmetric binary Ramsey numbers greater for $n > 3$ are intractable from all practical points of view. In Section 8, we state a number of conjectures and open problems related to Ramsey games.

2 Preliminaries and related work

Like many other combinatorial games, including Chess, Checkers, and Go, Sim is a two-player zero-sum perfect-information (no hidden information as in some card games, so there is no bluffing) game without chance moves (no rolling of dice). Zero-sum here means that the outcome of the game for the two players is restricted to either win-loss, loss-win, tie-tie, or draw-draw. The distinction between a draw and a tie is that a tie ends the game, whereas in a draw, the game would continue forever, both players being unable to force a win, following the terminology in the survey on combinatorial games by Fraenkel [24]. Sim is based on the simplest nontrivial example of

Ramsey theory [31, 47], the example being also known under the name of "party-puzzle": How many persons must be at a party so that either three mutual acquaintances or three persons that are not mutual acquaintances are present? More formally, classic binary Ramsey numbers are defined as follows:

Definition 2.1 $\mathrm{Ramsey}(n, m)$ *denotes the smallest number* r *such that any complete graph* K_r *(an undirected graph with r vertices and all possible edges between them) whose edges are all colored in red or in green either contains a red subgraph isomorphic to* K_n *or a green subgraph isomorphic to* K_m*. In classic* symmetric *binary Ramsey numbers,* n *equals* m*.*

Observation 2.1 *Another equivalent formulation says that* $\mathrm{Ramsey}(n, m)$ *is the smallest number of vertices such that an arbitrary undirected graph of that size either contains an* n*-clique (that is, a* K_n*) or an* m*-independent set (*m *isolated vertices, that is, with no edges between them).*

The classic result of F. P. Ramsey [54], a structural generalization of the pigeon-hole principle, tells us that these numbers always exist:

Theorem 1 (Ramsey [54]) $\forall (n, m) \in \mathbb{N}^2 \quad \mathrm{Ramsey}(n, m) < \infty.$

Ramsey used this result (that by itself was popularized only some years later by Erdős & Szekeres [17]) to prove that if ϕ is a first-order formula of the form $\exists x_1 \exists x_2 \cdots \exists x_n \forall y_1 \cdots \forall y_m \Phi$ where Φ is quantifier free, i.e., if ϕ is a Bernays-Schönfinkel formula, then the problem whether ϕ holds for every finite structure is decidable (see also Nešetřil [47]).

A simple combinatorial argument that $\mathrm{Ramsey}(3, 3) = 6$ is shown in Figure 2, and so the minimal number of persons satisfying above's "party-puzzle" question is six. Theoretically, Sim ends after a maximum of 15 moves since this is the number of edges in a complete graph with six vertices. If we define Sim such that monochromatic triangles are not allowed, and since Ramsey theory says that any edge-2-colored K_6 will contain at least one monochromatic triangle, we know that the second player will not be forced to give up simply because all edges are colored after 15 moves, as all games will end before the 15^{th} move. The game of Sim as it is usually described and played ends when one of the players completes a triangle in his color, whether forced or by mistake (this is called a 'misère-type' end condition: the last player to *move* loses, see e.g. Guy [33]), with no winner, that is, a tie, defined for the case when all edges are colored without a monochromatic triangle having been completed. For this misère-variant of Sim, the $\mathrm{Ramsey}(3, 3) = 6$ result implies that no game will ever end in a tie.

It is easy to see that in finite, two-player zero-sum perfect-information games with no ties and no chance moves, either the player who starts the game or his opponent must have the possibility to play according to a *winning strategy*: A player who follows such a strategy will always win no matter how well the opponent plays (for the existence of such a strategy, see for instance the fundamental theorem of combinatorial game theory in Fraenkel [23]). Clearly, this means that one of the players will have an a-priori upper-hand in Sim, so the answer to the following question is of central interest: Which of the two players has a winning strategy, the first or the second to move?

This question is the classic decision problem one can ask for any combinatorial game. In case of Sim, Mead *et al.* [44] have shown that the second player can always win. Nevertheless, a

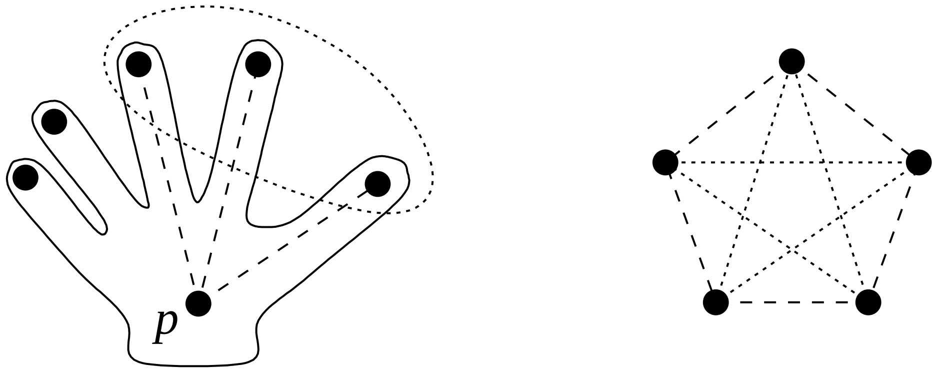

Figure 2: Visual proof that $\mathrm{Ramsey}(3,3) = 6$, as communicated by Ranan Banerji. The drawing on the left shows that six vertices are enough, as follows: Take any vertex p (as in 'palm') of an edge-2-colored K_6. At least three edges connected to p will have the same color. Without loss of generality, assume that this color is the dashed one. Consider the three vertices connected to p through these three edges: Either one of the edges that connect two of these vertices is of the dashed type (and then there is a dashed triangle with the edges connected to p), or not (and then the three top edges form a triangle in the other color). The edge-2-colored K_5 on the right serves as a counter-example, showing that five vertices are insufficient to force a monochromatic triangle. Thus, six is the smallest number with the required property.

winning strategy that is easy to memorize for human players has so far eluded us, despite much effort [9, 12, 25, 26, 27, 28, 44, 45, 46, 48, 57, 61, 62, 63, 64, 66, 69]. Knowing the strategy itself, especially if it can be stated in a concise form, might appear to be even better, but it is easy to see that knowing the answer to the decision problem for an arbitrary game situation, or at least being able to efficiently find out that answer, is equivalent to knowing the complete strategy.

A game being finite means that it should theoretically be possible to solve it. However, the trouble is that it might take an astronomical amount of time and memory (often even more as we will see in Section 7.4) to actually compute the winning strategy. Note that J. Schaeffer & Lake [60] have started trying to prove that a certain strategy for the game of Checkers is a winning one, using a massive amount of parallel hardware already running for several years. Their attempt requires the analysis of positions roughly equal in number to the square root of the size of the full game tree of Checkers (which in case of Checkers appears to be barely in reach of present day computing power) and thus can be substantially faster than finding a winning strategy from scratch, obviously for the price that in case their strategy is shown not to be a winning one, the game remains unsolved. Being usually unable to even prove that a strategy is a winning one, we turn to the next best thing, which is to classify the games in terms of computational complexity classes, that is, to find out how the function bounding the computational resources that are needed in the worst case to determine a winning strategy for the first player grows in relation to the size of the game description.

Here a technical problem becomes apparent, in that games must be scalable instead of having a fixed finite size in order to be classifiable. Generalizations to boards of size $n \times n$ of well-known games such as Chess, Checkers, and Go have been classified as **PSPACE**-complete and **EXPTIME**-complete [22, 21, 42]. **PSPACE** in particular is important for the analysis of these and large classes of more formal combinatorial games [23, 24, 29, 50, 59]. **PSPACE** is the class of problems that can be solved using memory space bounded by a polynomial in the size of the problem description. **PSPACE**-complete problems are the hardest problems in the class **PSPACE**: Solving one of these problems efficiently would mean that we could solve *any* other problem in **PSPACE** efficiently as well. While nobody so far was able to show that **PSPACE** problems are inherently difficult, despite much effort to show that the complexity class **P** containing the tractable problems solvable in polynomial time is different from **PSPACE**, it would be very surprising if they were not. Indeed, the well-known complexity class **NP** is included in **PSPACE**, so problems in **PSPACE** are at least as difficult as many problems believed to be very hard such as the satisfiability of boolean formulas or the traveling salesman problem. This means that it is rather unlikely that efficient general algorithms to solve **PSPACE**-complete combinatorial games do exist. For further details on computational complexity theory, consult Garey & Johnson [29] or Papadimitriou [50]. Obviously, the high complexity of such combinatorial games contributes to their attractiveness.

So the question is, what could be a generalization of Sim to game boards of arbitrary size? Let us first introduce some more notions from graph Ramsey theory that generalize the classic Ramsey numbers from Definition 2.1:

Definition 2.2 (see, e.g., [6, 14, 31, 58]) $G \to (A^r, A^g)$*: We say that a graph G* arrows *a graph-tuple (A^r, A^g) if for every edge-coloring with colors red and green, a red A^r or a green A^g occurs as a subgraph. In* symmetric *arrowing, $A^r = A^g = A$, and G is called a* Ramsey graph *of A if $G \to A$.*

Observation 2.2 $K_{\mathrm{Ramsey}(n,m)} \to (K_n, K_m)$.

The following generalization of Theorem 1 was proved in 1962 by Harary [37] after hearing a lecture on Ramsey theory given by Erdős and first published around 1973, independently by Chvátal & Harary [8], by Deuber [13], by Erdős *et al.* [18], and by Rödl [56]:

Theorem 2 (see, e.g., [14]) *Every graph has Ramsey graphs. In other words, for every graph A there exists a graph G that, for every edge-coloring with colors red and green, either contains a red or a green subgraph isomorphic to A.*

Note that the complexity of the arrowing relation has recently been determined:

Definition 2.3 ARROWING

Instance: (Finite) graphs G, A^r, and A^g.
Question: Does $G \to (A^r, A^g)$?

Theorem 3 (M. Schaefer [58]) ARROWING *is $\mathbf{\Pi_2^P}$-complete.*

We extend the arrowing relation for our purposes:

Definition 2.4 $(G, E^r, E^g) \rightarrow A$: *A* partly *edge-colored graph (G, E^r, E^g), where some edges E^r of G are precolored in red and some other edges E^g of G are precolored in green, arrows a graph A if every* complete *edge-coloring of (G, E^r, E^g) with colors red and green contains a monochromatic subgraph isomorphic to A.*

The game $G_{\text{Avoid-Ramsey}}$ is the generalization of Sim to graph Ramsey theory (exact definitions of all graph Ramsey game variants follow in Section 3). Similarly, $G_{\text{Achieve-Ramsey}}$ is a graph Ramsey achievement game. Harary [38] studied both $G_{\text{Achieve-Ramsey}}$ and $G_{\text{Avoid-Ramsey}}$ where G is restricted to complete graphs, A being an arbitrary graph. We call $G_{\text{Avoid-Ramsey}^+}$ the avoidance$^+$ variant where each player selects *at least one* so-far uncolored edge per move. For graph Ramsey achievement games, several tractable subcases are known:

Theorem 4 (Erdős & Selfridge [16]) *The first player has a winning strategy in $G_{\text{Achieve-Ramsey}}(K_n, K_k, \{\}, \{\})$ if*

$$k \leq \frac{1}{2} \log_2 n$$

and the game ends in a tie *if*

$$2^l > \binom{n}{k}, \quad \textit{where} \quad l = \binom{k}{2} - 1 ,$$

i.e., it is a tie *if*

$$k \geq 2\,(1 + o(1))\, \log_2 n .$$

While these results do not cover all cases with complete graphs such as for example $\text{Sim}_a = G_{\text{Achieve-Ramsey}}(K_6, K_3, \{\}, \{\})$, small instances of $G_{\text{Achieve-Ramsey}}$ generally seem to be very easy to analyze. Figure 3 shows, for instance, a trivial winning strategy for the first player in Sim_a. Beck [3] and Beck & Csirmaz [2] have generalized these results to games where the players alternate in choosing among previously unchosen elements of the complete k-uniform hypergraph of N vertices K_N^k, and the first player wins if he has selected all k-tuples of an n-set. For the case $k = 2$, their results subsumes Theorem 4. They also study infinite Ramsey games where the edges of the hypergraphs are required to be infinite but countable, for which they show that there always exist winning strategies for the first player. Several games of this kind are analyzed, all featuring simple winning strategies that imply their tractability. Further studies following the results of Erdős & Selfridge [16] can be found in [4, 5, 20, 34, 39, 40, 51, 55].

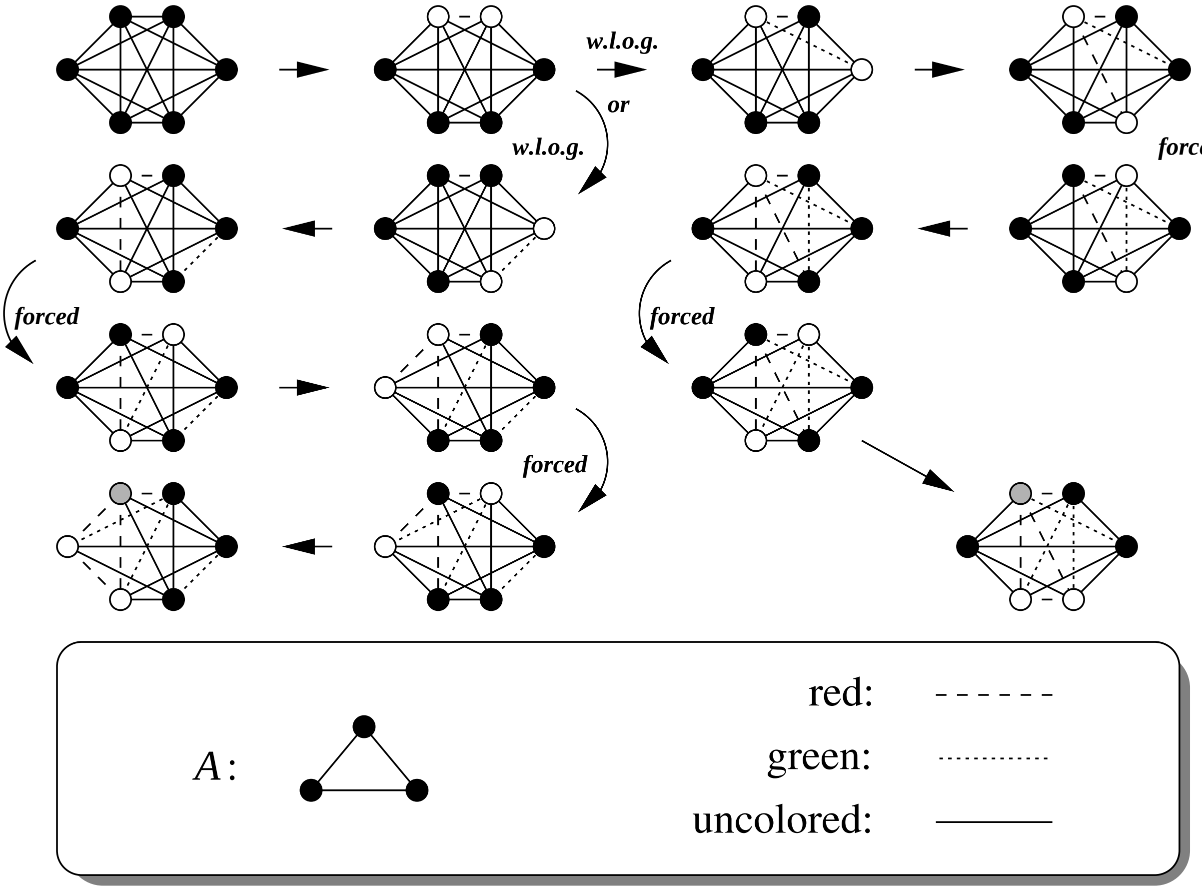

Figure 3: Trivial winning strategy for player Red in $\mathrm{Sim}_a = G_{\text{Achieve-Ramsey}}(K_6, K_3, \{\}, \{\})$.

3 Definitions of the graph Ramsey games

Let us now precisely define the introduced graph Ramsey games. Note that in the following definitions, the precolorings (E^r, E^g) are part of the input and are needed for the analysis of arbitrary game situations appearing in mid-game.

Definition 3.1 *The graph Ramsey* avoidance *game* $G_{\text{Avoid-Ramsey}}(G, A, E^r, E^g)$ *is played on a graph* $G = (V, E)$*, another graph* A*, and two nonintersecting sets* $E^r \cup E^g \subseteq E$ *that contain edges initially colored in red and green, respectively. Two players, Red and Green, take turns in selecting at each move one so-far uncolored edge from* E *and color it in red for player Red respectively in green for player Green. However, both players are forbidden to choose an edge such that* A *becomes isomorphic to a subgraph of the red or the green part of* G*. It is Red's turn. The first player unable to move loses.*

Definition 3.2 $G_{\text{Avoid'-Ramsey}}$ *is the misère-variant of* $G_{\text{Avoid-Ramsey}}$*, were completing a monochromatic subgraph isomorphic to graph* A *leads to immediate defeat. The game ends in a tie if no edges are left to color.*

Clearly, these two avoidance variants coincide whenever $(G, E^r, E^g) \to A$ (the proof is straightforward):

Corollary 5 *If* $(G, E^r, E^g) \to A$*, then player Red has a winning strategy in* $G_{\text{Avoid-Ramsey}}(G, A, E^r, E^g)$ *iff player Red has a winning strategy in* $G_{\text{Avoid'-Ramsey}}(G, A, E^r, E^g)$.

Observation 3.1 *Sim* $= G_{\text{Avoid-Ramsey}}(K_{\text{Ramsey}(3,3)}, K_3, \{\}, \{\})$
$= G_{\text{Avoid'-Ramsey}}(K_{\text{Ramsey}(3,3)}, K_3, \{\}, \{\})$.

The following avoidance$^+$ variant intuitively corresponds even closer to the spirit of Ramsey theory because any combination in the number of red and green edges is possible (in the other graph Ramsey avoidance and achievement games, red and green edges are added at the same rate):

Definition 3.3 $G_{\text{Avoid-Ramsey}^+}(G, A, E^r, E^g)$*: Everything is as in Definition 3.1, except that each player selects* at least one *so-far uncolored edge from* E *during one move.*

Observation 3.2 *Sim*$^+ = G_{\text{Avoid-Ramsey}^+}(K_{\text{Ramsey}(3,3)}, K_3, \{\}, \{\})$.

In the case of graph Ramsey achievement games, three major variants can be distinguished, as follows:

Definition 3.4 *In the graph Ramsey* achievement *game* $G_{\text{Achieve-Ramsey}}(G, A, E^r, E^g)$ *everything is as in Definition 3.1, except that the first player who builds a monochromatic subgraph isomorphic to* A wins.

Definition 3.5 *A simple strategy-stealing argument tells us that with optimal play on an uncolored board,* $G_{\text{Achieve-Ramsey}}$ *must be either a first-player win or a draw, so it is only fair to count a draw as a second-player win. Let us call this variant* $G_{\text{Achieve'-Ramsey}}$.

We know from the fundamental theorem of combinatorial game theory (see e.g. [23]) that there exists a winning strategy for this game. It is straightforward that when $(G, E^r, E^g) \to A$, $G_{\text{Achieve-Ramsey}}$ and $G_{\text{Achieve'-Ramsey}}$ are in fact the same game.

Definition 3.6 *Following the terminology of Beck & Csirmaz [2], let us call the variant of* $G_{\text{Achieve-Ramsey}}$ *where all the second player does is to try to prevent the first player to build* A*, without winning by building it himself, the "weak" graph Ramsey achievement game* $G_{\text{Achieve"-Ramsey}}$.

Again, it is straightforward that when the first player has a winning strategy or when there is no possibility for the second player to build a green subgraph isomorphic to A, $G_{\text{Achieve'-Ramsey}}$ and $G_{\text{Achieve"-Ramsey}}$ are in fact the same game.

Observation 3.3 *Sim*$_a = G_{\text{Achieve-Ramsey}}(K_{\text{Ramsey}(3,3)}, K_3, \{\}, \{\})$,
and from the point of view of a perfect first player,

Sim$_a = G_{\text{Achieve'-Ramsey}}(K_{\text{Ramsey}(3,3)}, K_3, \{\}, \{\})$
$= G_{\text{Achieve"-Ramsey}}(K_{\text{Ramsey}(3,3)}, K_3, \{\}, \{\})$.

4 Main complexity results

We have long believed that the problems of deciding whether the first players have winning strategies in the graph Ramsey avoidance games are complete for polynomial space. We show here that our intuition was indeed right, corroborating the apparent difficulty of Sim and its $G_{\text{Avoid-Ramsey}^+}$ variant Sim$^+$. All game variants mentioned below have been formally defined in Section 3. The proofs of these results are discussed in Section 5.

Theorem 6 $G_{\text{Avoid-Ramsey}}$ *is* **PSPACE**-*complete.*

Theorem 7 $G_{\text{Avoid'-Ramsey}}$ *is* **PSPACE**-*complete.*

In order to prove these results, a special gadget construction was needed that constrains the moves of the players in spite of their apparent freedom to choose any uncolored edge. Its construction was inspired from the similar notion of "illegitimate" moves introduced by Even & Tarjan [19] and further developed by T. Schaefer [59].

Theorem 8 $G_{\text{Avoid-Ramsey}^+}$ *is* **PSPACE**-*complete.*

Theorem 8 facilitates the matching between abstract problems and real life applications as it allows to drop the artificial requirement that players must move in a predetermined sequence. Let us observe, however, that **PSPACE**-completeness of avoidance games such as the avoidance games played on propositional formulas and on sets described in [59] do not automatically imply the **PSPACE**-completeness of their avoidance$^+$ variants: Most of these **PSPACE**-complete single-choice-per-move avoidance games have trivially decidable, and thus tractable, avoidance$^+$ variants. We also note that even in case both the avoidance$^+$ and the single-choice-per-move avoidance variant are **PSPACE**-complete, it is easy to see that the players having winning strategies can be different for the two games, and that even if in both games the first player has a winning strategy, completely new game situations requiring different playing behavior may arise in an avoidance$^+$ variant.

Observation 4.1 *Note that a misère-variant of* $G_{\text{Avoid-Ramsey}^+}$ *is easily imaginable, its* **PSPACE**-*completeness proof following the lines of the proof of Theorem 7 when applied to Theorem 8.*

Corollary 9 $G_{\text{Avoid-Ramsey}}$, $G_{\text{Avoid'-Ramsey}}$, *and* $G_{\text{Avoid-Ramsey}^+}$ *remain* **PSPACE**-*complete even if the avoidance graph* A *is restricted to a specific fixed graph.*

For achievement games, the situation is similar:

Theorem 10 $G_{\text{Achieve''-Ramsey}}$ *is* **PSPACE**-*complete.*

Theorem 11 $G_{\text{Achieve'-Ramsey}}$ *is* **PSPACE**-*complete.*

Corollary 12 $G_{\text{Achieve''-Ramsey}}$ *and* $G_{\text{Achieve'-Ramsey}}$ *remain* **PSPACE**-*complete even if the achievement graph* A *is degree-restricted.*

Theorem 13 $G_{\text{Achieve-Ramsey}}$ *is* **PSPACE**-*complete.*

The **PSPACE**-completeness of the achievement games came a bit as a surprise since tractable subcases are known [3, 4, 16, 20, 34, 38, 39, 40, 51], and the $G_{\text{Achieve-Ramsey}}$ game Sim$_a$ corresponding to Sim with respect to graphs G and A has a trivial winning strategy, in blatant contrast to Sim and Sim$^+$.

5 Proofs of the main complexity results

One way to prove **PSPACE**-completeness consists in showing that the problem is solvable in **PSPACE** ("membership") and that it is at least as difficult as any other problem in **PSPACE** ("hardness"). The intricate parts of the proofs of Theorems 6–13 will be found in their hardness parts. The following lemma establishes the membership parts of all proofs:

Lemma 14 *All graph Ramsey games defined in Section 3 are in* **PSPACE**.

Proof. Let $n \stackrel{\text{def}}{=} |(G, A, E^r, E^g)|$ denote the size of the input. The number of moves in any graph Ramsey game is limited by the number of initially uncolored edges in the graph G, so any game will end after at most $|E| - |E^r| - |E^g| < n$ edge colorings, and each game situation can be described as the edge that was just colored, so this information uses memory $O(\log n)$ which is bounded by $O(n)$. It is easy to enumerate in some lexicographic order all game situations that can originate from a particular game situation through the coloring of one edge. Altogether, this implies membership in **PSPACE** by the following argument: Given an initial game situation, a depth-first algorithm that checks all possible game sequences but keeps in memory only one branch of the game tree at a time, backtracking to unexplored branching points in order to scan through the whole game tree, can decide whether there is a winning strategy for player Red using memory bounded by the maximum stack size, which is $O(n \log n) < O(n^2)$ and thus polynomial in the size of the input. ∎

5.1 Proof of Theorem 6

Membership of $G_{\text{Avoid-Ramsey}}$ in **PSPACE** follows from Lemma 14. To show hardness, i.e., that a problem is at least as difficult as *any* other problem in the class, it is enough to show that it is at least as difficult as *one* complete problem from that class. Thus, it suffices to show that there exists a simple reduction from one known **PSPACE**-complete problem to $G_{\text{Avoid-Ramsey}}$. In the case at hand, the complete problem will be $G_{\text{Achieve-POS-CNF}}$, a game first described by T. Schaefer [59]. The definition of the game $G_{\text{Achieve-POS-CNF}}$ is restated below in Definition 5.1. The reduction will be a **LOGSPACE** transducer that transforms any instance of the $G_{\text{Achieve-POS-CNF}}$ game into an instance of the $G_{\text{Avoid-Ramsey}}$ game using only space logarithmic in the size of the $G_{\text{Achieve-POS-CNF}}$ instance for intermediate results, and such that the answer to the $G_{\text{Achieve-POS-CNF}}$ decision problem is the same as the answer to the corresponding $G_{\text{Avoid-Ramsey}}$ decision problem. This would allow to decide $G_{\text{Achieve-POS-CNF}}$, which is know to be complete and therefore by definition a most difficult problem in **PSPACE**, by doing a simple transformation and solving a $G_{\text{Avoid-Ramsey}}$ problem, thus establishing that deciding $G_{\text{Avoid-Ramsey}}$ has to be at least as difficult as the complete problem of deciding $G_{\text{Achieve-POS-CNF}}$, and therefore as difficult as *any* other problem in **PSPACE**.

The game that is reduced to $G_{\text{Achieve-POS-CNF}}$ is defined as follows:

Definition 5.1 (T. Schaefer [59]) $G_{\text{Achieve-POS-CNF}}(F)$*: We are given a positive CNF formula F. A move consists of choosing some variable of F which has not yet been chosen. Player I starts the game. The game ends after all variables of F have been chosen. Player I wins iff F is* `true` *when all variables chosen by player I are set to* `true` *and all variables chosen by player II are set to* `false`.

Observation 5.1 $G_{\text{Achieve-POS-CNF}}$ *by definition is a finite two-player zero-sum perfect-information game with no ties and no chance moves, so either one of its two players has a winning strategy.*

For example, on input $x_1 \wedge (x_2 \vee x_3) \wedge (x_2 \vee x_4)$ player II has a winning strategy, whereas on input $(x_1 \vee x_4) \wedge (x_2 \vee x_3) \wedge (x_2 \vee x_4)$ player I has a winning strategy.

To prove the hardness part of the proof of Theorem 6, formally we will show that

$$G_{\text{Achieve-POS-CNF}} \leq_{\log} G_{\text{Avoid-Ramsey}}.$$

The following result from T. Schaefer would then complete our proof:

Theorem 15 (T. Schaefer [59])
$G_{\text{Achieve-POS-CNF}}$ *is* **PSPACE**-*complete.*

Let us sketch here the idea of the proof using the small example in Figure 4. The exact description of the reduction follows later. Each gadget P_i, containing among other precolored edges (note the abbreviations in Figure 4) the three uncolored edges r_i , y_i , g_i, corresponds to the boolean variable x_i of F. Each gadget D_j, containing the uncolored edge d_j, corresponds to conjunct C_j of F. The links between the two types of gadgets correspond to the occurrence of the variables in the conjuncts. Player Red can only color edges r_i and y_i whereas player Green can only color edges g_i, y_i, and possibly one of the edges d_j if the g_i's connected to it are uncolored. By counting the number of possible moves, one sees that Green has a winning strategy if he succeeds in coloring one edge d_j at move $2n + 2$. Coloring edge y_i in a P_i gadget means that the other player can only color the remaining border edge. Thus, the players first will race to color all edges y_i, since by doing so, Red could possibly hinder Green from coloring any edge d_j at the end, whereas Green could possibly leave enough edges g_i uncolored so that he can color one edge d_j at the end.

Fact 5.2 *From the definition of $G_{\text{Achieve-POS-CNF}}$, we easily see that player I wins iff he succeeds in playing some variable in each conjunct. This is mirrored in $G_{\text{Avoid-Ramsey}}$ as follows: Player Red can win iff he succeeds in coloring some edge y_i so that player Green later on can only choose edges g_i in these particular triples, making it impossible for Green to color any edge d_j at the end.*

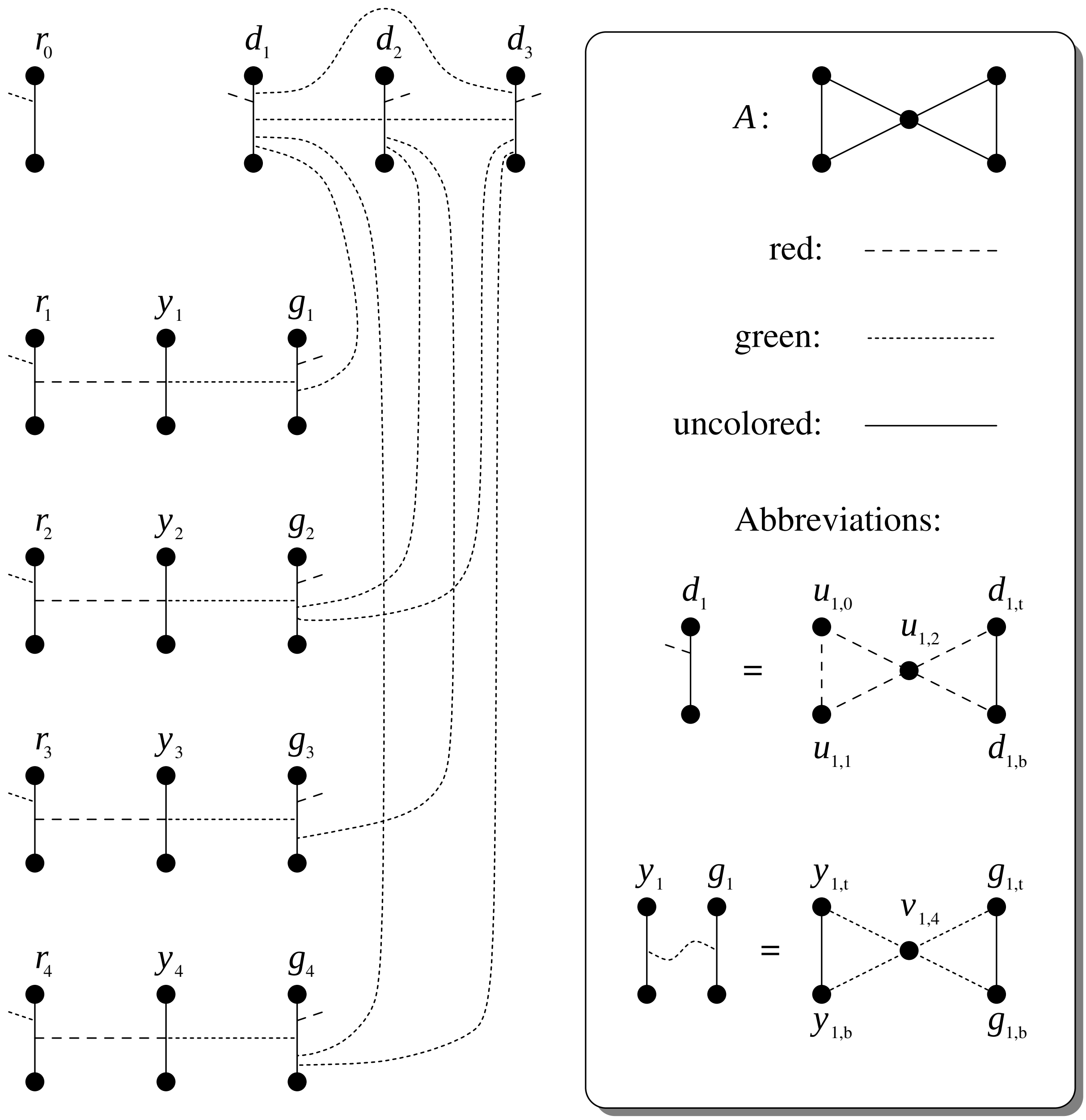

Figure 4: Example of the reduction $G_{\text{Achieve-POS-CNF}} \leq_{\log} G_{\text{Avoid-Ramsey}}$ from the proof of Theorem 6. The graph G is shown on the left and corresponds to the input formula $F = (x_1 \vee x_4) \wedge (x_2 \vee x_3) \wedge (x_2 \vee x_4)$, featuring a winning strategy for player I. The graph A (a 'bow-tie') that both players must avoid in their color is shown on the top right corner. The dashed (= red) and dotted (= green) lines are abbreviations as partly indicated on the bottom right corner, the rest following the same ideas.

The rest of the proof consists in an analysis of several cases showing that there is a winning strategy for player I in $G_{\text{Achieve-POS-CNF}}(F)$ iff there is a winning strategy for player Red in $G_{\text{Avoid-Ramsey}}(G, A, E^r, E^g)$. The detailed proof follows.

We first define the **LOGSPACE** reduction from $G_{\text{Achieve-POS-CNF}}(F)$ to $G_{\text{Avoid-Ramsey}}(G, A, E^r, E^g)$ that has already been illustrated through a small example in Figure 4: Let a positive CNF formula F be given. Assume without loss of generality that $F = C_1 \wedge \ldots \wedge C_m$ where each conjunct C_j is a disjunction of n_j positive literals, that is, $C_j = l_{j,1} \vee \ldots \vee l_{j,n_j}$ where $l_{j,k} \in \{x_1, \ldots, x_n\}$ and all n variables appear at least once in F. We then define the graphs $G \stackrel{\text{def}}{=} (V, E)$, $A \stackrel{\text{def}}{=} (V^A, E^A)$ and the edge-sets E^r, E^g, by

$$
\begin{aligned}
V &\stackrel{\text{def}}{=} \bigcup_{0\le i\le n} X_i \,, \\
X_0 &\stackrel{\text{def}}{=} \bigcup_{0\le j\le m} B_j \,, \\
B_0 &\stackrel{\text{def}}{=} \{u_{0,0}\,,\, u_{0,1}\,,\, u_{0,2}\,,\, r_{0,\text{t}}\,,\, r_{0,\text{b}}\} \,, \\
B_j &\stackrel{\text{def}}{=} \{u_{j,0}\,,\, u_{j,1}\,,\, u_{j,2}\,,\, d_{j,\text{t}}\,,\, d_{j,\text{b}}\} \cup \\
&\quad\ \bigcup_{1\le p<j} \{w_{j,p}\} \cup \bigcup_{1\le k\le n_j} \{f_{j,k}\} \quad \text{for} \quad 1 \le j \le m \,, \\
X_i &\stackrel{\text{def}}{=} \{v_{i,0}\,,\, v_{i,1}\,,\, v_{i,2}\,,\, r_{i,\text{t}}\,,\, r_{i,\text{b}}\,,\, v_{i,3}\,,\, y_{i,\text{t}}\,,\, y_{i,\text{b}}\,, \\
&\quad\ v_{i,4}\,,\, g_{i,\text{t}}\,,\, g_{i,\text{b}}\,,\, v_{i,5}\,,\, v_{i,6}\,,\, v_{i,7}\} \quad \text{for} \quad 1 \le i \le n \,, \\
E &\stackrel{\text{def}}{=} \bigcup_{0\le i\le n} P_i \,, \\
P_0 &\stackrel{\text{def}}{=} \bigcup_{0\le j\le m} D_j \,, \\
D_0 &\stackrel{\text{def}}{=} \triangle(u_{0,0}\,,\, u_{0,1}\,,\, u_{0,2}) \cup \triangle(u_{0,2}\,,\, r_{0,\text{t}}\,,\, r_{0,\text{b}}) \,, \\
&\qquad\qquad \text{where} \quad \triangle(\alpha\,,\,\beta\,,\,\gamma) \stackrel{\text{def}}{=} \Big\{\{\alpha\,,\,\beta\}, \{\alpha\,,\,\gamma\}, \{\beta\,,\,\gamma\}\Big\} \,, \\
D_j &\stackrel{\text{def}}{=} \triangle(u_{j,0}\,,\, u_{j,1}\,,\, u_{j,2}) \cup \triangle(u_{j,2}\,,\, d_{j,\text{t}}\,,\, d_{j,\text{b}}) \cup \\
&\quad\ \bigcup_{1\le p<j} \Big\{\{w_{j,p}\,,\, d_{p,\text{t}}\}, \{w_{j,p}\,,\, d_{p,\text{b}}\}, \{w_{j,p}\,,\, d_{j,\text{t}}\}, \{w_{j,p}\,,\, d_{j,\text{b}}\}\Big\} \cup \\
&\quad\ \bigcup_{1\le k\le n_j} \Big\{\{f_{j,k}\,,\, d_{j,\text{t}}\}, \{f_{j,k}\,,\, d_{j,\text{b}}\}, \{f_{j,k}\,,\, g_{h,\text{t}}\}, \{f_{j,k}\,,\, g_{h,\text{b}}\} \mid l_{j,k} = x_h\Big\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad 1 \le j \le m \,, \\
P_i &\stackrel{\text{def}}{=} \triangle(v_{i,0}\,,\, v_{i,1}\,,\, v_{i,2}) \cup \triangle(v_{i,2}\,,\, r_{i,\text{t}}\,,\, r_{i,\text{b}}) \cup \triangle(v_{i,3}\,,\, r_{i,\text{t}}\,,\, r_{i,\text{b}}) \cup \\
&\quad\ \triangle(v_{i,3}\,,\, y_{i,\text{t}}\,,\, y_{i,\text{b}}) \cup \triangle(v_{i,4}\,,\, y_{i,\text{t}}\,,\, y_{i,\text{b}}) \cup \triangle(v_{i,4}\,,\, g_{i,\text{t}}\,,\, g_{i,\text{b}}) \cup \\
&\quad\ \triangle(v_{i,5}\,,\, g_{i,\text{t}}\,,\, g_{i,\text{b}}) \cup \triangle(v_{i,5}\,,\, v_{i,6}\,,\, v_{i,7}) \qquad \text{for} \quad 1 \le i \le n \,,
\end{aligned}
$$

$$
\begin{aligned}
V^A &\stackrel{\text{def}}{=} \{a_0\,,\, a_1\,,\, a_2\,,\, a_3\,,\, a_4\}\,,\\
E^A &\stackrel{\text{def}}{=} \triangle(a_0\,,\, a_1\,,\, a_2) \cup \triangle(a_2\,,\, a_3\,,\, a_4)\,,\\
E^r &\stackrel{\text{def}}{=} \bigcup_{0\le i\le n} P_i^r\,,\\
P_0^r &\stackrel{\text{def}}{=} \bigcup_{1\le j\le m} \Big(\triangle(u_{j,0}\,,\, u_{j,1}\,,\, u_{j,2}) \cup \Big\{\{u_{j,2}\,,\, d_{j,\mathrm{t}}\}, \{u_{j,2}\,,\, d_{j,\mathrm{b}}\}\Big\}\Big)\,,\\
\\
P_i^r &\stackrel{\text{def}}{=} \Big\{\{v_{i,3}\,,\, r_{i,\mathrm{t}}\}, \{v_{i,3}\,,\, r_{i,\mathrm{b}}\}, \{v_{i,3}\,,\, y_{i,\mathrm{t}}\}, \{v_{i,3}\,,\, y_{i,\mathrm{b}}\},\\
&\qquad \{v_{i,5}\,,\, r_{i,\mathrm{t}}\}, \{v_{i,5}\,,\, r_{i,\mathrm{b}}\}\Big\} \cup \triangle(v_{i,5}\,,\, v_{i,6}\,,\, v_{i,7}) \qquad \text{for} \quad 1 \le i \le n\,,\\
E^g &\stackrel{\text{def}}{=} \bigcup_{0\le i\le n} P_i^g\,,\\
P_0^g &\stackrel{\text{def}}{=} \bigcup_{0\le j\le m} D_j^g\,,\\
D_0^g &\stackrel{\text{def}}{=} \triangle(u_{0,0}\,,\, u_{0,1}\,,\, u_{0,2}) \cup \Big\{\{u_{0,2}\,,\, r_{0,\mathrm{t}}\}, \{u_{0,2}\,,\, r_{0,\mathrm{b}}\}\Big\}\,,\\
D_j^g &\stackrel{\text{def}}{=} \bigcup_{1\le p<j} \Big\{\{w_{j,p}\,,\, d_{p,\mathrm{t}}\}, \{w_{j,p}\,,\, d_{p,\mathrm{b}}\}, \{w_{j,p}\,,\, d_{j,\mathrm{t}}\}, \{w_{j,p}\,,\, d_{j,\mathrm{b}}\}\Big\} \cup\\
&\qquad \bigcup_{1\le k\le n_j} \Big\{\{f_{j,k}\,,\, d_{j,\mathrm{t}}\}, \{f_{j,k}\,,\, d_{j,\mathrm{b}}\}, \{f_{j,k}\,,\, g_{h,\mathrm{t}}\}, \{f_{j,k}\,,\, g_{h,\mathrm{b}}\} \mid l_{j,k} = x_h\Big\}\\
&\qquad\qquad \text{for} \quad 1 \le j \le m\,,\\
P_i^g &\stackrel{\text{def}}{=} \triangle(v_{i,0}\,,\, v_{i,1}\,,\, v_{i,2}) \cup \Big\{\{v_{i,2}\,,\, g_{i,\mathrm{t}}\}, \{v_{i,2}\,,\, g_{i,\mathrm{b}}\},\\
&\qquad \{v_{i,4}\,,\, y_{i,\mathrm{t}}\}, \{v_{i,4}\,,\, y_{i,\mathrm{b}}\}, \{v_{i,4}\,,\, g_{i,\mathrm{t}}\}, \{v_{i,4}\,,\, g_{i,\mathrm{b}}\}\Big\} \quad \text{for} \quad 1 \le i \le n\,.
\end{aligned}
$$

Since printing and copying in color was not universally available when this paper was written, and to avoid confusion resulting from the large number of vertices and edges, the graph in Figure 4 uses certain conventions to represent colors, vertices and edges as indicated on its right-hand side. For instance, we use r_3 as a shortcut for the edge $\{r_{3,\mathrm{t}}\,,\, r_{3,\mathrm{b}}\}$, where "t" marks the vertex at the top of the edge and "b" the one at the bottom.

It immediately follows from the construction that there is a simple **LOGSPACE** transducer that computes (G, A, E^r, E^g) from input F. We still have to show that this construction ensures that there is a winning strategy for player I of $G_{\text{Achieve-POS-CNF}}(F)$ iff there is a winning strategy for player Red of $G_{\text{Avoid-Ramsey}}(G, A, E^r, E^g)$.

Observation 5.3 *By having a closer look at the construction of (G, A, E^r, E^g), we observe that edges r_i for $i = 0, \ldots, n$ can only be chosen by player Red. For instance, if player Green would color r_2 in green, this would complete a green subgraph made of $v_{2,0}\,,\, v_{2,1}\,,\, v_{2,2}\,,\, r_{2,\mathrm{t}}\,,$ and $r_{2,\mathrm{b}}$*

that would be isomorphic to A, which is forbidden according to Definition 3.1. Similarly, all edges d_j and g_i for $j = 1, \ldots, m$ and $i = 1, \ldots, n$ can only be chosen by player Green. Only the remaining edges y_i for $i = 1, \ldots, n$ can initially be chosen by both players. However, once player Red has chosen y_i for some $i = 1, \ldots, n$, he cannot choose edge r_i anymore, but Green can still play g_i, and vice versa for the reversed roles of Red and Green. Thus, for each triple r_i , y_i , g_i for $i = 1, \ldots, n$, the first player to move has the option to choose y_i, but the second player that colors an edge in that triple can only select, in the case of Red, edge r_i, and in the case of Green, edge g_i, once the middle edge y_i has been occupied. The remaining third edge of the triple always has to stay uncolored. In other words, each player can color one edge of each triple, but only the first to consider that particular triple has the possibility to occupy the central edge, the player coming second being left with the option to color his respective border edge once the other player effectively has already chosen the central edge.

Note that Green can only choose a single edge among edges d_j because of the green connections through vertices $w_{j,p}$. Additionally, Green cannot select both an edge d_j and an edge g_i when the variable x_i appears as literal $l_{j,k}$ in conjunct C_j of F because of the connection through $f_{j,k}$.

Fact 5.4 *As a result of what has been said above, player Red can color altogether exactly $n + 1$ edges: namely n edges out of the r_i , y_i for $i = 1, \ldots, n$, with no edges r_i and y_i from the same triple, plus r_0. Player Green can either color altogether at most n or at most $n + 1$ edges: namely at most n edges out of the y_i , g_i for $i = 1, \ldots, n$, with no edges y_i and g_i from the same triple, plus at most one edge among edges d_j, depending on the combination of edges g_i and edge d_j colored in green and on which edges g_i player Red could previously "force" player Green to color in green by occupying the corresponding edges y_i.*

Lemma 16 *For player Green being able to color less than $n + 1$ edges means that player Red has at least one more edge free to color at the end, so Red can win, whereas for Green to be able to color exactly $n + 1$ edges means that both players color the same number of edges, and since Red started, Green wins the game.*

Proof. Follows immediately from Fact 5.4. ∎

We constructed (G, A, E^r, E^g) in such a way as to constrain the moves of the players in spite of their apparent freedom to choose an uncolored edge, by punishing moves that are *illegitimate*. Illegitimate moves are those that do not follow the complementary notion of *legitimate* play, which is defined as follows:

Definition 5.2 Legitimate play: *We call a $G_{\text{Avoid-Ramsey}}(G, A, E^r, E^g)$ game sequence* legitimate *iff it has the following form: For moves $q = 1, 2, \ldots, n$ both players choose so far uncolored edges y_{i_q}, where $i_q \in \{1, \ldots, n\}$. For moves $q = n + 1, n + 2, \ldots, 2n + 2$ player Red chooses colorable edges of type r_i with $i \in \{0, \ldots, n\}$, and player Green chooses colorable edges of type g_i, with $i \in \{1, \ldots, n\}$ and, if possible, one edge d_j.*

We observe that the first n moves of a legitimate game sequence played on $G_{\text{Avoid-Ramsey}}(G, A, E^r, E^g)$ mimic those of $G_{\text{Achieve-POS-CNF}}(F)$ in an obvious way: Player Red on move q chooses edge y_{i_q} where player I chooses variable x_{i_q}, and similar for players Green and II. Let us note that in our construction, an illegitimate move may put the player who selects such a move into a less favorable position for the rest of the play.

Once at least one edge in every triple r_i, y_i, g_i for $i = 1, \ldots, n$ is colored, all remaining playable edges are *uncontested*, that is, only one of the two players can color each particular colorable edge that is left, or, said in another way, the two players cannot take away edges from each other anymore.

Definition 5.3 The racing phase: *We call the part of a game sequence until only uncontested moves remain the* racing phase *of the $G_{\text{Avoid-Ramsey}}$ game, because during that phase the two players race to occupy the 'right' edges y_i that ultimately will lead to the victory of one of them.*

After this racing phase, each player becomes preoccupied with his own set of edges that are left to play for him alone and tries to play a solitaire in it as long as possible. The order in which Red plays his remaining colorable edges is irrelevant, and Green can maximize the number of edges he can color by coloring all remaining colorable edges g_{i_q} and, if available, y_{i_q}, the order being again irrelevant, with the possible addition of one edge d_j, depending on which edges g_{i_q} Green colors during a game sequence.

Lemma 17 *If Green can color some d_j at one of his moves and also is able to color every g_i such that Red selected y_i during the racing phase, then Green could as well have chosen to color d_j as his last move.*

Proof. Independently of the notion of legitimate play, it does not matter during which move Green colors this one edge d_j, i.e., it needs not to be his last move, but Green still must be able to color every edge g_i where Red colored edge y_i during the racing phase. On the one hand, if Green can color some d_j at one of his moves and also is able to color every g_i such that Red selected y_i during the racing phase, then Green could as well have chosen to color d_j as his last move. On the other hand, if Green chooses to color some d_j at one of his moves such that he cannot color every g_i where Red selected y_i during the racing phase, then Green could as well have chosen to color one of these edges g_i instead of the edge d_j. ∎

As remarked in Lemma 16, win or loss depends only on the difference in the number of edges the two players can color, but not on the particular edges they color. In the following, we thus can assume without loss of generality that, if permitted at all by the coloring of edges g_i, Green always colors any edge d_j *after* coloring all possible edges of type y_i and g_i.

We will also need the following weaker variant of legitimate play:

Definition 5.4 Winner-legitimate play: *We call a game sequence* winner-legitimate *if the player with the winning strategy always has chosen legitimate moves.*

The following key lemma will allow us to decide in each case who can win the $G_{\text{Avoid-Ramsey}}$ game:

Lemma 18 *Consider the game situation just before move $2n + 2$ in a winner-legitimate game played on $G_{\text{Avoid-Ramsey}}(G, A, E^r, E^g)$. Player I has a winning strategy on the corresponding $G_{\text{Achieve-POS-CNF}}(F)$ game iff there exists no edge d_j that Green can choose after coloring all possible edges of type g_i.*

Proof. The statement follows straightforward from the following facts:

1. Observation 5.1 which says that either one of the two players of $G_{\text{Achieve-POS-CNF}}(F)$ has a winning strategy,
2. Definition 5.4 of winner-legitimate play on $G_{\text{Avoid-Ramsey}}(G, A, E^r, E^g)$ that forces the player with the winning strategy to first color edges y_i,
3. Lemma 17 which says we can always assume that Green first colors all possible edges y_i and g_i before coloring any edge d_j,
4. Lemma 16 and the previous fact, which together imply that Red and Green color in sum $2n + 1$ edges before Green colors any edge d_j, and
5. the one-to-one correspondence remarked in Fact 5.2 between,
   (a) in the case of $G_{\text{Achieve-POS-CNF}}(F)$, for player I to win iff he succeeds in playing some variable x_i in each conjunct C_j, and,
   (b) in the case of $G_{\text{Avoid-Ramsey}}(G, A, E^r, E^g)$, for the first player, Red, to win iff he succeeds in playing some edges y_i (note that there is an edge y_i for each variable x_i in F) so that his opponent, player Green, later on can only choose edges g_i in these particular triples, making it impossible for him to color any edge d_j (remember that there is an edge d_j for each conjunct C_j in F, and that edges g_i and d_j are connected through $f_{j,k}$ iff variable x_i appears in conjunct C_j). ∎

In the following, we show that player I can win $G_{\text{Achieve-POS-CNF}}(F)$ iff player Red can win $G_{\text{Avoid-Ramsey}}(G, A, E^r, E^g)$, which concludes the proof.

($\Rightarrow$) Assume that player I has a winning strategy for $G_{\text{Achieve-POS-CNF}}(F)$. We first claim that Red has a strategy for $G_{\text{Avoid-Ramsey}}(G, A, E^r, E^g)$ that wins any game in which Green plays legitimately. The strategy consists of playing legitimately and applying player I's winning strategy for $G_{\text{Achieve-POS-CNF}}(F)$ during the racing phase, via the correspondence between variables x_{i_q} and edges y_{i_q}. After the racing phase is over, that is in case of legitimate play, after n moves, player Red and Green alternate in coloring edges of the two disjoint sets of uncontested moves of each player. The uncontested moves of Red consist in edge r_0 and in all edges r_{i_q} such that Green played y_{i_q} during the racing phase, so Red altogether colors $n + 1$ edges during the game. The uncontested moves of Green can consist in all edges g_{i_q} such that Red played y_{i_q} during the racing phase, so Green altogether colors n edges before coloring any edge d_j. Because of Lemma 17, we can assume without loss of generality that Green colors any edge d_j only after coloring all possible edges g_i and y_i. In sum, this makes $2n+1$ moves for both players before Green colors any edge d_j. However, it is easy to check that after move $2n+1$, that is, when it would be again Green's turn to play, the sufficient conditions for Green to be unable to color any edge d_j after coloring all possible edges of type g_i stated in Lemma 18 do hold, and so Red wins, as described in Lemma 16.

It remains to show that Red can also win if Green does not play legitimately. Suppose Green makes any illegitimate move at some point, when all previous play was legitimate or at least winner-legitimate. We show that, whatever this move is, Red has a response such that the game continues with no disadvantage to Red but with a possible disadvantage for Green. In the following, we examine all possible illegitimate moves by Green. In light of Lemma 17, we can always assume without loss of generality that coloring any edge d_j is the last move player Green makes, which is in accordance to legitimate play, so the only illegitimate move that Green is free to make is to color some edge g_{i_q} during the racing phase when a legitimate move for him would be to color some edge y_{i_q} instead. Red responds by playing *as if Green just had chosen* y_{i_q}, and the game continues in winner-legitimate way as if no illegitimate move had been played. Red is none the worse off since the net result after the racing phase is that Green voluntarily colored at least one edge g_{i_q} more than necessary, thus making it only harder for Green to find an edge d_j that can be colored in the last move. Red's play is totally unaffected by Green's illegitimate play, and so again, after move $2n+1$ the sufficient conditions for Green to be unable to color any edge d_j after coloring all possible edges of type g_i stated in Lemma 18 do hold, and so Red wins, as described in Lemma 16.

Thus, no matter what illegitimate moves Green makes, Red can win. This completes the proof of the $(\Rightarrow)$ part.

$(\Leftarrow)$ Assume that player II has a winning strategy for $G_{\text{Achieve-POS-CNF}}(F)$. We first claim that Green has a strategy for $G_{\text{Avoid-Ramsey}}(G, A, E^r, E^g)$ that wins any game in which Red plays legitimately. The strategy again consists of playing legitimately and applying player II's winning strategy for $G_{\text{Achieve-POS-CNF}}(F)$ during the racing phase, via the correspondence between variables x_{i_q} and edges y_{i_q}. After the racing phase is over, that is in case of legitimate play, after n moves, player Red and Green again alternate in coloring edges of two disjoint sets of uncontested moves of each player. The uncontested moves of Red consist in edge r_0 and in all edges r_{i_q} such that Green played y_{i_q} during the racing phase, so Red again altogether colors $n+1$ edges during the game. The uncontested moves of Green consist in all edges g_{i_q} such that Red played y_{i_q} during the racing phase, so Green altogether colors n edges before he attempts to color some edge d_j. After move $2n+1$, that is, when it is again Green's turn to play, the necessary conditions for Green to be unable to color any edge d_j after coloring all possible edges of type g_i stated in Lemma 18 do not hold, therefore Green is able to color some d_j and so Green wins, there being no edge left to color for Red after move $2n+2$, as described in Lemma 16.

It remains to show that Green can also win if Red does not play legitimately. Suppose Red makes any illegitimate move at some point, when all previous play was legitimate or at least winner-legitimate. We show that, whatever this move is, Green has a response such that the game continues with no disadvantage to Green but also with no advantage for Red.

In the following, we examine all possible illegitimate moves by Red, assuming in each case that all previous play was legitimate or at least winner-legitimate.

Case 1: During the racing phase, Red plays some edge r_{i_q}, where $i_q \in \{1, \ldots, n\}$, when a legitimate move for him would be to color some edge y_{i_q} instead. In this case Green responds by playing *as if Red just had chosen* y_{i_q}, and the game continues in winner-legitimate way, Green's strategy staying just as if no illegitimate move had been played. Clearly, Green is none the worse off by Red's choice since Red's illegitimate play will not influence Green's capability to color some d_j as his last move. In summary, Green's play and strategy can remain totally unaffected by Red's illegitimate play. After move $2n + 1$, that is, when it is again Green's turn to play, the necessary conditions for Green to be unable to color any edge d_j after coloring all possible edges of type g_i stated in Lemma 18 do not hold, therefore Green is able to color some d_j and so Green wins, there being no edge left to color for Red after move $2n + 2$, as described in Lemma 16.

Case 2: During the racing phase, Red plays edge r_0, when a legitimate move for him would be to color some edge y_{i_q} instead. In this case, again, Green responds by playing *as if Red just had chosen* y_{i_q}, and the game continues in winner-legitimate way as if no illegitimate move had been played. There is a small technicality to be observed for Green, since the triple containing y_{i_q} is not really uncontested yet, but Green has to consider it as such, whereas Red does not care about it. At some later point of the play, Red will choose to color y_{i_q} or r_{i_q}. To see that this effectively will happen, remember that Red needs to color either edge r_i or y_i in every triple in any game, as remarked in Fact 5.4. After Red's coloring of y_{i_q} or r_{i_q}, Green has to differentiate between the following two subcases:

Subcase 2a: If Red's move ends the racing phase, Green continues as if the racing phase had ended already at Green's previous move and as if Red had only now chosen to color r_0 and previously had colored y_{i_q} when Red actually had colored r_0. Clearly, the game situations of the game really played so far and the game in which Red's moves in question would have been played the other way around are identical after Red's move, so the rest of the play can continue as if no exchange of Red's two moves had ever occurred. In summary, Green's play and strategy can remain totally unaffected by Red's illegitimate play. After move $2n + 1$, that is, when it is again Green's turn to play, the necessary conditions for Green to be unable to color any edge d_j after coloring all possible edges of type g_i stated in Lemma 18 do not hold, therefore Green is able to color some d_j and so Green wins, there being no edge left to color for Red after move $2n + 2$, as described in Lemma 16.

Subcase 2b: If Red's move does not end the racing phase, there must be at least one triple that is not uncontested left to color at Green's turn, with an edge $y_{i_{q'}}$. Green responds by playing *as if Red just had chosen* $y_{i_{q'}}$, and the game again continues in winner-legitimate way as if no illegitimate move had been played. That is, Green only replaces the now uncontested y_{i_q} by the not yet uncontested $y_{i_{q'}}$, and we are again in the situation of Case 2 from above, the only difference being that Green now imagines that Red played $y_{i_{q'}}$ instead of r_0, forgetting about any special treatment of y_{i_q}.

Thus, no matter what illegitimate moves Red makes, Green can win. This completes the proof of the $(\Leftarrow)$ part and also the proof of Theorem 6. ∎

5.2 Proof of Theorem 7

Membership is identical to its treatment in the proof of Theorem 6, again following from Lemma 14. Hardness follows from Definition 3.2 as well as Theorem 6 and the construction in its proof, which effectively makes sure that $G_{\text{Avoid'-Ramsey}}(G, A, E^r, E^g)$ never ends in a tie, by forcing $(G, E^r, E^g) \to A$. Indeed, as described in Observation 5.3, the construction features, among others, n triples r_i, y_i, g_i for $i = 1, \ldots, n$, such that coloring more than *one* edge in any triple would end a $G_{\text{Avoid'-Ramsey}}$ game for that player. Since each triple contains three edges but there are only two player, no $G_{\text{Avoid'-Ramsey}}$ game will ever end in a tie because all edges have been occupied. Therefore, Corollary 5 ensures that $G_{\text{Avoid'-Ramsey}}(G, A, E^r, E^g)$ will have the same winning strategy as $G_{\text{Avoid-Ramsey}}(G, A, E^r, E^g)$ for one of its players, and the proof of Theorem 6 carries over. ∎

5.3 Proof of Theorem 8

The membership of $G_{\text{Avoid-Ramsey}^+}$ in **PSPACE** follows again from Lemma 14.

For the hardness part, a careful analysis of the proof of Theorem 6 reveals that we can reuse the reduction of that proof to show the **PSPACE**-completeness of $G_{\text{Avoid-Ramsey}^+}$. Indeed, all arguments go through even when both players are allowed to color more than one edge per move. The difficulty here lies in the analysis of the cases when the opponent plays illegitimately, as described below:

Let us assume that player I has a winning strategy and player Red has so far played according to it, as explained in the proof of Theorem 6. In addition to everything which has already been said there, we notice that if all previous play was legitimate or at least winner-legitimate and player Green colors at least two edges in his current move, the best he can hope for is that he will have colored *one* edge d_j at the end of the game. At any rate, the number of edges Green has left to color altogether after his move will decrease by at least two. Red is none the worse off by Green's move and actually just needs to continue to choose *one* uncolored edge r_i after the other per move to win, since there is no urge now to force Green to color edges g_i. Conversely, let us assume player II has a winning strategy and player Green so far followed it. If player Red colors at least two edges in some move, the best he can hope for is that this will disable Green to color an edge d_j as his last move, so Green has only one edge less left to color during the rest of the game. However, the number of edges Red has left to color decreases by the number of edges he colored, so Green is none the worse off and still wins, now without having to worry to color one additional edge d_j at the end of the game. Thus, Green just needs to choose one uncolored edge g_i after the other per move to win, since he is the second player and he has now at least as many edges left to color as Red. ∎

5.4 Proof of Corollary 9

Follows directly from the bow-tie construction of A in the proof of Theorem 6. ∎

5.5 Proof of Theorem 10

We show that there is a **LOGSPACE** reduction from $G_{\text{Achieve-POS-DNF}}$ to the game $G_{\text{Achieve"-Ramsey}}$. By a result of T. Schaefer [59] showing that $G_{\text{Achieve-POS-DNF}}$ is **PSPACE**-complete and the obvious membership of the $G_{\text{Achieve"-Ramsey}}$ game in **PSPACE** (again following from Lemma 14), the result follows.

The definition of $G_{\text{Achieve-POS-DNF}}$ is restated here:

Definition 5.5 (T. Schaefer [59]) $G_{\text{Achieve-POS-DNF}}(F)$ *We are given a positive DNF formula F. A move consists of choosing some variable of F which has not yet been chosen. Player I starts the game. The game ends after all variables of F have been chosen. Player I wins iff F is* `true` *when all variables chosen by player I are set to* `true` *and all variables chosen by player II are set to* `false`. *In other words, player I wins iff he succeeds in playing all variables in at least one disjunct.*

Theorem 19 (T. Schaefer [59])
$G_{\text{Achieve-POS-DNF}}$ *is* **PSPACE**-*complete.*

We next describe the **LOGSPACE** reduction from $G_{\text{Achieve-POS-DNF}}(F)$ to $G_{\text{Achieve"-Ramsey}}(G, A, E^r, E^g)$. Figure 5 shows a small example. The exact definition follows: Let a positive DNF formula F be given. Assume without loss of generality that $F = D_1 \vee \ldots \vee D_q$ where each disjunct D_j is a conjunction of n_j positive literals, that is, $D_j = l_{j,1} \wedge \ldots \wedge l_{j,n_j}$ where $l_{j,k} \in \{x_1, \ldots, x_n\}$ and all n variables appear at least once in F. We then define the graphs $G \stackrel{\text{def}}{=} (V, E)$, $A \stackrel{\text{def}}{=} (V^A, E^A)$ and the edge-sets E^r, E^g, by

$$V \stackrel{\text{def}}{=} \bigcup_{0\le i\le n} X_i \,,$$

$$X_0 \stackrel{\text{def}}{=} \bigcup_{0\le j\le q} C_j \,,$$

$$C_0 \stackrel{\text{def}}{=} \bigcup_{1\le k\le p} \{r_{k,\mathrm{t}} \,,\, r_{k,\mathrm{b}}\} \,,$$

$$p \stackrel{\text{def}}{=} m - \min_{1\le j\le q} \{n_j\} \,,$$

$$m \stackrel{\text{def}}{=} \max_{1\le j\le q} \{n_j\} \,,$$

$$C_j \stackrel{\text{def}}{=} \{u_{j,0} \,,\, u_{j,1} \,,\, u_{j,2} \,,\, u_{j,3}\} \cup \bigcup_{1\le i< m} \{v_{i,j}\} \quad \text{for} \quad 1 \le j \le q \,,$$

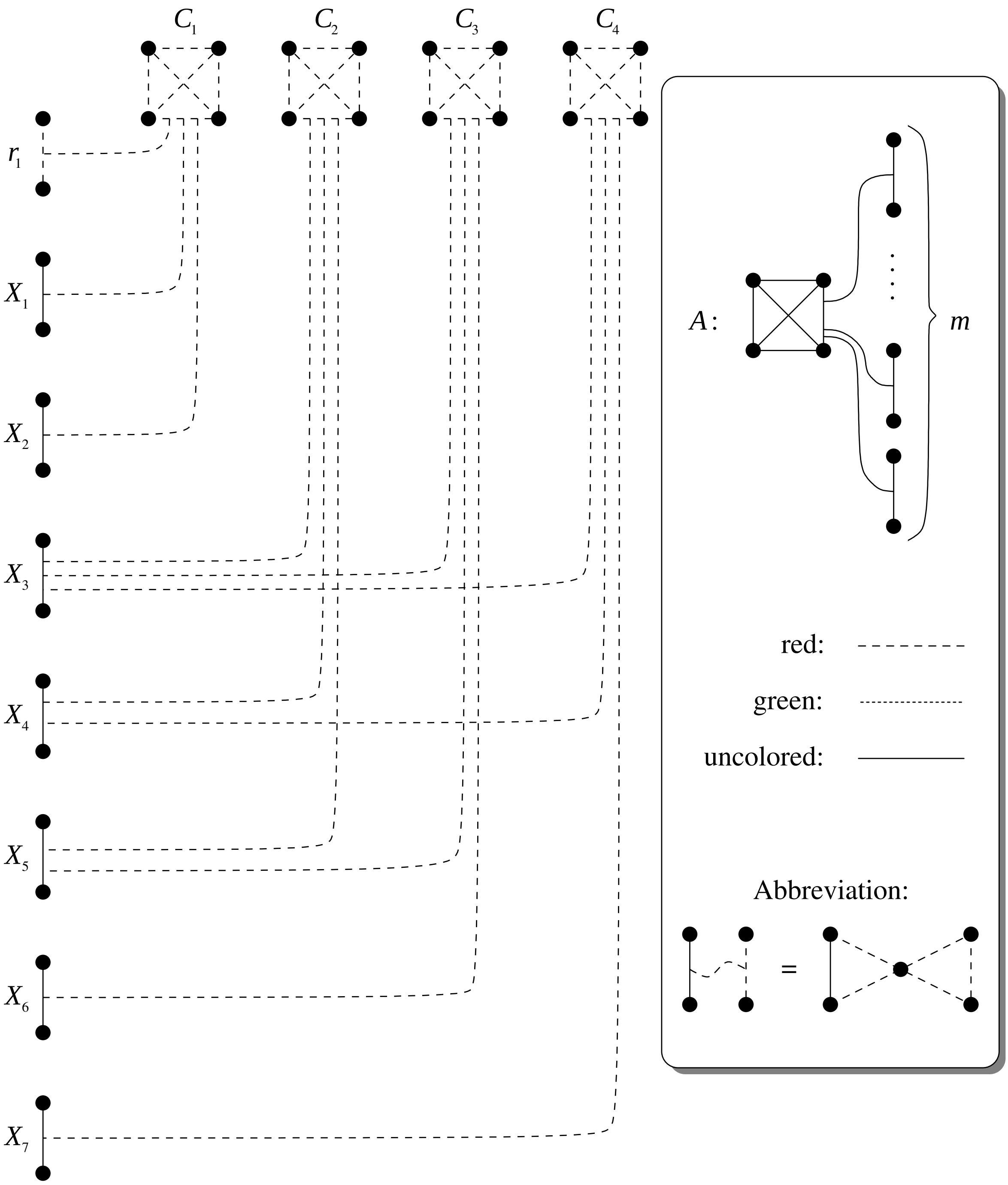

Figure 5: Instance of the $G_{\text{Achieve"-Ramsey}}$ game corresponding to the $G_{\text{Achieve-POS-DNF}}$ input formula $F = (x_1 \wedge x_2) \vee (x_3 \wedge x_4 \wedge x_5) \vee (x_3 \wedge x_5 \wedge x_6) \vee (x_3 \wedge x_4 \wedge x_7)$, which features a winning strategy for player II. The number m is the size of the largest disjunct in F.

$$
\begin{aligned}
X_i &\stackrel{\text{def}}{=} \{x_{i,\mathrm{t}}\,,\, x_{i,\mathrm{b}}\} \quad \text{for} \quad 1 \le i \le n\,,\\
E &\stackrel{\text{def}}{=} \bigcup_{0\le i\le n} P_i\,,\\
P_0 &\stackrel{\text{def}}{=} \bigcup_{0\le j\le q} D_j\,,\\
D_0 &\stackrel{\text{def}}{=} \bigcup_{1\le k\le p} \Big\{\{r_{k,\mathrm{t}}\,,\, r_{k,\mathrm{b}}\}\Big\}\,,\\
D_j &\stackrel{\text{def}}{=} \Big\{\{u_{j,0}\,,\, u_{j,1}\}\,,\, \{u_{j,0}\,,\, u_{j,2}\}\,,\, \{u_{j,0}\,,\, u_{j,3}\}\,,\\
&\quad \{u_{j,1}\,,\, u_{j,2}\}\,,\, \{u_{j,1}\,,\, u_{j,3}\}\,,\, \{u_{j,2}\,,\, u_{j,3}\}\Big\} \cup\\
&\quad \bigcup_{1\le k\le n_j} \Big\{\{u_{j,0}\,,\, v_{i,j}\}\,,\, \{u_{j,1}\,,\, v_{i,j}\}\,,\\
&\qquad \{v_{i,j}\,,\, x_{i,\mathrm{t}}\}\,,\, \{v_{i,j}\,,\, x_{i,\mathrm{b}}\} \mid l_{j,k} = x_i\Big\} \cup\\
&\quad \bigcup_{1\le k\le m-n_j} \Big\{\{u_{j,0}\,,\, v_{n_j+k,j}\}\,,\, \{u_{j,1}\,,\, v_{n_j+k,j}\}\,,\\
&\qquad \{v_{n_j+k,j}\,,\, r_{k,\mathrm{t}}\}\,,\, \{v_{n_j+k,j}\,,\, r_{k,\mathrm{b}}\}\Big\} \qquad \text{for} \quad 1 \le j \le q\,,\\
P_i &\stackrel{\text{def}}{=} \Big\{\{x_{i,\mathrm{t}}\,,\, x_{i,\mathrm{b}}\}\Big\} \qquad \text{for} \quad 1 \le i \le n\,,\\
V^A &\stackrel{\text{def}}{=} \{a_0\,,\, a_1\,,\, a_2\,,\, a_3\} \cup \bigcup_{1\le i<m} \{b_{i,0}\,,\, b_{i,1}\,,\, b_{i,2}\}\,,\\
E^A &\stackrel{\text{def}}{=} \Big\{\{a_0\,,\, a_1\}\,,\, \{a_0\,,\, a_2\}\,,\, \{a_0\,,\, a_3\}\,,\, \{a_1\,,\, a_2\}\,,\, \{a_1\,,\, a_3\}\,,\, \{a_2\,,\, a_3\}\Big\} \cup\\
&\quad \bigcup_{1\le i\le m} \Big\{\{a_0\,,\, b_{i,0}\}\,,\, \{a_1\,,\, b_{i,0}\}\,,\, \{b_{i,0}\,,\, b_{i,1}\}\,,\, \{b_{i,0}\,,\, b_{i,2}\}\,,\, \{b_{i,1}\,,\, b_{i,2}\}\Big\}\,,\\
E^r &\stackrel{\text{def}}{=} P_0\,,\\
E^g &\stackrel{\text{def}}{=} \{\}\,.
\end{aligned}
$$

It immediately follows from the construction that there is a simple **LOGSPACE** transducer that computes (G, A, E^r, E^g) from input F.

Since the number of variables that can be chosen in $G_{\text{Achieve-POS-DNF}}$ is equal to the number of edges that can be colored in $G_{\text{Achieve''-Ramsey}}$, it is easy to see that there is a one-to-one correspondence between variables and edges. A winning strategy from $G_{\text{Achieve-POS-DNF}}$ is directly translated into a winning strategy for the $G_{\text{Achieve''-Ramsey}}$ game by coloring edge X_i whenever variable x_i of F needs to be chosen, and vice versa.

The graph A that player Red has to complete in his color looks like an 'm-legged octopus', we therefore call it an 'm-topus' in the following. Each disjunct D_j of the positive DNF formula F is mirrored by an m-topus already partly precolored in red such that only the 'feet-edges' of the m-topus that correspond to the variables occurring in the disjunct are still uncolored.

In the following, we show that above construction ensures that there is a winning strategy for player I of $G_{\text{Achieve-POS-DNF}}(F)$ iff there is a winning strategy for player Red of $G_{\text{Achieve''-Ramsey}}(G, A, E^r, E^g)$. The strategy consists in copying the winning strategy for $G_{\text{Achieve-POS-DNF}}(F)$ via the correspondence between variables x_i and edges X_i. It is easy to see that player I can play all variables in at least one disjunct iff player Red can color the feet-edges of at least one m-topus. Since player I wins the game $G_{\text{Achieve-POS-DNF}}(F)$ iff he can play all variables in at least one disjunct, and since player Red wins the game $G_{\text{Achieve''-Ramsey}}(G, A, E^r, E^g)$ iff he completes to color the feet-edges of at least one m-topus, thereby building a red subgraph isomorphic to A, this concludes the proof. ∎

5.6 Proof of Theorem 11

Since the construction used in the proof of Theorem 10 leaves no possibility open for Green to construct a green subgraph isomorphic to A, we can reinterpret the whole proof according to the rules of $G_{\text{Achieve'-Ramsey}}$. It is easy to see that $G_{\text{Achieve''-Ramsey}}$ and $G_{\text{Achieve'-Ramsey}}$ are in fact the same game when the second player cannot build a green subgraph isomorphic to A, and so the **PSPACE**-completeness proof remains true if we replace every occurrence of $G_{\text{Achieve''-Ramsey}}$ by an occurrence of $G_{\text{Achieve'-Ramsey}}$. Therefore, the **PSPACE**-completeness of $G_{\text{Achieve''-Ramsey}}$ directly carries over to the $G_{\text{Achieve'-Ramsey}}$ game. ∎

5.7 Proof of Corollary 12

Follows directly from the m-topus construction of A in the proof of Theorem 10 and the restriction to DNF formulas having at most 11 variables in each disjunct in the **PSPACE**-completeness proof of $G_{\text{Achieve-POS-DNF}}$ (T. Schaefer [59, Theorem 3.6 and Corollary 3.7]), which limits the maximum degree of the vertices in the achievement graph A to 14. ∎

5.8 Proof of Theorem 13

Membership in **PSPACE** again easily follows again from Lemma 14. To show hardness, we will adapt the reduction from the proof of Theorem 10 for the present proof. Indeed, all we have to make sure is that player Red has a winning strategy in $G_{\text{Achieve''-Ramsey}}$ iff Red also wins the corresponding $G_{\text{Achieve-Ramsey}}$ game. The modifications we describe below ensure that, on the one hand, if player Red has a winning strategy in $G_{\text{Achieve''-Ramsey}}$ and thus can construct a red subgraph isomorphic to graph A, this winning strategy will carry over to $G_{\text{Achieve-Ramsey}}$ without change. On the other hand, if player Green can prevent Red from constructing such a red subgraph, player Green has a winning strategy in $G_{\text{Achieve''-Ramsey}}$ but not (yet) in $G_{\text{Achieve-Ramsey}}$, so for the (modified) latter game we need to add a gadget that makes sure Green can build a green subgraph isomorphic to A, of course only in case Red cannot build one earlier in red. This would make sure that Green would be given a winning strategy in $G_{\text{Achieve-Ramsey}}$ in case Green had a winning strategy in $G_{\text{Achieve''-Ramsey}}$.

Let us first define the modified reduction in detail. Almost everything is defined as in the proof of Theorem 10 besides the redefinition of m, of G, and of E^g. The changes add the new

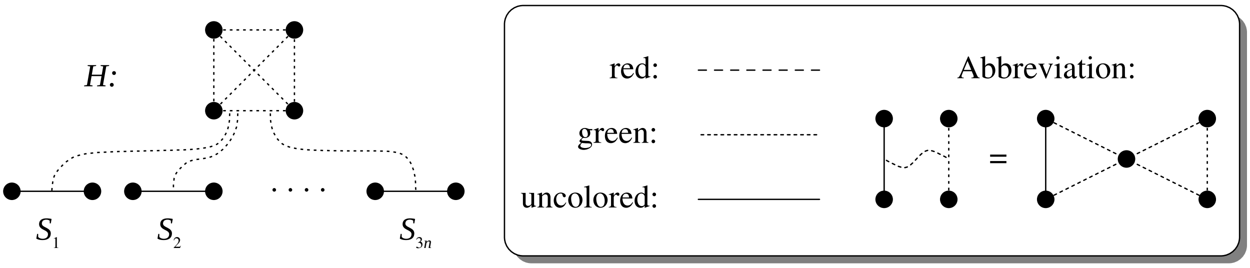

Figure 6: $3n$-topus gadget that must be added to the construction in Figure 5 besides the redefinition of m to the number of variables in order to transform the reduction used in the proof of Theorem 10 to one that allows to prove Theorem 13. Conventions are similar to those in Figure 4.

graph $H \stackrel{\text{def}}{=} (V^H, E^H)$, namely a $3n$-topus partly precolored in green with all $3n$ feet-edges still uncolored, as depicted in Figure 6, to the initial game situation, increase the number of 'legs' in A from the size of the largest clause to the number of variables n, and add legs already precolored completely in red to each gadget C_i corresponding to disjunct D_i such that in case Red had a winning strategy in $G_{\text{Achieve"-Ramsey}}$, it can be reused without change in the new $G_{\text{Achieve-Ramsey}}$ game. Below we only state the formulas that were changed with respect to the formulas in the proof of Theorem 10:

$$
\begin{aligned}
m &\stackrel{\text{redef}}{=} n\,, \\
G &\stackrel{\text{redef}}{=} (V \cup V^H\,,\, E \cup E^H)\,, \\
V^H &\stackrel{\text{def}}{=} \{h_0\,,\, h_1\,,\, h_2\,,\, h_3\} \cup \bigcup_{1 \le i < 3n} \{s_{i,0}\,,\, s_{i,1}\,,\, s_{i,2}\}\,, \\
E^H &\stackrel{\text{def}}{=} \Big\{\{h_0\,,\, h_1\}\,,\, \{h_0\,,\, h_2\}\,,\, \{h_0\,,\, h_3\}\,,\, \{h_1\,,\, h_2\}\,,\, \{h_1\,,\, h_3\}\,,\, \{h_2\,,\, h_3\}\Big\} \cup \\
&\qquad \bigcup_{1 \le i \le 3n} \Big\{\{h_0\,,\, s_{i,0}\}\,,\, \{h_1\,,\, s_{i,0}\}\,,\, \{s_{i,0}\,,\, s_{i,1}\}\,,\, \{s_{i,0}\,,\, s_{i,2}\}\,,\, \{s_{i,1}\,,\, s_{i,2}\}\Big\}\,, \\
E^g &\stackrel{\text{redef}}{=} \Big\{\{h_0\,,\, h_1\}\,,\, \{h_0\,,\, h_2\}\,,\, \{h_0\,,\, h_3\}\,,\, \{h_1\,,\, h_2\}\,,\, \{h_1\,,\, h_3\}\,,\, \{h_2\,,\, h_3\}\Big\} \cup \\
&\qquad \bigcup_{1 \le i \le 3n} \Big\{\{h_0\,,\, s_{i,0}\}\,,\, \{h_1\,,\, s_{i,0}\}\,,\, \{s_{i,0}\,,\, s_{i,1}\}\,,\, \{s_{i,0}\,,\, s_{i,2}\}\Big\}\,.
\end{aligned}
$$

Similar to the proof of Theorem 6, we will use S_i as a shortcut for the edge $\{s_{i,1}\,,\, s_{i,2}\}$. The new (G, A, E^r, E^g) is constructed as in the proof of Theorem 6 in such a way as to constrain the moves of the players in spite of their apparent freedom to choose an uncolored edge, by punishing moves that are *illegitimate*. Again, an illegitimate move may put the player who selects such a move into a less favorable position for the rest of the play and is defined via the complementary notion of *legitimate* play:

Definition 5.6 Legitimate play: *We call a $G_{\text{Achieve-Ramsey}}(G, A, E^r, E^g)$ game sequence* legitimate *iff it has the following form: For moves $q = 1, 2, \ldots, n$ both players choose so far uncolored edges X_{i_q}, where $i_q \in \{1, \ldots, n\}$. In case Red builds a red subgraph isomorphic to A during the first n moves, the game ends, in accordance with the rules of the $G_{\text{Achieve-Ramsey}}$ game. In case Red does not build a red subgraph isomorphic to A during the first n moves, the game continues for at most $2n$ more moves during which both players take turns to color edges S_i in the added $3n$-topus H.*

The following weaker variant of legitimate play corresponds to its definition in the proof of Theorem 6:

Definition 5.7 Winner-legitimate play: *We call a game sequence* winner-legitimate *if the player with the winning strategy always has chosen legitimate moves.*

As in the proof of Theorem 6, the following notion will be of help:

Definition 5.8 The racing phase: *We call the first n moves of a game sequence the* racing phase *of the $G_{\text{Achieve-Ramsey}}$ game, because during that phase the two players race to occupy the 'right' edges X_i that ultimately will lead to the victory of one of them.*

Let us observe that the moves of the racing phase of a legitimate game sequence played on $G_{\text{Achieve-Ramsey}}(G, A, E^r, E^g)$ mimic those of $G_{\text{Achieve-POS-DNF}}(F)$ in an obvious way: Player Red on move q chooses edge X_{i_q} where player I chooses variable x_{i_q}, and similar for player Green and player II.

We will show that this modified reduction will make sure that there is a winning strategy for player I of $G_{\text{Achieve-POS-DNF}}(F)$ iff there is a winning strategy for player Red of $G_{\text{Achieve-Ramsey}}(G, A, E^r, E^g)$:

($\Rightarrow$) Assume that player I has a winning strategy for $G_{\text{Achieve-POS-DNF}}(F)$. We first claim that Red has a strategy for $G_{\text{Achieve-Ramsey}}(G, A, E^r, E^g)$ that wins any game in which Green plays legitimately. The strategy consists of playing legitimately and applying player I's winning strategy for $G_{\text{Achieve-POS-DNF}}(F)$, via the correspondence between variables x_{i_q} and edges X_{i_q}. Player Red will be able to complete a red subgraph isomorphic to A and thus win, the argument being the same as in the proof of Theorem 10.

It remains to show that Red can also win if Green does not play legitimately. Suppose Green makes any illegitimate move at some point, when all previous play was legitimate or at least winner-legitimate. We show that, whatever this move is, Red has a response such that the game continues with no disadvantage to Red but also with no advantage for Green. The only illegitimate move that Green is free to make is to color some edge S_{i_q} during the racing phase when a legitimate move for him would be to color some edge X_{i_q} instead. Red responds by playing *as if Green just had chosen X_{i_q}*, and the game continues in winner-legitimate way as if no illegitimate move had been played. If Green later during the game, say at move q', chooses to actually color this edge X_{i_q}, Red continues as if Green had colored X_{i_q} already at move q and another uncolored edge $X_{i_{q'}}$ at the present move. Red is none the worse off since the net result after the racing phase is that Green voluntarily renounced to color some edges X_i, thus making it even easier for Red to complete a red subgraph isomorphic to A. Since Red needed at most $\lceil n/2 \rceil$ moves to construct a red subgraph isomorphic to A, Green cannot yet have finished to color the necessary n feet-edges of H that are needed to construct a green subgraph isomorphic to A. Altogether, Red's play is totally unaffected by Green's illegitimate play, and so again, after at most n steps, Red will be able to complete a red subgraph isomorphic to A and thus Red again will be able to win the game.

Thus, no matter what illegitimate moves Green makes, Red can win. This completes the proof of the $(\Rightarrow)$ part.

$(\Leftarrow)$ Assume that player II has a winning strategy for $G_{\text{Achieve-POS-DNF}}(F)$. We first claim that Green has a strategy for $G_{\text{Achieve-Ramsey}}(G, A, E^r, E^g)$ that wins any game in which Red plays legitimately. The strategy again consists of playing legitimately and applying player II's winning strategy for $G_{\text{Achieve-POS-DNF}}(F)$ during the racing phase, via the correspondence between variables x_{i_q} and edges X_{i_q}. After the racing phase is over, that is in case of legitimate play, after n moves, Green will have hindered Red from constructing a red subgraph isomorphic to A, the argument being the same as in the proof of Theorem 10. It is easy to see that from then on, Red cannot hinder Green from building a green subgraph isomorphic to A by coloring n feet-edges of the $3n$-topus H, and so Green will win the game at move $3n$.

It remains to show that Green can also win if Red does not play legitimately. Suppose Red makes any illegitimate move at some point, when all previous play was legitimate or at least winner-legitimate. We show that, whatever this move is, Green has a response such that the game continues with no disadvantage to Green but also with no advantage for Red. The only illegitimate move that Red is free to make is to color some edge S_{i_q} during the racing phase when a legitimate move for him would be to color some edge X_{i_q} instead. Green responds by playing *as if Red just had chosen* X_{i_q}, and the game continues in winner-legitimate way as if no illegitimate move had been played. If Red later during the game, say at move q', chooses to actually color this edge X_{i_q}, Green continues as if Red had colored X_{i_q} already at move q and another uncolored edge $X_{i_{q'}}$ at the present move. Green is none the worse off since the net result after the racing phase is that Red voluntarily renounced to color some edges X_i, thus making it even easier for Green to hinder Red from completing a red subgraph isomorphic to A. Since only n moves have been played so far, Red will have colored at most $\lceil n/2 \rceil$ edges S_i in the $3n$-topus H, and so there are still more than $3n - \lceil n/2 \rceil \geq 2n$ uncolored feet-edges of H that Green can color to complete a green subgraph isomorphic to A. Again, it is easy to see that Green will win the game at move $3n$.

Thus, no matter what illegitimate moves Red makes, Green can win. This completes the proof of the $(\Leftarrow)$ part and also the proof of Theorem 13. ∎

6 Further complexity results for graph Ramsey games

We next study the complexities of two natural generalizations of $G_{\text{Avoid-Ramsey}}$.

The first game, $G_{\text{Avoid-Ramsey}^n}$, is based on generalizing Definitions 2.1 and 2.2 to more arguments. Indeed, classic Ramsey numbers for more than two colors such as $\text{Ramsey}(3, 3, 3) = 17$

(Greenwood & Gleason [32]) also lead to interesting combinatorial multi-player games. Note that $\mathrm{Ramsey}(3,3,3) = 17$ is the only nontrivial case for which, to the best of our knowledge, the answer is known so far for n-ary Ramsey numbers with $n > 2$ [53].

Definition 6.1 $G_{\text{Avoid-Ramsey}^n}(G, A, E^1, \ldots, E^n)$: *We are given a graph $G = (V, E)$, another graph A, and $n \geq 2$ non-intersecting sets $\cup_{i=1}^{n} E^i \subseteq E$ that contain edges initially colored in colors c_1 to c_n, respectively. One after the other, the n players select at each move one so-far uncolored edge from E and color it in their respective color. However, all players are forbidden to choose an edge such that A becomes isomorphic to a monochromatic subgraph of G. Player one starts. The first player unable to move, say player l, loses and withdraws from the rest of the game. The winner of $G_{\text{Avoid-Ramsey}^n}(G, A, E^1, \ldots, E^n)$ is recursively defined as follows:*

Case $n = 2$: *the winner is the sole player that is still in the game.*

Case $n > 2$: *the winner is the winner of the game $G_{\text{Avoid-Ramsey}^{n-1}}(G, A, \underbrace{\{\}, \ldots, \{\}}_{n-1})$, where the new player i is the old player $(l + i - 1 \bmod n) + 1$.*

Corollary 20 $G_{\text{Avoid-Ramsey}^n}$ *is* **PSPACE**-*complete.*

Proof. Easily follows from Theorem 6 and the observation that the recursion in Definition 6.1 ends after $n - 1$ steps. Membership in **PSPACE** follows again from Lemma 14. ∎

The second game, $G_{\text{Asymmetric-Avoid-Ramsey}}$, is based on the unchanged Definitions 2.1 and 2.2 but considers different avoidance graphs for the two players. Again, only few results are known, for instance $\mathrm{Ramsey}(4,5) = 25$ proved recently by McKay & Radziszowski [43] using a massive amount of computing power, reportedly a cluster of workstations running for more than 10 cpu-years (consult [53] for a survey on known small Ramsey numbers).

Definition 6.2 $G_{\text{Asymmetric-Avoid-Ramsey}}(G, A^r, A^g, E^r, E^g)$: *We are given a graph $G = (V, E)$, two more graphs A^r and A^g, and two non-intersecting sets $E^r \cup E^g \subseteq E$ that contain edges initially colored in red and green, respectively. The two players, Red and Green, select at each move one so-far uncolored edge from E and color it in red for player Red respectively in green for player Green. However, player Red is forbidden to choose an edge such that A^r becomes isomorphic to a red subgraph of G, and player Green is forbidden to choose an edge such that A^g becomes isomorphic to a green subgraph of G. It is Red's turn. The first player unable to move loses.*

Corollary 21 $G_{\text{Asymmetric-Avoid-Ramsey}}$ *is* **PSPACE**-*complete.*

Proof. Follows directly from Theorem 6. ∎

Of course, appropriate combinations of $G_{\text{Avoid-Ramsey}^n}$ and $G_{\text{Asymmetric-Avoid-Ramsey}}$ as well as similar variants of avoidance$^+$ and achievement games are conceivable and are also **PSPACE**-complete, following from Corollaries 20 and 21.

7 Solving concrete instances

In Section 5, we proved that deciding graph Ramsey games is **PSPACE**-complete. Besides determining the theoretical computational resources needed to solve a game's decision problem asymptotically, it is possible to differentiate the following four levels of how to solve a particular instance of a combinatorial game in practice:

Definition 7.1 Levels of solving a combinatorial game in practice [1, 60]:

Ultra-weakly solved: *The game-theoretic value for the initial position has been determined.*

Weakly solved: *The game is ultra-weakly solved and a strategy exists for achieving the game-theoretic value from the opening position, assuming reasonable computing resources.*

Strongly solved: *For all possible positions, a strategy is known for determining the game-theoretic value for both players, assuming reasonable computing resources.*

Ultra-strongly solved: *For all positions in a strongly solved game, a strategy is known that improves the chances of achieving more than the game-theoretic value against a fallible opponent.*

Let us look at what can be achieved in the case of concrete instances of $G_{\text{Avoid-Ramsey}}$ and $G_{\text{Avoid-Ramsey}^+}$, starting with Sim, that is, $G_{\text{Avoid-Ramsey}}(K_{\text{Ramsey}(3,3)}, K_3, \{\}, \{\})$.

7.1 Strongly solving Sim

In accordance with Definition 3.1, we assume that player Red always makes the first move, even if this is not necessarily so in real games, in particular in our implementation of Sim, where player Red is always the human player, independently of who starts. As mentioned in Section 3, in Sim the longest game sequence lasts 15 moves. If none of the two players builds a monochromatic triangle in the first 14 steps, it is the first player, Red, whose turn it would be to make the 15^{th}, final move. That move, however, for certain is fatal, for there must be a monochromatic triangle according to the party-puzzle result from Ramsey theory mentioned in Section 3 (see Figure 2). Intuitively, if both players can delay building a triangle up to the 15^{th} move, Red will loose, so Green might have a winning strategy in Sim.

Strongly solving a game requires that the complete game tree is known. For most games this tree is far too large to be generated and evaluated backwards from the terminal game positions to determine this strategy. Is it possible to generate the whole game tree for Sim?

In order to approximate the size of Sim's game tree, let us assume for the moment that all play sequences will last 15 steps, thus including successors of terminal positions that actually are impossible in real games. Proceeding with this assumption, we can construct a game tree with 15 alternatives on the first level, 15×14 on the second, $15 \times 14 \times 13$ on the third and so on. That makes altogether $15!$, around 1.3×10^{12}, leaves in the last level. We would need memory in the order of 150 Gigabytes for a search tree of that size even if we could manage to use only one bit for each position. While that seems feasible, we wondered whether we could do with less.

Observation 7.1 *By looking at an arbitrary game sequence, we immediately notice that at his starting move the first player actually has, instead of 15 possibilities, only one possibility to choose from, because all game positions are isomorphic at this level modulo a permutation of the six vertices. Similarly, the second player does not really choose between 14 edges, but instead can either select an edge with a vertex in common to the edge colored by the first player, or unconnected to that edge, summing up to two possibilities.*

It is easy to see that this isomorphism between game positions dramatically reduces the size of the directed acyclic graph corresponding to a compressed version of the fully expanded game tree. We found that the number of non-isomorphic legal game positions, including positions containing monochromatic triangles, was thus reduced to a mere 3728. Actually, it is not even necessary to save the positions that contain monochromatic triangles, since they can be easily detected during play. Thus, the number can be further reduced to 2309 positions. We defined a normalization based on choosing the smallest member from the set of isomorphic positions in a convenient lexicographic ordering as the representative of the isomorphism class to convert the game positions in the numerical range [0, $3^{15} - 1$] to these 2309 positions. In our implementation, we stored these positions in a hash table with $2^{12} = 4096$ entries that can be downloaded as a whole by the Java applet mentioned in footnote 1 so that no further contact with the server-side nor any lengthy computations are necessary during play.

A position in the game tree is labeled R-WIN if the first player, Red, has a winning strategy from that position, and R-LOSS otherwise. Terminal positions of the implemented misère variant of Sim are labeled R-WIN if Green closed a triangle, and R-LOSS otherwise. The positions that do not contain triangles are recursively labeled from bottom-up as follows. If it is Red's turn, and at least one of the positions following the current one in the game tree is labeled R-WIN, then the current position is also labeled R-WIN, since selecting that move would lead to Red's victory. If, however, all of them are R-LOSS, then Green will be able to beat Red whatever move Red chooses, so the current position must be labeled R-LOSS. If it is Green's turn, at least one move leading to an R-LOSS position is enough to label the current one R-LOSS, and only if all possibilities are R-WIN must it be labeled R-WIN as well. Of course isomorphic game positions have to be labeled only once, so it is enough to label the 2309 normalized positions in the directed acyclic graph. Our results coincide with those of Mead *et al.* [44]:

Theorem 22 *The second player, Green, has a winning strategy in Sim =* $G_{\text{Avoid-Ramsey}}(K_{\mathrm{Ramsey}(3,3)}, K_3, \{\}, \{\})$.

The proof of this statement being more or less a very long enumeration of cases for the game $G_{\text{Avoid'-Ramsey}}(K_{\mathrm{Ramsey}(3,3)}, K_3, \{\}, \{\})$, we do not include it here. It can easily be reconstructed from the data available together with the Java-applet on the author's home-page. Because of Corollary 5, $G_{\text{Avoid'-Ramsey}}(K_{\mathrm{Ramsey}(3,3)}, K_3, \{\}, \{\})$ is equivalent to $G_{\text{Avoid-Ramsey}}(K_{\mathrm{Ramsey}(3,3)}, K_3, \{\}, \{\})$, so the statement is true.

Accordingly, the second player has a winning strategy which we implemented in our Java applet.

Theorem 22 established that we *ultra-weakly solved* Sim in the sense of Definition 7.1. However, since we are also able to quickly generate the game-theoretic value of all possible game positions from our data, we effectively both *weakly* and *strongly solved* the Sim game.

An interesting problem that remained was to find a strategy when our program plays as the first player, i.e., to additionally *ultra-strongly solve* the game. Our solution is presented in the next section.

7.2 Ultra-strongly solving Sim

We have already compressed the game tree and saved it in a hash table. If the second player is perfect, the first player has no chance to win. However, it may be extremely difficult for a human second player to choose a perfect move at every single step, unless one memorizes the directed acyclic graph with the 3728 non-isomorphic game positions and additionally is able to identify the current position with the corresponding normalized one. As soon as the human player by mistake chooses a non-perfect move, the program is able to follow the information in the game tree and from that point on has a winning strategy. Thus, what the program playing as the first player could do is to maximize the probability that the second player makes a mistake.

At the beginning, all moves lead to R-LOSS successors. However, some of these successors may themselves have R-WIN successors that could be chosen by Green carelessly in the next move. Red can look one step ahead and choose a move that leads to a position having more R-WIN successors than the others. Then, for a human playing as Green, it might be more difficult not to choose an R-WIN successor at the next step, thus leading to the program's victory.

Let us assume Red is using this method to make his choice. He definitely has a higher chance to win than choosing blindly by just avoiding triangles, under the condition that Green is not perfect and sometimes chooses his moves randomly. Is there a way to make it even easier for Green to make a mistake? One way could be to take into account small perceptive preferences of the human higher visual system. Empirically, the six edges in the square at the center of the hexagon seem to be slightly more eye-catching than the other four side-edges, and these seem easier to perceive than the remaining five edges. Apart from the quantitative value (the number of R-WIN successors), each move thus has its qualitative value (whether the R-WIN successor is easily reached), the latter depending on the current game layout. Red has a good chance of winning against a human playing as Green by selecting with higher probability a move that scores higher in this combined heuristic function. Next, we explain how this function is further refined through learning to take counter-measures against the learning capabilities of human players.

Learning

We now have a heuristic strategy for Red which works as follows: The program will choose its best move by considering both the number of R-WIN positions following this move and whether such R-WIN positions are likely to be chosen by Green. This approach sounds reasonable, but once a human player has found a sequence of moves leading to his victory, he can always replay this sequence and win again, assuming that he can restart the game in case the program randomly chooses a move that does not follow his sequence. As a counter-strategy, the first player was made able to learn from experience.

Whenever the program played as the first player, Red, it analyzes its moves at the end of the game. If it won by forcing the human to give up, that is, the human did not lose by closing a triangle merely by mistake, without being forced to do so, the program's moves were apparently well chosen and the sequence should be memorized and chosen again with increased likelihood. With this knowledge, the program will tend to behave identically if the same or an isomorphic situation is encountered in the future. On the other hand, if the program has lost the game, it will know that its moves may not have been good enough ones, and thus will decrease the probability of selecting such moves or isomorphic ones in later plays.

All the R-WIN/R-LOSS information for each move is stored in a position table. Each entry of the table is a number representing an R-WIN or R-LOSS. All R-WIN entries have the value 0 because no additional information is necessary: Once reached, Red's strategy is to always select an R-WIN move. The value of an R-LOSS entry changes according to its learned desirability for Red from -128 to 127, the range of a byte in Java. Each entry is initially set to 1 to distinguish it from R-WIN entries. This value will be modified according to the analysis made after each play. If the program has won, the value of the corresponding R-LOSS entries will be increased by a learning factor, and decreased if it lost, with a maximum and minimum value. Additionally, value 0 is always avoided when learning as it is reserved for R-WIN. The values held by R-LOSS entries are used during heuristic play to probabilistically guide the selection of the moves by the computer, together with the static heuristic function already described. At the very beginning, no experience is available, so the selection of moves is only based on the static heuristic function. As more and more experience accumulates, the learned information becomes the dominant factor in the selection of moves. Since changes are memorized in the compressed position table, learning is done very efficiently for a multitude of isomorphic game sequences.

Because the program is available in form of a Java applet, it can easily recontact the Internet server from whom it was originally downloaded. Through a pre-specified Internet port, it hands back the acquired playing experience. On the server side, a Java daemon is running which registers all information communicated through Java applets on client sides. If no contact can be established because the Internet connection is down, the information is lost. However, through the same client-server connection, human players who have won are allowed to enter their nickname into a hall-of-fame stored at the server-side, which may be an incentive to allow the Internet connection to be reestablished at the end of the play (relevant for non-permanent Internet access only). To make it more interesting, this hall-of-fame is sorted according to the amount of time that was needed to win against the program. In order to avoid that a clever person places his or her nickname on top of the list by quickly letting two Java applets play against each other, entering one's nickname is only possible in the "Allow shaking" mode: In this mode, an animated random permutation of the six board vertices is performed after each move of the program, thus resulting in an isomorphic game situation which nevertheless looks quite different to the human eye and makes it more difficult to let two applets play against each other. Of course, these permutations have to be taken already into account when selecting the next move according to the static heuristic function described in the previous section.

The combination of perfect information with the static heuristic function alone makes the program already very strong. The learning additionally ensures that it will be unlikely that someone will win repeatedly against the program. According to our experience, it is now counter-intuitively hard to win against the program, and even harder to win again. We therefore assert:

Claim 7.1 *Following the described adaptive strategy together with our complete game tree data means that we have come as close as possible to* ultra-strongly solving *Sim in the sense of Definition 7.1.*

7.3 Ultra-strongly solving Sim$^+$

What happens if we allow that each player colors more than one edge during his turn to move? This variant which we called $G_{\text{Avoid-Ramsey}^+}$ intuitively corresponds even closer to the results from Ramsey theory than $G_{\text{Avoid-Ramsey}}$ does, since the relation between red and green edges can vary arbitrarily. Obviously, this game is more difficult to analyze as we are confronted with a larger game tree. For Sim$^+$, the first move no longer consists in choosing one edge from fifteen but an arbitrary selection out of the 15 possible edges.

In the original Sim game, it is clear at each step who should move next because each player colors only one edge at a time. Thus, one table is enough for both players since it is always clear from the number of edges which player's turn it is. In this new variant, however, it is impossible to tell which one should move next merely from the game position. We have to use two position tables to store the strategies of the first and second player. Only a few modifications were needed to extend Sim to allow more edges to be colored at each player's turn. The same methods were used to generate the whole game tree (the directed acyclic game graph has 13158 entries and thus is more than five times as large as in the standard variant Sim), classify the positions and then save them in two tables with different R-WIN/R-LOSS information. This game variant is available from the same Internet address by choosing option "Allow more moves each time" in the applet's control panel.

Theorem 23 *The second player, Green, has a winning strategy in Sim$^+$ = $G_{\text{Avoid-Ramsey}^+}(K_{\text{Ramsey}(3,3)}, K_3, \{\}, \{\})$.*

As with Theorem 22, the proof of this statement is more or less a very long enumeration of cases which we do not include here. It can easily be reconstructed from the data available together with the Java-applet on the author's home-page.

Accordingly, it is again the second player who has a winning strategy. In our Java applet, we implemented both this winning strategy and a heuristic learning counter-strategy almost identical to the one for Sim, thus allowing us again to assert the following:

Claim 7.2 *We have come as close as possible to* ultra-strongly solving *the game $G_{\text{Avoid-Ramsey}^+}(K_{\text{Ramsey}(3,3)}, K_3, \{\}, \{\})$.*

7.4 Sim$_4$ and above

Another challenge may be a similar game based on $\mathrm{Ramsey}(4,4)$. With this variant, we need $\mathrm{Ramsey}(4,4) = 18$ (Greenwood & Gleason [32]) vertices with $\binom{18}{2} = 153$ edges between them to play a game analogous to Sim. Since the structure to avoid would be K_4, i.e., a tetrahedron (a pyramid with four triangular faces), one could play this game in simulated three-dimensional space. The crucial question is again the number of possible non-isomorphic game positions. The exact number can be bounded from above by counting the number of non-isomorphic game positions including non-legal positions containing several monochromatic tetrahedra. This task can be solved by straightforward application of Pólya's Theorem [52] (or 'Pólya's enumeration formula') on counting orbits under group actions using Harary's cycle index for the group S_{18}^2 of edge permutations of K_{18} [36] which enables us to count colorings which are distinct with respect to the action of above permutation group (for a survey see Harary & Palmer [35], for a gentle introduction Tucker [68, Chap. 9]; the following special case is proved and explained in detail in Gessel & Stanley [30]):

$$\begin{aligned} Z(S_n^2) \;=\; & \sum_{\substack{(m_1,m_2,\ldots,m_n)\in[1,n]^n \\ \sum_{i=1}^n i m_i = n}} \frac{1}{\prod_{k=1}^n (k^{m_k} m_k!)} \prod_{k=1}^{\lfloor n/2\rfloor} (p_k p_{2k}^{k-1})^{m_{2k}} \times \\ & \prod_{k=1}^{\lfloor (n-1)/2\rfloor} p_{2k+1}^{k m_{2k+1}} \prod_{k=1}^{n} p_k^{k\binom{m_k}{2}} \prod_{\substack{(i,j)\in[1,n]^2 \\ i<j}} p_{\mathrm{lcm}(i,j)}^{\gcd(i,j)\, m_i m_j} \end{aligned}$$

where lcm and gcd denote the least common multiple and greatest common divisor. The number of non-isomorphic edge-red-green-colorings of K_{18} with r red and g green edges is the coefficient of $x^r y^g$ in $Z(S_{18}^2)$ when we replace each p_i with $1 + x^i + y^i$, denoted by

$$|K_{18}^{(r,g)}| \;\stackrel{\mathrm{def}}{=}\; [x^r y^g]\; Z(S_{18}^2)\Big|_{p_i \;\to\; 1+x^i+y^i}$$

Summing up over all legal (r,g) tuples, we found that the number of non-isomorphic game positions is thus bounded by a number larger than 10^{56} ‡.

A fifty times smaller number, 2×10^{54}, came out by probabilistically counting only legal non-isomorphic game positions through sampling using Monte Carlo methods. The corresponding experiments were conducted in two ways, using two methodically completely independent ways to sample. As one sees below, the confidence intervals of the two methods overlapped, boosting our trust in the results:

‡The exact value, computed with Pólya's enumeration formula from above, of partial edge-red-green-colorings of K_{18} such that the number of red edges is always equal or one larger than the number of green edges is

$$122817954504260150325481627994395745196940238595512818831.$$

1. method, where the mean is computed by the following formula:

$$L_1(18,4) \stackrel{\text{def}}{=} \sum_{\substack{(r,g)\,\in\,\mathbb{N}^2 \\ r\,=\,g \text{ or } r\,=\,g+1 \\ r+g\,\leq\,\binom{18}{2}}} \frac{1}{M}\sum_{k=1}^{M}\left(\frac{\binom{\binom{18}{2}}{r}\binom{\binom{18}{2}-r}{g}}{E(\mathcal{G}_k(r,g))}\,T_4(\mathcal{G}_k(r,g))\right)$$

   where $M = 1000$ was the constant sampling size for each (r, g) class, $\mathcal{G}_k(r, g)$ being the k-th randomly and uniformly drawn edge-red-green-coloring of K_{18} with r red edges and g green edges, $E(\mathcal{G}_k(r, g))$ the size of the isomorphism class of $\mathcal{G}_k(r, g)$ with respect to edge permutations, and $T_n(\mathcal{G}_k(r, g))$ being defined as zero if $\mathcal{G}_k(r, g)$ contained a monochromatic K_n, and one otherwise. By the formula, we adjust the ratio of graphs found to contain no monochromatic tetrahedron when sampling 1000 times over each class by multiplying with the complete population and dividing by the size of the isomorphism class, thus taking into account the various sizes of isomorphism classes and correcting the sampling error, and then sum up over all legal (r, n) tuples. The resulting .99 confidence interval, with the mean value inserted in the middle, is:

$$[1.7 \times 10^{54}\,,\ 2.2 \times 10^{54}\,,\ 2.7 \times 10^{54}]$$

2. method, using Pólya's enumeration formula from above to compute the number of non-isomorphic graphs for an (r, g) coloring with the same restrictions as above on (r, g), this time using different sampling sizes for different (r, g) values, using the following formula for the mean value:

$$L_2(18,4) \stackrel{\text{def}}{=} \sum_{\substack{(r,g)\,\in\,\mathbb{N}^2 \\ r\,=\,g \text{ or } r\,=\,g+1 \\ r+g\,\leq\,\binom{18}{2}}} \frac{\left(\sum_{k=1}^{M_{r+g}} T_4(\mathcal{G}_k(r,g))\right)}{M_{r+g}}\,|K_{18}^{(r,g)}|$$

   where the sampling rate M_{r+g} was 100 for low (r, g) tuples and increased up to 50000 for larger values of (r, g), the other variables and functions being defined as before. This formula computes the ratio of valid colorings that do not contain monochromatic tetrahedra, multiply it with the number of non-isomorphic (r, g) colorings of K_{18}, and then again sum up over all legal (r, n) tuples. The resulting .99 confidence interval, with the mean value inserted in the middle, is:

$$[1.9 \times 10^{53}\,,\ 2.0 \times 10^{54}\,,\ 5.0 \times 10^{54}]$$

We thus can safely assume that the number 2×10^{54} is not off by too many orders of magnitude compared to the real size of the directed acyclic game graph. Generating a graph of this size would require around 10^{21} centuries even if 300 trillion nodes could be generated during each second by each one of one billion computers running in parallel all the time, sloppily assuming for the sake of the argument that the computation lends itself to such massive parallelization. To put this time span into perspective, this is ten trillion times the currently estimated amount of time [41] since the Big Bang.

Claim 7.3 *In view of above numbers and the* **PSPACE**-*completeness of the general* $G_{\text{Avoid-Ramsey}}$ *game, we claim that to compute a winning strategy for* $G_{\text{Avoid-Ramsey}}(K_{\text{Ramsey}(n,n)}, K_n, \{\}, \{\})$ *(as well as for the* $G_{\text{Avoid-Ramsey}^+}(K_{\text{Ramsey}(n,n)}, K_n, \{\}, \{\})$ *variant) is, from all practical points of view, intractable for* $n > 3$.

These result deterred us from attempting to compute the compressed game tree and thus the winning strategy for the bigger variants of Sim, but a heuristic strategy might still be of interest. As mentioned already earlier, Ramsey numbers (n, n) with n greater than four are open research problems that recently generated a lot of interest [53], and plausibly lead to more and more complicated games.

8 Further observations

In the previous section, we showed that it is very likely that we never will be able to weakly solve $G_{\text{Avoid-Ramsey}}(K_{\text{Ramsey}(n,n)}, K_n, \{\}, \{\})$ for $n > 3$. However, it might be possible to at least ultra-weakly solve these games, as described below for the $G_{\text{Avoid-Ramsey}}(K_{18}, K_4, \{\}, \{\})$ case:

From $\text{Ramsey}(4, 4) = 18$, we know that all

$$|K_{18}^{(77,76)}| = 114722035311851620271616102401 > 10^{29}$$

non-isomorphic $(r, g) = (77, 76)$ edge-red-green-colorings of K_{18} contain at least one monochromatic tetrahedron. We also know that there exists one $(76, 76)$ edge-red-green-coloring of K_{18} (one edge remaining uncolored) that contains no monochromatic tetrahedron, through the following fact:

Fact 8.1 (adapted from Staszek Radziszowski) *There is a unique edge-red-green-coloring of* K_{17} *without monochromatic tetrahedron, call it* C*, where the number of edges of the same color leaving any vertex is equal to 8. Take any vertex* v *of* C*, make its duplicate* u*, i.e., edges* $\{x\,,\,v\}$ *and* $\{x\,,\,u\}$ *have the same color, for all* x*.* C *extended by* u *is the desired* $(76, 76)$ *edge-red-green-coloring of* K_{18} *containing no monochromatic tetrahedron, where edge* $\{u\,,\,v\}$ *is not colored, and it is unique up to isomorphism.*

Because of above observations and the fact that there are $\binom{18}{2} = 153$ edges, that is, an odd number, that can be colored altogether, a natural question would be whether the second player has a winning strategy in $G_{\text{Avoid-Ramsey}}(K_{18}, K_4, \{\}, \{\})$. This question as well as Fact 8.1 (without the uniqueness, and by being careful in the choice of the vertex to be duplicated) can easily be generalized to larger symmetric binary Ramsey numbers:

Open Problem 1 *Consider $G_{\text{Avoid-Ramsey}}(K_k, K_n, \{\}, \{\})$ where $k = \mathrm{Ramsey}(n, n)$. Is it always true that the first player has a winning strategy in this game iff $\binom{k}{2}$ is even?*

If we could show that a player with a winning strategy in some small game instance of $G_{\text{Avoid-Ramsey}}(K_k, K_n, \{\}, \{\})$ where $k = \mathrm{Ramsey}(n, n)$, such as Sim, can always force a win already *before* move $\binom{k}{2}$, this obviously would strongly indicate that the answer to the question in Open Problem 1 is no. We tested in on Sim's directed acyclic game graph but found that the second player cannot force a win before move 15, so the problem remains open. Open Problem 1 can be generalized to $G_{\text{Avoid-Ramsey}}(G, A, E^r, E^g)$ games as follows:

Open Problem 2 *Consider $G_{\text{Avoid-Ramsey}}(G, A, E^r, E^g)$, where*

$$c \stackrel{\text{def}}{=} \min_{\substack{(r,g) \in \mathbb{N}^2 \\ r = g \text{ or } r = g+1 \\ r+g \leq |E(G)| - |E^r| - |E^g|}} \{r + g \mid (G, E^r, E^g)^{(r,g)} \to A\} ,$$

and where $(G, E^r, E^g)^{(r,g)}$ denotes an (r, g) edge-red-green-coloring of the uncolored edges of the precolored graph (G, E^r, E^g). Is it always true that the first player has a winning strategy in this game iff c is even?

It is, however, rather unlikely that Open Problem 2 has a positive answer since together with Theorem 3 and Theorem 6, this would imply that **PSPACE**= $\mathbf{\Pi_2^P}$ and that the polynomial hierarchy collapses to its second level, which would be very surprising.

Regarding our asymptotic results, we could prove that all unrestricted graph Ramsey games are **PSPACE**-complete. However, a word of caution might be appropriate. **PSPACE**-completeness implies that, under the condition that **P** $\neq$ **PSPACE**, there exists no efficient algorithm to decide whether a *general* position allows a forced win for the first player. Unfortunately, such a proof says nothing about the most important position of all, namely the uncolored graph. It could be that there is a simple winning strategy for player Red given an uncolored graph, and all situations proved hard never occur during optimal play. This leads us to the following conjecture:

Conjecture 1 *Graph Ramsey games played on $(G, A, \{\}, \{\})$ are* **PSPACE**-*complete.*

However, given the evidence of tractable subcases (cf. Theorem 4) we also believe that:

Conjecture 2 *Graph Ramsey achievement games played on (K_n, A, E^r, E^g) are tractable.*

We still believe that graph Ramsey avoidance games restricted to graphs based on classic symmetric binary Ramsey numbers are difficult problems:

Conjecture 3 *Graph Ramsey avoidance games played on (K_k, K_n, E^r, E^g) where $k \geq \mathrm{Ramsey}(n, n)$ are* **PSPACE**-*complete.*

In contrast, the complexity of avoidance games based on Ramsey numbers where we specify only the size n of the monochromatic complete graph $A = K_n$ is conjectured to lie well beyond **PSPACE**, since it requires the computation of explicit Ramsey numbers (for exponential lower *and* upper bounds of classic symmetric binary Ramsey numbers, see [31, 47]; as mentioned, only the first two nontrivial of these numbers are known so far [53]) and manipulations on graphs of the size of these numbers, thus suggesting doubly exponential space requirements because of the succinct input representation. The combination of Conjectures 1 and 3 then leads to:

Conjecture 4 *The graph Ramsey avoidance games played on* $(K_{\mathrm{Ramsey}(n,n)}, K_n, \{\}, \{\})$ *are 2-***EXPSPACE***-complete.*

Note that if Open Problem 1 is answered positively, the complexity in Conjecture 4 would have to be changed to **EXPSPACE**-complete.

We could not show meaningful restrictions on the achievement graph A as in Corollaries 9 and 12 for the $G_{\text{Achieve-Ramsey}}$ game. We also could not find any meaningful restriction on the game graphs G for any of the graph Ramsey games, even though the construction in the proof of Theorem 10 looks promising: Despite the easily proved fact that QUANTIFIED BOOLEAN FORMULA(3CNF) is **PSPACE**-complete even if each variable occurs less than 6 times in the propositional formula, the construction used by T. Schaefer [59] to prove the **PSPACE**-completeness of $G_{\text{Achieve-POS-DNF}}$ adds variables occurring linearly in the number of clauses of the original 3CNF formula, and there seems to be no way to get rid of these occurrences since only positive literals are allowed in $G_{\text{Achieve-POS-DNF}}$.

Open Problem 3 *Show that* $G_{\text{Achieve-Ramsey}}$ *remains* **PSPACE***-complete even if the achievement graph* A *is restricted to a meaningful subclass of graphs such as fixed, bipartite or degree-restricted graphs.*

Open Problem 4 *Show that Theorems 6–13 hold even if the game graph* G *is restricted to a meaningful subclass of graphs such as bipartite or degree-restricted graphs.*

Other interesting directions of research include graph Ramsey games played on directed graphs. Plausibly, they will be as difficult as their undirected versions, but might be useful for the analysis of different real world applications. Also, transfinite graph Ramsey avoidance games in the spirit of [3, 4, 34, 39, 40, 51] where players must color $\aleph_0$ many edges per move are conceivable, their decision problems likely being questions of computability rather than of complexity.

9 Concluding remarks

Sim and Sim$^+$ are very easy to learn and can be played on a small piece of paper, a typical game taking only a few minutes. Nevertheless, they are fascinating to play because they are much more difficult than it first appears while at the same time being simple and elegant. In this paper, we proved that these games belong to a family of graph Ramsey games that are **PSPACE**-complete, implying that they are the most difficult problems in a class of problems generally believed to be intractable, though a formal proof that $\mathbf{P} \neq \mathbf{PSPACE}$ is lacking. At the very least, our results imply that the studied games are equivalent from the point of view of structural complexity theory to a large number of well-known games (e.g., Go [42]) and problems of industrial relevance (e.g., decision-making under uncertainty such as stochastic scheduling [49]) generally recognized as very difficult. The new characterization of **PSPACE**-complete problems as graph Ramsey games might help in studying competitive situations from industry, economics or politics where opposing parties try to achieve or to avoid a certain pattern in the structure of their commitments, in particular situations that may arise in distributed networks, maybe in a future not too far away (cf. for instance the mobile Internet agent warfare scenarios described in [67]).

We also explained how we constructed a perfect second and an ultra-strong heuristic first player for Sim and Sim$^+$ that can now be played with an attractive graphical interface on any computer for which a Java compatible browser is available. Our program is able to learn persistently from past experience by playing with different persons through the Internet. Similar self-improving techniques with client-server style learning over the Internet could be applied to other games, but also to more down-to-earth applications such as tutoring systems, intelligent language tools such as intelligent dictionaries or grammar and style checkers, intelligent agents, or distributed manufacturing systems.

Additionally, we showed that it is highly unlikely that similar games based on symmetric binary Ramsey numbers for $n > 3$ will ever be even weakly solved. Finally, we listed a number of open problems and conjectures related to graph Ramsey games.

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