To appear in *Journal of the ACM*

Translating Default Logic Into Standard Autoepistemic Logic*

Georg Gottlob
Technische Universität Wien, Vienna, Austria

Abstract

Since Konolige's translation of default logic into strongly grounded autoepistemic logic *(Artificial Intelligence 35,1988,pp.343-382)* several other variants of Moore's original autoepistemic logic that embody default logic have been studied. All these logics differ significantly from Moore's autoepistemic logic (standard AEL) in that expansions are subject to additional groundedness-conditions. Hence, the question naturally arises whether default logic can be translated into standard AEL at all. We show that a modular translation is not possible. However, we are able to construct a faithful polynomial-time translation from default logic into standard AEL which is nonmodular. Our translation exploits the self-referentiality of AEL. It uses as important intermediate step an embedding of Marek's and Truszczyński's nonmonotonic logic **N** into standard AEL. It follows from our results that the expressive power of standard AEL is strictly greater than that of default logic.

Categories and Subject Descriptors: I.2.3. [**Artificial Intelligence**]: Deduction and Theorem Proving – *Nonmonotonic reasoning and belief revision*; I.2.4. [**Artificial Intelligence**]: Knowledge Representation Formalisms and Methods – *representation languages*; I.2.0. [**Artificial Intelligence**]: General – *philosophical foundation*; F.4.1. [**Mathematical Logic and Formal Languages**]: Mathematical Logic – *Computational logic*

General Terms: Theory

Additional Key Words and Phrases: default logic, autoepistemic logic, nonmonotonic modal logic, reasoning, translation.

*Short preliminary versions containing only proof sketches of most theorems were presented at an ECAI-1992 workshop and at the IJCAI-93 conference; see reference [8]. This work was sponsored by the Christian Doppler Expert Systems Laboratory. Author's address: Institut für informationssysteme, TU Wien, Paniglgasse 16, A-1040 Wien, Austria. Email: `gottlob@vexpert.dbai.tuwien.ac.at`.

1 Introduction

Reiter's default logic [26] and Moore's autoepistemic logic [20] are among the most relevant formalizations of nonmonotonic logic. A first investigation into the relationship between default and autoepistemic logic was carried out by Konolige [11].

Konolige [11] encountered some groundedness-problems in Moore's original version of autoepistemic logic (standard AEL), which impeded a straightforward translation of default logic into standard AEL. In particular, the autoepistemic theory A that would most intuitively correspond to a given default theory T in many cases admits some additional stable expansions that do not correspond to any default extensions of T. These additional expansions are insufficiently grounded in the initial premises in the sense that they contain sentences p whose inclusion is merely based on the agent's belief in p. Consequently, Konolige defined a more restrictive version of autoepistemic logic that we will here call *strongly grounded autoepistemic logic (SGAEL)*. In SGAEL, only specific *strongly grounded* expansions are admissible. Each strongly grounded expansion is a stable expansion in the sense of standard AEL, but not vice-versa. Konolige succeeds in showing that SGAEL exactly corresponds to default logic and exhibits bidirectional translations between these two formalisms [11].

SGAEL differs significantly from standard AEL. In particular, in SGAEL it may be the case that two logically equivalent but syntactically different sets of premises Σ_1 and Σ_2 have a different semantics, i.e., different respective sets of SGAEL-expansions. Furthermore, all strongly grounded extensions of a theory Σ are *stable-set minimal* for Σ, which is not the case in standard AEL.

The second major approach of translating default logic into variants of AEL has been taken by Marek, and Truszczyński [15, 34, 33]. Their approach is based on the concept of nonmonotonic modal logics as introduced by McDermott and Doyle [19, 18]. They show that default logic is faithfully embeddable in a wide range of different nonmonotonic modal logics. The simplest of these logics is the nonmonotonic counterpart of the *pure logic of necessitation* **N** – the modal logic consisting of propositional calculus augmented by the necessitation rule. We will make intensive use of this logic in the present paper. Note that just as SGAEL, nonmonotonic **N** is considered a stronger logic than standard AEL, since each **N**-expansion of a set of premises Σ is also a standard AEL-expansion of Σ, but not vice-versa.

In [27] it is shown that standard AEL corresponds to the nonmonotonic version of the modal logic **KD45** based on the modal axioms **K**, **4**, **5**, and **D** : $L\phi \rightarrow \neg L \neg \phi$. Unfortunately, the proposed translations schemes which allow the embedding of default logic into a large number of other nonmonotonic modal logics fail to apply to nonmonotonic **KD45**, hence they are not applicable to standard AEL.

Other methods for translating default logic into formalisms close to AEL have been developed by Niemelä [23], Lin and Shoham [12], Siegel [29], and Kaminski [10]. Each approach introduces a different version of AEL which captures default logic. Each of these logics is *more restrictive* than standard AEL in the sense that, in general, a set Σ of given premises in these logics admits fewer expansions than in standard AEL.

In summary, all previous methods translate default logic into formalisms that are different from standard AEL. The reason seems to be that standard AEL allows for expansions that are not sufficiently grounded in the premises. Consequently, these approaches are based on more restrictive formalisms that admit fewer expansions.

The question whether default logic can be translated into standard AEL has remained open so far.

In the present paper, we solve this problem by giving both a negative and a positive answer.

First we show that there exists no *modular* translation between default logic and AEL. This means that after adding a new fact F to the formula set W of a default theory $\langle D, W\rangle$, a complete recomputation of the translation becomes necessary. In other words, it is not possible to translate facts and defaults separately and taking the union of these separate translations. This is probably the reason why all previous attempts to a faithful translation between default logic and autoepistemic logic failed. In fact, all previous attempts tried to establish modular translations.

An exception are prerequisite-free default theories, which are modularly translatable to standard AEL.

In this paper we limit ourselves to consider *propositional* default logic and AEL. Note that the impossibility result for modular translations extends trivially from propositional default logic to the more general first order case.

After deriving this negative result, we show that it is possible to polynomially translate general default theories to standard AEL if one gives up on modularity. We construct a polynomial transformation mapping each default theory $\langle D, W\rangle$ into a set of AE-formulas $e(\langle D, W\rangle)$, such that the extensions of $\langle D, W\rangle$ are in a one-to-one and onto relationship with the standard-AE-expansions of $e(\langle D, W\rangle)$. More specifically, the extensions of $\langle D, W\rangle$ and the objective (i.e., non-modal) parts of the standard-AE-expansions of $e(\langle D, W\rangle)$ coincide

The main step in our construction is to establish a translation from nonmonotonic **N** into standard AEL. Since, by results of Marek and Truszczyński [15, 34], default logic is translatable into nonmonotonic **N**, the composition of these two translations yields the desired translation from default logic in standard AEL.

The rationale of our translation from nonmonotonic **N** into standard AEL is as follows. Let Σ be a set of autoepistemic premises. Recall that any **N**-expansion of Σ is also a standard-AE-expansion, but Σ may have some undesired (insufficiently grounded) standard-AE-expansions that are not **N**-expansions of Σ. We force these undesired expansions to disappear by adding a formula $LG(\Sigma)$ to Σ. Intuitively, $G(\Sigma)$ encodes the fact that every expansion is sufficiently grounded, and $LG(\Sigma)$ forces $G(\Sigma)$ to be valid. $G(\Sigma)$ is called the *grounding condition* for Σ.

While default logic can be faithfully embedded into standard AEL, the converse is not possible. Indeed, one can easily find an autoepistemic theory Σ such that the objective parts of the expansions of Σ do not coincide with the extensions of any default theory. Thus standard AEL is strictly more expressive than default logic.

Note that autoepistemic logic is a logic for expressing *beliefs*, while the stronger default logic or nonmonotonic **N** are logics for reasoning about *knowledge* or, at least, for expressing strongly grounded beliefs such as default knowledge. Therefore, our translatability result means that knowledge (or strongly grounded beliefs) can be emulated by pure (autoepistemic) beliefs. This interesting, but somewhat speculative interpretation will be discussed at the end of this paper.

The rest of this paper is structured as follows. In the next section we recall the basic concepts of default logic and of autoepistemic logic. We also give a formal definition of the concept of faithful translation. In Section 3, we prove that a modular translation beteen default logic and AEL is impossible. In Section 4 we review a number of different concepts of groundedness. Section 5 presents the rationale and a survey of our nonmodular translation. In Section 6 we give an overview of both the monotonic version and the nonmonotonic version of modal logic **N** and quote a number of relevant previous results as well as a useful lemma. In Section 7, we show how to translate inferencing in the modal monotonic logic **N** to validity checking in classical propositional calculus. In Section 8, we exhibit our faithful translation from nonmonotonic **N** to standard AEL. The translation from default logic to standard AEL is carried-out in Section 9. Section 10 concludes the paper with a philosophical interpretation of our translation.

2 Default Logic, AEL, and Faithful Translations

In this section we define the basic concepts related to default and autoepistemic logic. We also recall some recent complexity results concerning these logics and define the concept of *faithful translation.* We start by introducing our notation for pure propositional logic.

Propositional Logic

Let $\mathcal{L}$ denote an ordinary propositional language over a countably infinite alphabet of propositional variables, the syntactic operators $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\top$, and $\bot$ (where $\top$ is a constant for truth and $\bot$ is a constant for falsity). The classical propositional logic over $\mathcal{L}$ is denoted by $\mathbf{PL}(\mathcal{L})$.

For the sake of a simpler notation, we will sometimes implicitly identify *finite sets* and *conjunctions* of propositional formulas. So, when a set S of formulas occurs in the context of a logical formula, we intend the conjunction $\bigwedge_{\phi\in S}\phi$ of its elements.

Default Logic

Default logic was introduced by Reiter [26]. Excellent surveys are given in [1, 17]. A *default* d is a confifguration of the form

$$\frac{\alpha\ :\ M\beta_1,\ M\beta_2,\ldots,M\beta_n}{\omega}$$

where $\alpha,\beta_1,\ldots,\beta_n$, and ω are propositional sentences. Usually α is called the *prerequisite* of d and is referred to as $p(d)$, the $\beta_1\ldots\beta_n$ are called the *justification* of d and the set $\{\beta_1,\ldots,\beta_n\}$ is referred to as $j(d)$, and ω is called the *consequent* of the default d and is referred to as $c(d)$. For a set H of defaults, we define $c(H)=\{c(d)|d\in H\}$. For convenience, we sometimes write a default in the following form: $\alpha : M\beta_1,\ M\beta_2,\ldots,M\beta_n/\omega$.

A default d is satisfied by a deductively closed set of sentences $\mathcal{E}$ if, whenever α is an element of $\mathcal{E}$ and $\beta_1\ldots\beta_n$ are all consistent with $\mathcal{E}$, then also ω is an element of $\mathcal{E}$.

A *propositional default theory* is a pair $\langle D,W\rangle$, where W is a finite set of propositional sentences and D is a finite set of defaults. Whenever we speak about *default theory* in the rest of this paper, we mean *propositional default theory.*

Informally, a default extension of $\langle D,W\rangle$ is a grounded minimal deductively closed set of propositional formulas containing W and satisfying all defaults of D. The formal definition of an extension is as follows.

Definition 2.1 (Reiter [26]) *Let $\mathcal{D}=\langle D,W\rangle$ be a default theory. For any set S*

of propositional formulas, $\Gamma_{\mathcal{D}}(S)$ is the smallest set U satisfying the following three properties:

1. *$W \subseteq U$*

2. *U is deductively closed*

3. *If $\frac{\alpha : M\beta_1, M\beta_2, \ldots, M\beta_n}{\omega} \in D$ and $\alpha \in U$ and $\neg\beta_1 \notin S$ … and $\neg\beta_n \notin S$, then $\omega \in U$.*

An extension of $\mathcal{D} = \langle D, W\rangle$ is a fixed point of $\Gamma_{\mathcal{D}}$, i.e., a set $\mathcal{E}$ of propositional formulae satisfying $\Gamma_{\mathcal{D}}(\mathcal{E}) = \mathcal{E}$.

A given default theory $\langle D, W\rangle$ may admit zero, one, or more extensions. It is often useful to consider default theories whose defaults are subject to certain syntactic restrictions.

Definition 2.2 *A default is normal iff it is of the form $\frac{\alpha : M\omega}{\omega}$. A default is semi-normal iff it is of the form $\frac{\alpha : M(\gamma \wedge \omega),}{\omega}$. A default d is prerequisite-free iff $p(d) = \top$. A default d is called* justification-free *iff $j(d) = \emptyset$. A default theory is called normal (semi-normal, prerequisite-free, justification-free) iff all its defaults are normal (semi-normal, prerequisite-free, justification-free).*

In the case of prerequisite-free defaults, one often omits the sign $\top$ and writes the default with an empty prerequisite. Similarly, justification-free defaults are usually written with an empty (i.e, blank) list of justifications.

Reiter [26] has shown that each normal default theory has at least one extension. However, there exist semi-normal default theories with no extension.

Autoepistemic Logic

The language $\mathcal{L}_L$ of autoepistemic logic extends $\mathcal{L}$ by a unary modal operator L. L is an "introspective" operator, i.e., an operator that refers to the theory itself. Intuitively, if ϕ is a formula (possibly containing occurrences of L), then $L\phi$ means that *ϕ is believed.*

A *set of autoepistemic premises* Σ is a finite set $\Sigma \subset \mathcal{L}_L$. We will use the following notation. If $\Delta \subseteq \mathcal{L}_L$ then $L\Delta = \{L\phi \mid \phi \in \Delta\}$, $\neg\Delta = \{\neg\phi \mid \phi \in \Delta\}$, and $\overline{\Delta} = \mathcal{L}_L - \Delta$.

Classical propositional formulas with no occurrences of L are called *objective formulas.* Propositional interpretations are extended to autoepistemic logic by taking autoepistemic formulas of the form $L\phi$ to be atoms. In particular, the atoms of a formula are all propositional and modal atoms of form $L\psi$ occurring *at the surface*, i.e., outside the scope of any L-operator. For example, the atoms of the formula $r \rightarrow (p \wedge LL(q \vee r))$ are r, p, and $LL(q \vee r)$. An interpretation consists of an assignment of truth-values to all atomic objective formulas and to all formulas of the form $L\phi$. An interpretation assigns a truth value to each formula of $\mathcal{L}_L$ by the classical rules of truth recursion. The classical consequence relation $\models$ on $\mathcal{L}$ is extended to $\mathcal{L}_L$ as follows. If $\Sigma \subseteq \mathcal{L}_L$ and $\phi \in \mathcal{L}_L$ then $\Sigma \models \phi$ iff ϕ is true in every interpretation in which Σ is true. The consequence operation *cons* is defined as usual by $cons(\Delta) = \{\phi \mid \Delta \models \phi\}$ for $\Delta \subseteq \mathcal{L}_L$. Thus, classical propositional logic $\mathbf{PL}(\mathcal{L})$ is extended in a natural way to the language $\mathcal{L}_L$. We denote this extension by $\mathbf{PL}(\mathcal{L}_L)$.

The main objective of autoepistemic logic is to determine, for a given initial set $\Sigma \subseteq \mathcal{L}_L$ of premises, all *stable extensions of* Σ, i.e., all alternative possible sets of total beliefs that an ideally rational agent may adopt. Formally, a stable expansion of an autoepistemic theory Σ is defined as follows.

Definition 2.3 ([20]) *Δ is a stable expansion of Σ iff Δ satisfies the following fixed point equation.*

$$\Delta = cons(\Sigma \cup L\Delta \cup \neg L\overline{\Delta}).$$

The above definition establishes *standard AEL*. In the rest of this paper, whenever we write AEL, we refer to standard AEL. Whenever we speak about *stable expansions* or stable AE-expansions, we refer to stable expansions in the sense of standard AEL, Def. 2.3.

Definition 2.4 ([31]) *A theory $T \subseteq \mathcal{L}_L$ is stable if T satisfies the following three conditions:*

(St1) *$Cn(T) = T$; i.e., T is closed under propositional consequence.*

(St2) *(Positive Introspection) For every $\phi \in \mathcal{L}_L$, if $\phi \in T$ then $L\phi \in T$.*

(St3) *(Negative Introspection) For every $\phi \in \mathcal{L}_L$, if $\phi \notin T$ then $\neg L\phi \in T$.*

Each set stable expansion of a set of premises Σ is a stable set [20]. Moreover, it is well-known that each set $I \subseteq \mathcal{L}$ of objective formulas has exactly one stable AE-expansion [20, 17]. We denote this expansion by $E(I)$.

There are several approaches for finitely characterizing the extensions of a given set $\Sigma \subseteq \mathcal{L}_L$ [16, 27, 22, 24, 30]. In the following, we briefly describe Niemelä's approach [24].

Let Σ be an autoepistemic theory. Then Σ^L denotes the set of all subformulas of the form $L\phi$ of each formula of Σ at all levels of nesting. For example, if $\Sigma = \{L(L(p) \wedge L(q)), L(L(r))\}$ then $\Sigma^L = \{L(L(p) \wedge L(q)), L(p), L(q), L(L(r)), L(r)\}$. The elements of Σ^L are also called *L-subformulas* of Σ. For single formulas $\phi \in \mathcal{L}_L$ we write ϕ^L instead of $\{\phi\}^L$. Furthermore, for $\Sigma \subseteq \mathcal{L}_L$, let $\Sigma^{\pm L} = \Sigma^L \cup \neg(\Sigma^L)$.

Definition 2.5 *For an autoepistemic theory Σ a set $\Lambda \subseteq \Sigma^{\pm L}$ is Σ-full iff the following two conditions hold for each $L\phi \in \Sigma^L$:*

1. $\Sigma \cup \Lambda \models \phi$ *iff* $L\phi \in \Lambda$
2. $\Sigma \cup \Lambda \not\models \phi$ *iff* $\neg L\phi \in \Lambda$

Note that by this definition, if Λ is a Σ-full set, then for each $L\phi \in \Sigma^L$, either $L\phi \in \Lambda$ or $\neg L\phi \in \Lambda$.

Proposition 2.1 (Niemelä [24]) *For each autoepistemic theory Σ, there is a one-to-one correspondence between the stable expansions of Σ and the Σ-full sets. In particular, for the AE-expansion E corresponding to a Σ-full set Λ it holds that $\Lambda = \Sigma^{\pm L} \cap (\{L\phi \mid L\phi \in E\} \cup \{\neg L\phi \mid \phi \notin E\})$*

The stable AE-expansion corresponding to a Σ-full set Λ is denoted by $SE_\Sigma(\Lambda)$. Λ is called the *kernel* of $SE_\Sigma(\Lambda)$ w.r.t. Σ.

Recent Complexity Results

It was recently shown that propositional default reasoning is presumably harder than classical propositional reasoning because the computational complexity of determining whether a default theory has an extension is $\mathbf{\Sigma_2^P}$-complete, while it is well-known that checking whether a classical propositional theory is satisfiable is **NP**-complete [7, 25, 32].

$\mathbf{\Sigma_2^P} = \mathbf{NP^{NP}}$ is the class of all decision problems solvable in nondeterministic polynomial time with access to an oracle in **NP**. The complementary class is $\mathbf{\Pi_2^P}$. These classes are the central classes of the second level of the Polynomial Hierarchy (see [6]).

For AEL, it was shown in [7] that determinining whether a set of premises Σ has a stable expansion is $\mathbf{\Sigma_2^P}$-complete. There are two basic forms of reasoning with default logic and AEL: credulous and skeptical reasoning. Credulous reasoning consists in checking whether a formula ϕ is contained in *at least one* extension/expansion of a given theory. Skeptical reasoning consists in checking whether a formula ϕ is contained in *all* extensions/expansions. In [7] it was shown that for both default logic and AEL, credulous reasoning is $\mathbf{\Sigma_2^P}$-complete, while skeptical reasoning is $\mathbf{\Pi_2^P}$-complete. For a survey of related complexity results for several types of nonmonotonic reasoning, see [2].

From these results it follows immediately that there are polynomial transformations from the basic reasoning tasks in default logic to the corresponding in autoepistemic reasoning and vice-versa. In fact, such transformations can be easily derived from the respective completeness proofs for $\mathbf{\Sigma_2^P}$ or $\mathbf{\Pi_2^P}$. However, these transformations are at a nonlogical algorithmic level. They translate decision problems into equivalent decision problems but not, say, default theories into equivalent autoepistemic theories.

In this paper we are interested in a much tighter and much more logical notion of translation.

Faithful Translations

The kind of translations we are interested in is usually referred to as *faithful translations* or *faithful embeddings*[1]. The rationale is that a default theory $\langle D, W\rangle$ is translated to an autoepistemic theory Σ such that $\langle D, W\rangle$ and Σ are in some sense equivalent.

Equivalence between a default theory $\langle D, W\rangle$ and an autoepistemic theory Σ is defined via the respective sets of extensions and stable expansions. $\langle D, W\rangle$ is equivalent to Σ if the set of all extensions of $\langle D, W\rangle$ is equal to the set of all objective parts of all stable expansions of Σ. It is well-known that two different stable expansions of an autoepistemic theory Σ have different objective parts. Therefore, if $\langle D, W\rangle$ is equivalent to Σ, then the extensions of $\langle D, W\rangle$ are in a one-to-one correspondence with the expansions of Σ. In summary, we define the notion of faithful translation as follows.

Definition 2.6 *A* faithful *translation from default logic to AEL is a mapping tr which transforms each default theory $\langle D, W\rangle$ into an autoepistemic theory $\Sigma = tr(\langle D, W\rangle)$ such that the objective parts of the autoepistemic expansions of Σ are identical with the default extensions of $\langle D, W\rangle$.*

3 Modularity

The concept of *modular translation* in the context of default logic was introduced by T. Imielinski in [9]. Loosely speaking, a translation scheme is modular if adding new facts (not defaults) to a default theory is reflected by the translation through adding these

[1]We use the terms *translation* and *embedding* as synonyms throughout the paper.

new facts to the result of the translation. Imielinski [9] considered translations between default logic and circumscription. We adapt his formal definition of modularity to the context where AEL is the target system as follows.

Definition 3.1 *A translation tr from default logic to AEL is modular iff for each set D of defaults and each set $W \subseteq \mathcal{L}$ it holds that $tr(\langle D, W\rangle) = tr(\langle D, \emptyset\rangle) \cup W$.*

As Imielinski points out, modular translations are highly desirable both from the conceptual and the computational point of view. Indeed, changes to a default theory $\langle D, W\rangle$, due to changes in the underlying "real world", will in most cases affect W but not D, as the default rules in D usually represent time-invariant properties. With a modular translation, when W is changed, we do not have to recompute the result of the translation, but we may just add the new elements of W to the old result.

Note that Konolige's translation from default logic to SGAEL [11] and Marek and Truszczyński's translations from default logic to nonmonotonic **N** [15, 34] are modular.

It is natural to ask whether there exists a faithful modular translation from default logic to standard AEL. Actually, it is likely that all researchers who have until recently attempted to translate default logic into standard AEL have tried to construct a modular translation. This is probably the reason why their attempts failed.

In the rest of this section, we will show that a faithful modular translation from default logic to standard AEL is not possible. In particular, such a translation is not possible even if the defaults are restricted to very simple classes such as normal defaults. We will, however, identify one important class of defaults admitting a faithful modular translation to standard AEL, namely the prerequisite-free defaults.

Theorem 3.1 *There exist no faithful modular translation from default logic into standard AEL. Such a translation does not exist even if the defaults are restricted to be normal.*

PROOF. Consider the normal default theories

$\langle D, W_0\rangle$, $\langle D, W_1\rangle$, and $\langle D, W_2\rangle$, where $W_0 = \emptyset$, $W_1 = \{a\}$, $W_2 = \{a \rightarrow b\}$, and

$$D = \{\frac{a \rightarrow b \,:\, Ma}{a}, \frac{a \,:\, Mb}{b}\}.$$

Each of these three default theories has exactly one extension. $\langle D, W_0\rangle$ has as unique extension $cons(\emptyset)$, while $\langle D, W_1\rangle$ and $\langle D, W_2\rangle$ both have the set $cons(\{a, b\})$ as unique extension.

Assume that there exists a modular faithful translation tr from default logic to standard AEL. We will show that this assumption implies that $cons(\{a, b\})$ is an extension of $\langle D, W_0\rangle$, an obvious contradiction.

Let $tr(\langle DW_0\rangle) = \Sigma$. Since tr is modular, it must hold that $tr(\langle D, W_1\rangle) = \Sigma \cup \{a\}$. Since tr is faithful, the unique stable AE-expansion Δ of $\Sigma \cup \{a\}$ is given by $\Delta = E(\{a, b\})$. By Def. 2.3 we thus have

$$\Delta = cons(\Sigma \cup \{a\} \cup L\Delta \cup \neg L\overline{\Delta}). \tag{1}$$

Since $b \in \Delta$, it follows that

$$\Sigma \cup \{a\} \cup L\Delta \cup \neg L\overline{\Delta} \models b \tag{2}$$

By applying the deduction theorem, we get

$$\Sigma \cup L\Delta \cup \neg L\overline{\Delta} \models a \rightarrow b \tag{3}$$

In other words,

$$a \rightarrow b \in cons(\Sigma \cup L\Delta \cup \neg L\overline{\Delta}). \tag{4}$$

It thus holds that

$$cons(\Sigma \cup L\Delta \cup \neg L\overline{\Delta}) = cons(\Sigma \cup \{a \rightarrow b\} \cup L\Delta \cup \neg L\overline{\Delta}). \tag{5}$$

Now, since tr is modular, it must hold that $tr(\langle D, W_2\rangle) = \Sigma \cup \{a \rightarrow b\}$. Furthermore, since $\langle D, W_1\rangle$ and $\langle D, W_2\rangle$ both have the same unique default extension $cons(\{a, b\})$, their respective translations $tr(\langle D, W_1\rangle)$ and $tr(\langle D, W_2\rangle)$ both have the same unique stable expansion Δ. In particular, since Δ is a stable expansion of $tr(\langle D, W_2\rangle) = \Sigma \cup \{a \rightarrow b\}$, it holds that

$$cons(\Sigma \cup \{a \rightarrow b\} \cup L\Delta \cup \neg L\overline{\Delta}) = \Delta. \tag{6}$$

By Equations (5) and (6), it follows that

$$cons(\Sigma \cup L\Delta \cup \neg L\overline{\Delta}) = \Delta. \tag{7}$$

But this means that Δ is a stable AE-expansion of Σ. By the faithfulness of tr, it follows that $cons(\{a, b\}) = \Delta \cap \mathcal{L}$ is a default extension of $\langle D, W_0\rangle$, which is obviously not true. Contradiction. A modular and faithful translation from default logic to standard AEL is thus impossible. $\Box$

Note that the above negative result also holds for the class of *justification-free* defaults, i.e., defaults with empty justifications. To see this, just replace $\frac{a \to b : Ma}{a}$ by $\frac{a \to b :}{a}$ and $\frac{a : Mb}{b}$ by $\frac{a :}{b}$ in the above proof. Note that justification-free defaults are ordinary inference rules. Hence, even ordinary rule systems cannot be modularly translated into autoepistemic logic. This shows where the real problem lies and hints at how we may restrict default theories in order to obtain modular translatability: to prerequisite-free defaults.

For prerequisite-free defaults, we may consider Konolige's most intuitive translation-scheme τ:

$$\tau(\frac{\top \,:\, M\beta_1, \ldots, M\beta_n}{\gamma}) \;=\; \neg L \neg \beta_1 \wedge \ldots \wedge \neg L \neg \beta_n \to \gamma$$

and, as usual,

$$\tau(\langle D, W \rangle) = W \,\cup\, \bigcup_{d \in D} \{\tau(d)\}.$$

Theorem 3.2 *τ is a faithful and modular translation from the class of prerequisite-free default theories to standard AEL.*

PROOF. (Sketch) The proof follows directly from previous results (Theorems 4.1.2 and 4.1.4) by Marek and Truszczyński [14]; the result is also implicitly present in the work of Lin and Shoham [12]. In [14], the concept of weak default extension is defined and it is shown that each default theory, in particular, each prerequisite-free default theory $\langle D, W \rangle$ can be transformed into an autoepistemic theory $\tau(\langle D, W \rangle)$ by the transformation τ. The weak extensions of $\langle D, W \rangle$ exactly correspond to the objective parts of $\tau(\langle D, W \rangle)$. Now, it is easy to see that for prerequisite-free default theories, the set of extensions extactly coincides with the set of weak extensions. τ is obviously modular and polynomial. $\Box$

Some authors argue that prerequisite-free defaults are the only natural defaults. Delgrande and Jackson [3], for instance, define *P-Default Logic* based on prerequisite-free defaults. By the above result, P-Default Logic is modularly translatable into standard-AEL.

Note that in Definition 3.1 we defined a rather weak concept of modularity. Stronger types of modularity would require that each single default be *separately* translatable. Since we proved that even weakly modular translations in the sense of Definition 3.1 are impossible, the impossibility of any stronger type of modular translation follows.

We conclude this section with two remarks.

Remark 1. In his definition of modular translation [9], Imielinski actually distinguishes between *general facts* and *specific facts* and restricts W to contain only specific facts, while general facts are added to D. We do not make this distinction here, because it does not make much sense in the present setting (we are not concerned with abnormality predicates that interfere with our translation). However, even if we adopted this refinement, the main theorem of this section (Thm. 3.1) would still remain valid (although it would require a slightly more involved proof).

Remark 2. One may define a yet weaker notion of modularity. A translation τ from default logic to AEL is *weakly modular* iff τ can be decomposed into two translations τ_1 and τ_2 such that for each default theory $\langle D, W\rangle$, $\tau(\langle D, W\rangle) = \tau_1(W) \cup \tau_2(D)$. This definition is very similar to the one adopted by us, except for the translation of W. In our definition, W must be translated to W while here W can be changed to $\tau_1(W)$. However, it is easily seen that weakly modular translations from default logic to AEL are impossible, just as modular translations in our sense. In fact, this is readily seen by observing that for each weakly modular translation it must hold that $\tau_1(W)$ is logically equivalent to W, and by adapting our impossibility proof (proof of Theorem 3.1) accordingly.

4 Notions of Groundedness

In order to make our subsequent translation results more accessible, we discuss in this section different notions of groundedness. One particular concept of groundedness, which we call *appropriate groundedness* turns out to be suitable for representing the kind of groundedness inherent in default logic. We informally describe how appropriate groundedness is embodied by nonmonotonic modal logic **N**.

Various concepts of groundedness have been described in the literature. We will discuss four concepts here. For related discussions see [11, 14, 15, 17, 23].

• **Weak Groundedness.** In his seminal paper [20], Moore required that *every* autoepistemic stable expansion Δ of a set of premises Σ must be grounded in Σ in the sense that $\Delta \subseteq cons(\Sigma \cup L\Delta \cup \neg L\overline{\Delta})$. Note that this groundedness condition is precisely one direction of the equation used here for defining a stable expansion (Definition 2.3). We will refer to this type of groundedness as *weak groundedness*. Moore has shown that weak groundedness of Δ in Σ is equivalent to the *soundness* of Δ with respect to Σ which holds iff every interpretation of Δ in which all formulas of Σ are true is an autoepistemic model of Δ (i.e, satisfies also all formulas of Δ).

Weak groundedness is an important feature of AEL. However, as pointed out earlier, it is too liberal for directly expressing default knowledge. For instance, the default theory $\mathcal{D} = \langle\{\frac{\mathrm{p}:}{\mathrm{p}}\}, \emptyset\rangle$ has only one default extension, namely, the set of all propositional tautologies $cons(\emptyset)$. Intuitively, $\mathcal{D}$ would be most appropriately translated to the AEL set $\Sigma = \{Lp \rightarrow p\}$. However, Σ has two stable AE-expansions, $E(\emptyset)$ and $E(\{p\})$. Only the first one corresponds to the extension of $\mathcal{D}$. An agent adopting the second expansion believes in p and in Lp, and it is clear from the fixed point equation in Definition 2.3 that the agents belief in p is grounded in the assumption of Lp, i.e. in the assumption that the agent beliefs p. Such circularities were recognized by Konolige [11] as a major impediment towards capturing default logic via AEL. Consequently, he proposed to investigate stronger notions of groundedness that eliminate such anomalies.

The following three concepts of groundedness are restrictions of weak groundedness. These restrictions lead to nonstandard versions of AEL where a set of premises Σ may admit fewer expansions than in standard AEL.

• **Moderate Groundedness.** In his first attempt to capture default logic by specific versions of AEL, Konolige [11] defined the concept of *moderately grounded expansion.* Let us denote by $\Delta \models_{\mathcal{SS}} \phi$ that ϕ is an element of *each* stable set containing Δ. Moreover, denote by Δ_0 the objective part of Δ and by $\overline{\Delta}_0$ the objective part of $\mathcal{L}_L - \Delta$. A set of formulas $\Delta \subseteq \mathcal{L}_L$ is *moderately grounded* in Σ iff $\Delta = \{\phi \mid \Sigma \cup L\Sigma \cup \neg L\overline{\Delta}_0 \models_{\mathcal{SS}} \phi\}$. Konolige [11] has shown that every set Δ that is moderately grounded in Σ is also an AEL expansion of Σ; however, the converse is not true. Konolige proved that an AEL expansion Δ of Σ is moderately grounded iff its objective part $obj(\Delta)$ is minimal in the following sense: there does not exist any stable set containing Σ whose objective part is strictly included in $obj(E)$.

For example, only the stable expansion $E(\emptyset)$ of the set $\Sigma = \{Lp \rightarrow p\}$ is moderately grounded, while the "circular" expansion $E(\{p\})$ is *not* moderately grounded. This seems to match our intuition about the "right" restriction of AEL for capturing default logic. However, Konolige [11] pointed out that moderately grounded expansions do not completely eliminate the problems due to circularity. Let $\Sigma_1 = \{\neg Lp \rightarrow q,\, Lp \rightarrow p\}$. Σ_1 has *two* moderately grounded expansions: $T = E(\{q\})$ and $T' = E(\{p\})$. As explained in [11], the second one is again circular, in the sense that Lp is derived before p. This shows that moderately grounded AEL is not well-suited for modeling default rules.

The causes of undesired circularities in moderately grounded AEL are not completely obvious. In the last example (Theory Σ_1), one does not need to *assume* Lp in order to get the "circular" expansion T', since Lp can be derived by use of $\models_{SS}$ from Σ_1 and the assumption $\neg Lq$. In fact, it holds that $\neg Lp \rightarrow q \models_{SS} \neg Lq \rightarrow Lp$. Thus $\Sigma_1 \cup \{\neg Lq\} \models_{SS} Lp$ and hence $\Sigma_1 \cup \{\neg Lq\} \models_{SS} p$. This shows that the undesired expansion arises due to an overstrained *interaction* of the formulas in Σ_1 that could not happen if the knowledge were represented in form of default *rules* instead of implicational *formulas*, or if a notion of inference weaker than $\models_{SS}$ were used. In moderately grounded AEL, kinds of weak contrapositions (such as $\neg Lq \rightarrow Lp$) of the formulas representing inference or default rules can be derived via $\models_{SS}$. Such contrapositions interact with other formulas and generate undesired, insufficiently grounded knowledge. Such overstrained interactions of formulas are also discussed in [15].

Yet another reason makes us refrain from using moderately grounded AEL as an intermediate step in the translation from default logic to standard AEL. In [4] it is shown that the main reasoning tasks in this formalism are one degree harder in the Polynomial-time Hierarchy than the analogous reasoning tasks in AEL or default logic. Namely, using moderately grounded AEL expansions, credulous reasoning is $\mathbf{\Sigma_3^P}$-complete and skeptical reasoning is $\mathbf{\Pi_3^P}$-complete, while the analogous reasoning tasks in standard AEL and in default logic are $\mathbf{\Sigma_2^P}$-complete and $\mathbf{\Pi_2^P}$-complete, respectively [7]. Thus, unless the Polynomial-time Hierarchy collapses, there cannot exist any polynomial-time translation (modular or not) from moderately-grounded AEL into standard AEL.

• **Appropriate Groundedness for Default Reasoning.** How can one remove the drawbacks of weakly and moderately grounded AEL that lead to inappropriate expansions? The previous examples suggest that these drawbacks are due to mainly two features: (1) making overly bold nonmonotonic assumptions (such as assuming Lp in order to obtain the expansion $E(\{p\})$ of $\Sigma = \{Lp \rightarrow p\}$), and (2) using a too liberal inference relation ($\models_{SS}$). Both features can be avoided by suitably modifying the basic fixed point equation of autoepistemic logic (see Def. 2.3). We will refer to the solutions of the modified fixed point equations as *appropriately grounded* expansions, keeping in mind that the term *appropriate* means here *appropriate for modelling default knowledge.* The main rationale of appropriate groundedness is that nonmonotonic assumptions are limited to assumptions of the form $\neg L\phi$. In fact, the nonmonotonic assumptions made in default reasoning are precisely of this form. To see this, consider any default rule $\frac{\phi : \psi}{\xi}$. When firing this rule, one makes the nonmonotonic assumption that ψ is *possible*, i.e., ψ is consistent with the premises and previously made assumptions. In the language of autoepistemic logic, this corresponds to assuming $\neg L \neg \psi$.

In particular, the following modifications are made to the basic fixed point equation of AEL (see Def. 2.3):

1. Eliminate $L\Delta$ from the right part of the equation. By this change we suppress the possibility of making nonmonotonic assumptions of positive knowledge, and restrict the nonmonotonic assumptions to the form $\neg L\phi$.

2. Replace the consequence operator *cons* by the consequence operator $cons_{\mathbf{N}}$ of (monotonic) logic **N**. Logic **N** consists of the propositional calculus augmented by the necessitation rule $\phi/L\phi$. Since we eliminated $L\phi$ from the fixed point equation, strengthening *cons* to $cons_{\mathbf{N}}$ becomes necessary, otherwise, when ϕ is among the premises, $L\phi$ would not necessarily belong to the fixed points of the equation, and thus, the fixed points would, in general, not be stable sets[2]. The important point here is that $cons_{\mathbf{N}}$ is still a rather restricted consequence operator. If we replaced *cons* by the more liberal consequence operator $cons_{SS}$ corresponding to $\models_{SS}$, then we would open the door for circular expansions, similar as in moderately grounded AEL.

The modified fixed point equation is: $\Delta = cons_{\mathbf{N}}(\Sigma \cup \neg L\overline{\Delta})$. It can be seen that all solutions to this fixed-point equation are stable expansions of Σ. As mentioned before, we call these expansions *appropriately grounded*. In summary, appropriate groundedness basically limits the "jump" to new nonmonotonic conclusions to sentences which are obtained by *negative* introspection. Positive introspection is limited to be used in a monotonic fashion, namely, via the (monotonic) necessitation rule inherent in the $cons_{\mathbf{N}}$ consequence operator. In our opinion, this is a very natural type of groundedness.

The modified fixed point equation is precisely the defining equation of a well-known nonmonotonic logic called *nonmonotonic* **N**, which will be discussed in detail later on. Thus, the appropriately grounded expansions of a set of premises Σ are precisely the **N**-expansions of Σ. Marek and Truszczyński [15, 34] have shown that it is indeed possible to faithfully translate default logic into nonmonotonic **N**.

• **Strong Groundedness.** Konolige [11][3] found a way of defining default extensions through epistemic fixed point equations. By a fixed point equation similar to the one defining moderately grounded expansions, he defined the notion of *strongly grounded* expansions for a fragment of AEL that syntactically corresponds to default logic. Basically,

[2] In fact, the fixed point equation $\Delta = cons(\Sigma \cup \neg L\overline{\Delta})$ precisely corresponds to the first nonmonotonic logic (often referred to as NM1) described by McDermott and Doyle [19]. This logic was later recognized to be counterintuitive [18].

[3] The correct definition of strong groundedness is given in the corrigendum to [11].

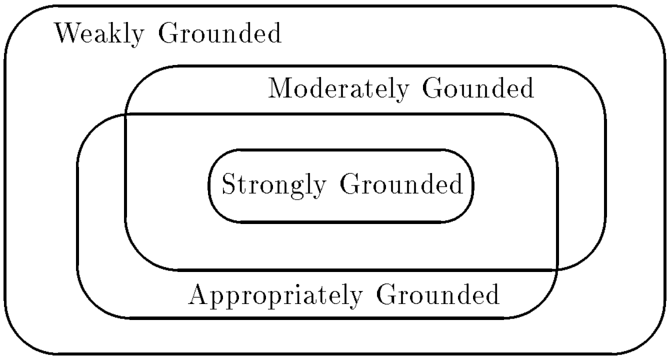

Figure 1: Different notions of groundedness

he succeeded to avoid the circular expansions arising from the use of $\models_{SS}$ by imposing certain syntactic conditions both on the form of the premises and on the knowledge one may assume while constructing the fixpoint. In particular, these conditions avoid the derivation of (kinds of) contrapositions of the represented default rules that may interact illegitimately with other formulas. These syntactic restrictions, however, make moderately grounded AEL more similar to a rule based system such as default logic, than to a formula based nonmonotonic logic such as, say, standard AEL. For this reason, we found it difficult to use strongly grounded AEL as a tool in our translation from default logic to standard AEL. On the other hand, the translation from default logic to standard AEL we describe in the subsequent sections can be trivially modified to become a translation from strongly grounded AEL to standard AEL.

The relationship between the different notions of groundedness is depicted in figure 1. Each strongly grounded expansion of a set of premises Σ is both a moderately grounded expansions of Σ and an appropriately grounded expansions of Σ (see [11, 15]). It is interesting to observe that appropriate groundedness is not a restriction of moderate groundednes, but these two notions of groundedness are actually incomparable. In fact, we have already seen that there are sets of premises, such as $\Sigma_1 = \{\neg Lp \rightarrow q,\, Lp \rightarrow p\}$ from above, which admit moderately grounded expansions ($T' = E(\{p\})$) that are not appropriately grounded. On the other hand, and this is more curious, one can also find sets of premises with expansions that are appropriately grounded but not moderately grounded (see also [15]). For instance, let $\Sigma_2 = \{\neg L \neg Lp \rightarrow p\}$. Then Σ_2 has two appropriately grounded expansions (i.e., $\mathbf{N}$-expansions): $T_1 = E(\emptyset)$ and $T_2 = E(\{p\})$. T_2 is not moderately grounded. Note, however, that Σ_2 is not the translation of any default theory according to any usual translation scheme. If we restrict our attention to sets of premises Σ that are syntactic counterparts of default theories according to an appropriate translation scheme, then each $\mathbf{N}$-expansion of Σ is also moderately grounded. (More details, and a formal proof of the last statement can be found in [15].)

5 Rationale of the Nonmodular Translation

Though there is no modular faithful translation from default logic to AEL, we are able to construct a nonmodular faithful translation which, in addition, is polynomial-time computable. As one could expect, this translation is rather involved. Therefore, in this section, we give a global overview of how we construct the nonmodular translation. Details are deferred to later sections.

Marek and Truszczyński [15, 34, 33] have exhibited different faithful polynomial translations from default logic to nonmonotonic $\mathbf{N}$, a variant of AEL which we have already mentioned in the previous section and which will be formally defined in the next section. We will use a translation t from default logic to nonmonotonic $\mathbf{N}$ as an intermediate step in our development. The translation t we use is just a slightly modified version of the translation introduced by Truszczyński in [34]. Our modification just makes sure that the translation also works for the inconsistent expansion, which is disregarded in [15, 34, 33]. The translation t will be described in Section 9. Since default logic is translatable via t to nonmonotonic $\mathbf{N}$, it suffices to establish a faithful translation h from nonmonotonic logic $\mathbf{N}$ to standard AEL. Then, our desired translation e from default logic to standard AEL is obtained by composing t and h, i.e., $e = h \circ t$. This is illustrated in Figure 2.

$$
\begin{tikzcd}
\text{Default Theory} & \text{Nonmon.-}\mathbf{N}\text{ Theory} & \text{AEL Theory} \\
\langle D, W\rangle \arrow[r, "t"] \arrow[rr, bend right, "e = h \circ t"'] & t(\langle D, W\rangle) \arrow[r, "h"] & h(t(\langle D, W\rangle))
\end{tikzcd}
$$

Figure 2: Schema of transformation composition

The more difficult part of the translation is the translation h from nonmonotonic logic $\mathbf{N}$ to standard AEL. This translation will be described intuitively in the rest of this section and more formally in the next three sections.

As pointed out in Section 4, nonmonotonic $\mathbf{N}$ is stronger than standard AEL in the following sense. Each $\mathbf{N}$-expansion of a set of premises Σ is also a standard-AEL-expansion. On the other hand, Σ may have several additional standard-AEL expansions which are not $\mathbf{N}$-expansions. Since, as explained, these additional expansions are not appropriately grounded, they should be eliminated.

In order to get rid of the undesired expansions, we define, for each theory Σ a formula $G(\Sigma)$, the so called *grounding formula*. In the context of a particular standard-AEL expansion, $G(\Sigma)$ has the following intuitive meaning:

$$\mathbf{G}(\boldsymbol{\Sigma}) \;\equiv\; \text{“I am an } \mathbf{N}\text{-expansion of } \boldsymbol{\Sigma}\text{”}.$$

In other terms, in the context of each AEL-expansion E of Σ, the formula $G(\Sigma)$ is true if E is a $\mathbf{N}$-expansion, i.e., if E is appropriately grounded, and false otherwise.

It is then easy to cut off the inappropriately grounded expansions. The trick is to add the formula $LG(\Sigma)$ to Σ. $LG(\Sigma)$ will be a tautology in the context of an $\mathbf{N}$-expansion of Σ and therefore harmless. In the context of a consistent inappropriately grounded expansion of Σ, however, $LG(\Sigma)$ evaluates to $L\mathit{false}$, which is inconsistent and forces the expansion to vanish if added to it. It therefore holds that the $\mathbf{N}$-expansions of Σ exactly correspond to the standard-AEL-expansions of $\Sigma \cup \{LG(\Sigma)\}$. Our transformation h is thus simply obtained by adding $LG(\Sigma)$ to Σ.

We are thus able to enforce appropriate groundedness by just adding a single belief $LG(\Sigma)$ to a theory, where $LG(\Sigma)$ informally states “I believe that I am an appropriately grounded (i.e., $\mathbf{N}$-) expansion of Σ”. This strong power of beliefs has interesting philosophical interpretations that we will briefly discuss in Section 10.

The most difficult part of the translation process consists in the construction of the formula $G(\Sigma)$ for each given set of premises Σ. Recall that $G(\Sigma)$ should be true in each AEL-expansion of Σ which is also an $\mathbf{N}$-expansion of Σ, and false in each AEL-expansion of Σ that is not an $\mathbf{N}$-expansion of Σ. By adding $LG(\Sigma)$ to Σ, we get a new set of premises whose expansions are precisely the $\mathbf{N}$-expansions of Σ. We conclude this section by describing informally how $G(\Sigma)$ is constructed.

In order to construct $G(\Sigma)$, we exploit a finite characterization of $\mathbf{N}$-expansions that is related to the finite characterization of AEL-expansions described in Section 2. Recall from Proposition 2.1 that for each set of premises Σ there is a one-to-one correspondence between the stable expansions of Σ and the Σ-full sets. Each Σ-full set Λ corresponds to a stable expansion $SE_\Sigma(\Lambda)$ of Σ and vice-versa. Recall that Λ, being a subset of the finite set Σ^L, consists of a finite set of modal atoms. Let Λ^+ denote the positive modal atoms in Λ and let Λ^- denote the negative modal atoms atoms in Λ. The following characterization of $\mathbf{N}$-expansions is known and will be stated more formally in section 6.

$SE_\Sigma(\Lambda)$ is an $\mathbf{N}$-expansion of Σ iff $\Lambda^+ \subseteq cons_\mathbf{N}(\Sigma \cup \Lambda^-)$.

Note that this characterization – in the finite microcosm of Σ^L – precisely reflects the basic requirements of appropriate groundedness that the (infinite) solutions of the fixed-point equation of nonmonotonic logic $\mathbf{N}$ must fulfill. In fact, the condition states that the positive epistemic knowledge must be derivable via $cons_\mathbf{N}$ from the set of premises plus the negative assumptions.

$G(\Sigma)$ will be defined in a way such that for each stable expansion E of Σ that corresponds to a Σ-full set Λ, $G(\Sigma)$ is true in E iff $\Lambda^+ \subseteq cons_\mathbf{N}(\Sigma \cup \Lambda^-)$, i.e., iff E is an $\mathbf{N}$-expansion of Σ.

We use two main steps in order to find a formula $G(\Sigma)$ with this property.

First, in Section 7, we show that entailment in logic $\mathbf{N}$ can be expressed within classical propositional logic. More precisely, we prove that the problem of deciding $\phi \in cons_\mathbf{N}(S)$ can be translated in polynomial time to a purely propositional formula $F(S, \phi)$ which is a propositional tautology iff $\phi \in cons_\mathbf{N}(S)$.

Second, in Section 8, we define $G(\Sigma)$ as a formula of the form $F(\alpha, \beta)$, where α and β are particular, rather involved, modal formulas depending on Σ. The formulas α and β are constructed in a way such that, in the context of any expansion $SE_\Sigma(\Lambda)$ of Σ, $F(\alpha, \beta)$ becomes equivalent to $F(\Sigma \cup \Lambda^-, \Lambda^+)$. Thus, in each expansion $E = SE_\Sigma(\Lambda)$ of Σ, $G(\Sigma)$ is true iff $\Lambda^+ \subseteq cons_\mathbf{N}(\Sigma \cup \Lambda^-)$, i.e., iff E is an $\mathbf{N}$-expansion of Σ.

6 Modal logic N: Overview

In this section we present an overview of both the monotonic version and the nonmonotonic version of the modal logic $\mathbf{N}$. We also present a number of previous results that will be used in the following sections.

Several researchers [11, 21, 15, 23, 12] have pointed-out that the notion of stable AE-expansion is too permissive since stable AE-expansions are not sufficiently "grounded" in the premises. For example, the set $\Sigma = \{Lp \rightarrow p\}$ has two stable AE-expansions, $E(\emptyset)$ and $E(\{p\})$. It can be argued that the second AE-expansion is not enough grounded in the premises, since Lp is introduced as an assumption in the fixed-point equation of Definition 2.3. Hence the belief of p is based solely on the assumption that p is believed.

Consequently, various more restricted versions of AEL have been proposed. These logics can be regarded as logics of *knowledge* rather than of belief, since they all tend to avoid ungroundedness. One of these systems, nonmonotonic logic **N** by Marek and Truszczyński [15] will be discussed here. This logic is the nonmonotonic version of the modal logic **N**, which is also called the *pure logic of necessitation* [5].

The (monotonic) modal logic **N** is defined over the language $\mathcal{L}_L$ just as standard AEL. **N** is defined by the axioms of propositional calculus, modus ponens and the necessitation rule $\phi/L\phi$. **N** can be considered as the simplest modal logic. A detailed study of **N** is carried-out in [5], where also an appropriate Kripke semantics for this logic is defined. The consequence relation according to **N** is denoted by $\models_{\overline{\mathbf{N}}}$. More precisely, for $S \subseteq \mathcal{L}_L$ and $\phi \in \mathcal{L}_L$ we write $S \models_{\overline{\mathbf{N}}} \phi$ iff there is a an **N**-proof of ϕ from S, i.e., a sequence of formulas $\phi_1, \phi_2, \ldots, \phi_m$ such that $\phi_m = \phi$ and for all $i \leq m$, either ϕ_i is in S, or ϕ_i is obtained from the preceding formulas in the proof by modus ponens or the necessitation rule. The consequence set operator $cons_{\mathbf{N}}$ of **N** is defined in the usual way: if $S \subseteq \mathcal{L}_L$, then $cons_{\mathbf{N}}(S) = \{\phi \mid S \models_{\overline{\mathbf{N}}} \phi\}$. Clearly, $cons_{\mathbf{N}}$ is a monotonic operator.

The following proposition shows that the relevant atoms for proving $S \models_{\overline{\mathbf{N}}} \phi$ are the subformulas of type $L\psi$ of S and ϕ. The proposition is easy to verify; a formal proof is given in [7].

Definition 6.1 *An L-proof of ϕ from S is a finite sequence $L\psi_1, L\psi_2, \ldots, L\psi_n$ of formulas from $S^L \cup \phi^L$ such that for $1 \leq i \leq n$, $S \cup \{L\psi_j | 1 \leq j < i\} \models \psi_i$ and $S \cup \{L\psi_1, \ldots, L\psi_n\} \models \phi$.*

Proposition 6.1 *$S \models_{\overline{\mathbf{N}}} \phi$ iff there is an L-proof of ϕ from S.*

The *nonmonotonic* version of **N** (also called *iterative AEL*) is defined from monotonic **N** according to McDermott's general definition scheme for nonmonotonic modal logics [18]. The basic concept is the one of an **N**-expansion.

Definition 6.2 ([15]) *$\Delta \subseteq \mathcal{L}_L$ is an **N**-expansion (also called iterative expansion) of Σ iff Δ satisfies the following fixed point equation.*

$$\Delta = cons_{\mathbf{N}}(\Sigma \cup \neg L\overline{\Delta}).$$

The following proposition was was shown in [15]:

Proposition 6.2 *Each **N**-expansion of a set $\Sigma \subseteq \mathcal{L}_L$ is also a stable AEL-expansion of Σ.*

The converse does not hold. In particular, the stable AEL-expansions which are not **N**-expansions are inappropriately grounded. For example, the stable AE-expansion $E(\{p\})$ of the theory $\Sigma = \{Lp \rightarrow p\}$ is not an **N**-expansion. As noted earlier, $E(\{p\})$ is not appropriately grounded in Σ.

The following proposition, showing that **N**-expansions can be characterized by special properties of their kernels, was shown in [7], and, in a slightly weaker form, in [13] (Theorems 3.2 and 3.5).

Proposition 6.3 *Let Σ be a set of premises and let Λ be a Σ-full set. $SE_\Sigma(\Lambda)$ is an* **N**-*expansion of Σ iff*

$$\Sigma \cup \Lambda^- \models_{\mathbf{N}} \Lambda^+$$

where $\Lambda^- = \{\neg L\phi \,|\, \neg L\phi \in \Lambda\}$ and $\Lambda^+ = \{L\phi \,|\, L\phi \in \Lambda\}$.

The relationship between default logic and nonmonotonic logic **N** will be discussed in Section 9.

Before concluding this section we state a simple technical lemma on propositional logic which will be used later on.

Lemma 6.1

a. *If $F_1, F_2, \ldots, F_n$ are propositional formulas such that for $i \neq j$, F_i and F_j have no propositional variable in common, then $F_1 \vee F_2 \cdots \vee F_n$ is a propositional tautology iff for at least one i, $1 \leq i \leq n$, F_i is a propositional tautology.*

b. *If F is a propositional formula and F^* is obtained from F by uniformly replacing each propositional variable by a new distinct propositional variable, then F is a propositional tautology iff F^* is a propositional tautology.*

c. *If $F \cup G \models H$ and G is consistent and has no propositional variables (atoms) in common with F and with H, then $F \models H$*

7 Expressing N-inference in Classical Propositional Logic

In this section we show how inference problems in modal logic **N** can be polynomially transformed into the problem of tautology-checking in ordinary propositional logic.

Definition 7.1 *Let $S \subset \mathcal{L}_L$ be a finite set formulas and let $\phi \in \mathcal{L}_L$ be a formula. A set $K \subseteq S^L \cup \phi^L$ is called S-ϕ-*closed *if $S \cup K \not\models \phi$ and for each $L\psi \in (S^L \cup \phi^L) - K$, $S \cup K \not\models \psi$.* [4]

Example 7.1 *Let $S = \{p,\, Lp \rightarrow q,\, Lr \rightarrow p\}$. Then there exist two S-r-closed sets, namely $\{Lp\}$ and $\{Lp, Lr\}$. The only S-Lr-closed set is $\{Lp\}$. There are no S-p-closed sets, no S-Lp-closed sets and no S-Lq-closed sets.*

Lemma 7.1 *$S \models_{\mathbf{N}} \phi$ iff there is no S-ϕ-closed set.*

PROOF. *(Only-if.)* Assume that $S \models_{\mathbf{N}} \phi$ and that there exists a S-ϕ-closed set K. By Proposition 6.1 there exists an L-proof $L\psi_1, \ldots, L\psi_n$ of ϕ from S. We show by induction that for $1 \leq i \leq n$, $L\psi_i \in K$.
Induction Base. Since $S \models \psi_1$, it must hold that $L\psi_1 \in K$, for otherwise we would have $L\psi_1 \in (S^L \cup \phi^L) - K$ and $S \cup K \models \psi_1$, which is a contradiction to the assumption that K is S-ϕ-closed.
Induction Step. Assume $L\psi_j \in K$ for $1 \leq j \leq i < k$. We show that $L\psi_{i+1} \in K$. Indeed, $S \cup \{L\psi_j | 1 \leq j \leq i\} \models \psi_{i+1}$. Therefore it must hold that $L\psi_{i+1} \in K$, for otherwise we would have $L\psi_{i+1} \in (S^L \cup \phi^L) - K$ and $S \cup K \models \psi_{i+1}$, which is a contradiction to the assumption that K is S-ϕ-closed.
Thus, for $1 \leq i \leq n$, $L\psi_i \in K$. But then, since $L\psi_1, \ldots, L\psi_n$ is an L-proof of ϕ from S, it holds that $S \cup \{L\psi_1 \ldots L\psi_n\} \models \phi$, and so $S \cup K \models \phi$. But this is a contradiction to our assumption that K is a S-ϕ-closed set. Therefore, if $S \models_{\mathbf{N}} \phi$, there exists no S-ϕ-closed set.

(If.) Now assume $S \not\models_{\mathbf{N}} \phi$. We show that there exists a S-ϕ-closed set. Let $K = \{L\theta \in S^L \cup \phi^L \mid S \models_{\mathbf{N}} L\theta\}$. Clearly, $S \cup K \not\models \phi$, for otherwise it would hold that $S \models_{\mathbf{N}} \phi$. Furthermore, if $L\psi \in (S^L \cup \phi^L) - K$, then $S \cup K \not\models \psi$ because otherwise, by application of the necessitation rule it would hold that $S \cup K \models_{\mathbf{N}} L\psi$ (contradiction). Thus, K is a S-ϕ-closed set. $\square$

The next theorem settles the complexity problem for derivability in logic **N**, which was left open in [5][5].

[4]The term *S-ϕ-closed* has been chosen because any set K having this property is closed under the operator C defined as follows: For each $X \subseteq S^L$, $C(X) = \{Y | S \cup X \models_{\mathbf{N}} Y\} \cap (S^L \cup \phi^L)$. Note that it even holds $C(K) = K$.

[5]Note that this result was recently independently established by means of a completely different proof by Schwarz and Truszczyński [28].

Theorem 7.2 *For $\phi \in \mathcal{L}_L$ and finite $S \subset \mathcal{L}_L$, the problem to decide whether $S \models_{\overline{\mathbf{N}}} \phi$ holds is* **coNP***-complete.*

PROOF. The problem is obviously **coNP**-hard, since the tautology problem for ordinary propositional logic can be immediately transformed into the validity problem in logic **N**. Indeed, it is easy to see that a propositional (i.e., objective) formula F is a propositional tautology iff it is valid in **N**, i.e., iff $\models_{\overline{\mathbf{N}}} F$. It remains to show that the problem of testing whether $S \models_{\overline{\mathbf{N}}} \phi$ holds is in **coNP**. We show this by proving that the complementary problem, i.e., testing whether $S \not\models_{\overline{\mathbf{N}}} \phi$ is in **NP**. To prove that $S \not\models_{\overline{\mathbf{N}}} \phi$, by Lemma 7.1, it suffices to show that there exists a S-ϕ-closed set K. This can be done in nondeterministic polynomial time as follows. Guess a subset K of $S^L \cup \phi^L$ and guess for each formula $L\psi \in (S^L \cup \phi^L) - K$ a truth value assignment σ_ψ to all (modal and non-modal) atoms occurring in $S \cup \{\phi\}$ such that σ_ψ makes $S \cup K$ true and falsifies ψ. Furthermore, guess a truth value assignment σ_ϕ which makes $S \cup K$ true and falsifies ϕ. Each such truth value assignment σ_θ expresses the fact that $S \cup K \not\models \theta$. Hence K is S-ϕ-closed iff these assignments exist. Clearly, the number of assignments σ_ψ we have to guess is bounded by the number of L-subformulas of $S \cup \{\phi\}$ which is at most linear in the size of the decision problem. Furthermore, for each assignment σ_θ, checking that it satisfies $S \cup K$ and falsifies θ is clearly feasible in polynomial time. Thus, proving that $S \not\models_{\overline{\mathbf{N}}} \phi$ is in **NP** and hence proving that $S \models_{\overline{\mathbf{N}}} \phi$ is in **coNP**. $\square$

Theorem 7.2 guarantees that the modal inference problem $S \models_{\overline{\mathbf{N}}} \phi$ can be translated in polynomial time into the problem of checking whether an ordinary propositional formula $F(S, \phi)$ is a tautology of propositional logic. To show how such a formula can be *effectively* constructed, we proceed in two steps. In the first step we construct a formula $F^+(S, \phi)$ over the language $\mathcal{L}_L$ such that $S \models_{\overline{\mathbf{N}}} \phi$ iff $F^+(S, \phi)$ is a propositional tautology in $\mathbf{PL}(\mathcal{L}_L)$. In the second step, we transform $F^+(S, \phi)$ by a simple renaming of variables into an ordinary propositional formula $F(S, \phi)$ (over language $\mathcal{L}$) such that $F(S, \phi)$ is a tautology in $\mathbf{PL}(\mathcal{L})$ iff $F^+(S, \phi)$ is a tautology in $\mathbf{PL}(\mathcal{L}_L)$

Since we will need a number of additional propositional variables for constructing $F^+(S, \phi)$, let us adopt the following convention. If $p \in \mathcal{L}$ is an ordinary propositional variable, then for each formula $\gamma \in \mathcal{L}_L$, p^γ denotes a new distinct propositional variable from $\mathcal{L}$. In particular, if $\gamma \neq \delta$, then p^γ and p^δ shall denote different propositional variables; if p and q are different propositional variables and η and θ are (possibly identical) formulas from $\mathcal{L}_L$, then p^η and q^θ shall denote different propositional variables. For $\phi, \gamma \in \mathcal{L}_L$, ϕ^γ denotes the formula obtained from ϕ by uniformly replacing each occurrence (not just on the surface, but at all levels of L-nesting) of any ordinary (i.e., non-modal) propositional variable p in ϕ by p^γ. For example, if $\phi = Lp \to (p \vee q)$, then $\phi^{Lq\to q} = Lp^{Lq\to q} \to (p^{Lq\to q} \vee q^{Lq\to q})$. If $S \subset \mathcal{L}_L$ is a finite set of formulas and $\gamma \in \mathcal{L}_L$, then $S^\gamma = \bigwedge_{\psi\in S} \psi^\gamma$.

Definition 7.2 *Let $S \subset \mathcal{L}_L$ be finite and $\phi \in \mathcal{L}_L$. For each $L\psi \in S^L \cup \phi^L$, let u_ψ denote a new distinct propositional variable. Let z be a propositional variable not occurring in S or ϕ. Then $F^+(S,\phi)$ is defined to be the formula*

$$\left(\left(S^z \wedge \bigwedge_{L\psi\in S^L\cup\phi^L} (u_\psi \to L\psi^z)\right) \to \phi^z\right) \quad \vee$$
$$\bigvee_{L\gamma\in S^L\cup\phi^L} \left(\neg u_\gamma \wedge \left(\left(S^\gamma \wedge \bigwedge_{L\psi\in S^L\cup\phi^L} (u_\psi \to L\psi^\gamma)\right) \to \gamma^\gamma\right) \right).$$

Theorem 7.3 *$S \models_{\mathbf{N}} \phi$ iff $F^+(S,\phi)$ is a tautology in $\mathbf{PL}(\mathcal{L}_L)$*

PROOF. By Lemma 7.1 it is sufficient to show that there exists a S-ϕ-closed set K iff $F^+(S,\phi)$ is not a tautology in $\mathbf{PL}(\mathcal{L}_L)$.

Assume K is a S-ϕ-closed set. Define a truth value assignment σ to the variables u_ψ for $L\psi \in S^L \cup \phi^L$ as follows: $\sigma(u_\psi) = \mathit{true}$ if $L\psi \in K$ and $\sigma(u_\psi) = \mathit{false}$ if $L\psi \notin K$. Let $F^+_\sigma(S,\phi)$ be the formula resulting from $F^+(S,\phi)$ by replacing each u_ψ by $\top$ if $\sigma(u_\psi) = \mathit{true}$ and by $\bot$ if $\sigma(u_\psi) = \mathit{false}$. Clearly, $F^+(S,\phi)$ is a tautology only if $F^+_\sigma(S,\phi)$ is a tautology. Let $K^{\mathbf{c}} = (S^L \cup \phi^L) - K$. $F^+_\sigma(S,\phi)$ is equivalent to the formula

$$\left((S^z \wedge \bigwedge_{L\psi\in K} L\psi^z) \to \phi^z\right) \;\vee \bigvee_{\gamma\in K^{\mathbf{c}}} \left((S^\gamma \wedge \bigwedge_{L\psi\in K} L\psi^\gamma) \to \gamma^\gamma \right). \tag{8}$$

By suitable rewriting we get

$$F^+(S,\phi) \;\equiv\; ((S^z \wedge K^z) \to \phi^z) \vee \bigvee_{L\gamma\in K^{\mathbf{c}}} ((S^\gamma \wedge K^\gamma) \to \gamma^\gamma)). \tag{9}$$

Since the disjuncts of this formula have no propositional variables in common, by Lemma 6.1.a, $F^+(S,\phi)$ is a tautology iff either $S^z \wedge K^z \to \phi^z$ holds, or for some $L\gamma \in K^{\mathbf{c}}$ it holds that $(S^\gamma \vee K^\gamma) \to \gamma^\gamma)$. By Lemma 6.1.b and the deduction theorem, $F^+(S,\phi)$ is a tautology iff either $S \wedge K \models \phi$ or for some $L\gamma \in K^{\mathbf{c}}$, $S \wedge K \models \gamma$. But since K is S-ϕ-closed, this cannot be the case. Thus $F^+(S,\phi)$ is not a tautology. Conversely, assume that $F^+(S,\phi)$ is not a tautology. Then there exists a truth value assignment σ falsifying $F^+(S,\phi)$. Let $K = \{L\psi \mid \sigma(u_\psi) = true\}$. As before let $F^+_\sigma(S,\phi)$ be the formula resulting from $F^+(S,\phi)$ by replacing each u_ψ by $\top$ if $\sigma(u_\psi) = true$ and by $\perp$ if $\sigma(u_\psi) = false$. Clearly, $F^+_\sigma(S,\phi)$ is not a tautology. If, as before, we define $K^{\mathbf{c}} = (S^L \cup \phi^L) - K$, then $F^+_\sigma(S,\phi)$ is identical to formula (9) from above. Since the disjuncts of this formula have no propositional variables in common, and since the formula is not a tautology, by Lemma 6.1.a, it neither holds that $S^z \wedge K^z \to \phi^z$ is a tautology nor for some $L\gamma \in K^{\mathbf{c}}$ does it hold tht $(S^\gamma \vee K^\gamma) \to \gamma^\gamma)$ is a tautology. By Lemma 6.1.b and the deduction theorem, we conclude that $S \wedge K \not\models \phi$ and for each $L\gamma \in K^{\mathbf{c}}$, $S \wedge K \not\models \gamma$. Thus K is S-ϕ-closed. □

Finally, we transform $F^+(S,\phi)$ into a formula $F(S,\phi)$ which does not contain any modal atoms. If $L\psi$ is a modal atom, then $[L\psi]$ denotes a new distinct propositional variable from $\mathcal{L}$. If U is a formula of $\mathbf{PL}(\mathcal{L}_L)$, then $[U]$ is the formula obtained from U by uniformly replacing each occurrence of any subformula of the form $L\psi$ of U not appearing in the scope of some L-operator by $[L\psi]$. For example, $[(Lp \to LLp) \vee (Lp \to q)]$ stands for $([Lp] \to [LLp]) \vee ([Lp] \to q)$, where $[Lp]$ and $[LLp]$ denote two distinct propositional variables different from p, q and all other previously used propositional variables.

Definition 7.3 *$F(S,\phi)$ is defined to be the formula $[F^+(S,\phi)]$.*

Note that $F(S,\phi) \in \mathcal{L}$, since this formula is obtained from $F^+(S,\phi)$ by uniformly replacing all modal atoms $L\psi$ by ordinary propositional atoms $[L\psi]$.

Theorem 7.4 *$S \models_{\overline{\mathbf{N}}} \phi$ iff $F(S,\phi)$ is a tautology in $\mathbf{PL}(\mathcal{L})$.*

PROOF. By Lemma 6.1.b, any formula F is a tautology in $\mathbf{PL}(\mathcal{L}_L)$ iff $[F]$ is a tautology in $\mathbf{PL}(\mathcal{L}_L)$. Furthermore, since $[F] \in \mathcal{L}$, $[F]$ is a $\mathbf{PL}(\mathcal{L}_L)$-tautology iff it is a tautology in $\mathbf{PL}(\mathcal{L})$. Since $F(S,\phi) = [F^+(S,\phi)]$, our theorem follows immediately by Theorem 7.3. □

Note that the size of the formula $F(S,\phi)$ is at most quadratic in the size of S plus the size of ϕ. Thus, the transformation given by Theorem 7.4 is polynomial.

8 Translating Nonmonotonic N into standard AEL

Our aim is, given a set of premises Σ, to find a formula $G(\Sigma)$ such that the consistent stable AE-expansions of $\Sigma \cup LG(\Sigma)$ exactly coincide with the consistent $\mathbf{N}$-expansions of Σ.

The formula $G(\Sigma)$ will enforce any stable AE-expansion to be grounded. More specifically, $G(\Sigma)$ will make sure that for each stable expansion Δ of $\Sigma \cup LG(\Sigma)$ with kernel Λ it holds that $\Sigma \cup \Lambda^- \models_{\overline{\mathbf{N}}} \Lambda^+$, hence, by Proposition 6.3, Δ is an $\mathbf{N}$-expansion.

If $\Sigma \subseteq \mathcal{L}_L$ is a set of premises, then we denote by $\mathbf{CEXP}(\Sigma)$ the set of all consistent standard-AE-expansions of Σ.

The following theorem – fundamental to our transformation – elucidates the relationship between $\mathbf{CEXP}(\Sigma)$ and $\mathbf{CEXP}(\Sigma \cup \{L\alpha\})$for autoepistemic formulas α of a certain type. In particular, the theorem shows, how by adding $L\alpha$ to Σ certain expansions of Σ can be suppressed. Therefore we will refer to it as the *Expansion Cutting Theorem.*

Theorem 8.1 (Expansion Cutting Theorem) *Given a set $\Sigma \subseteq \mathcal{L}_L$ and a formula $\alpha \in \mathcal{L}_L$ fulfilling the following conditions*

1. *$L\alpha \notin \Sigma^L$.*
2. *Any atom of α is either a propositional variable not occurring in Σ or a modal atom $L\psi \in \Sigma^L$.*

it holds that

a.) $\mathbf{CEXP}(\Sigma \cup \{L\alpha\}) \subseteq \mathbf{CEXP}(\Sigma)$

b.) *Let $\Delta \in \mathbf{CEXP}(\Sigma)$ be a stable AE-expansion of Σ with kernel Λ, i.e., $\Delta = SE_\Sigma(\Lambda)$. Let α_Λ be the propositional formula obtained from α by replacing each atom $L\psi$ of α by $\top$ if $L\psi \in \Lambda$ and by $\bot$ if $\neg L\psi \in \Lambda$. Then $\Delta \in \mathbf{CEXP}(\Sigma \cup \{L\alpha\})$ iff α_Λ is a propositional tautology.*

PROOF. Let Σ and α fulfill conditions 1 and 2. Let us abbreviate $\Sigma \cup \{L\alpha\}$ by Σ_* throughout the proof. If Σ is inconsistent, then both Σ and Σ_* have as unique stable AE-expansion the inconsistent expansion $\mathcal{L}_L = E(\{\bot\})$. In this case, $\mathbf{CEXP}(\Sigma) = \mathbf{CEXP}(\Sigma_*) = \emptyset$, and he theorem is vacuously fulfilled. For the rest of this proof we thus assume that Σ is consistent.

Assume $\Delta \in \mathbf{CEXP}(\Sigma_*)$. By Definition 2.3, since $L\alpha \in \Sigma_*$, it must hold that $L\alpha \in \Delta$. It follows that $\alpha \in \Delta$, and hence $L\alpha \in L\Delta$. It thus holds that $\Delta = cons(\Sigma_* \cup L\Delta \cup \neg L\overline{\Delta}) = cons(\Sigma \cup L\Delta \cup \neg L\overline{\Delta})$ and so Δ is a stable AE-expansion of Σ, which proves part a) of the theorem.

Now let Λ_* be the kernel of Δ w.r.t. Σ_*. Then $\Lambda = \Lambda_* - \{L\alpha\}$ is the kernel of Δ w.r.. Σ. Since Λ_* is a Σ_*-full set, it holds that $\Sigma_* \cup \Lambda_* \models \alpha$ (see Definition 2.5). Since $\Sigma_* \cup \Lambda_* = \Sigma \cup \Lambda \cup \{L\alpha\}$, it follows that $\Sigma \cup \Lambda \cup \{L\alpha\} \models \alpha$. Note that the atom $L\alpha$ does neither occur in α nor as a modal atom in $\Sigma \cup \Lambda$ (condition 1). Thus, by simple interpolation properties (see Proposition 6.1.c), it follows that $\Sigma \cup \Lambda \models \alpha$. Since each atom $L\psi$ of α is an element of Σ^L, either $L\psi$ r its negation $\neg L\psi$ occurs in the kernel Λ. Hence, in the formula $\Sigma \cup \Lambda \models \alpha$, we may replace each atom $L\psi$ in α by $\top$ if $L\psi \in \Lambda$ and by $\perp$ if $\neg L\psi \in \Lambda$, getting $\Sigma \cup \Lambda \models \alpha_\Lambda$. Note that α_Λ is a purely propositional formula having no propositional variables in common with $\Sigma \cup \Lambda$. Thus α_Λ must be a propositional tautology. This proves the *only-if* direction of part b) of the theorem.

In order to show the *if* direction of part b), assume $\Delta \in \mathbf{CEXP}(\Sigma)$. Then there exists a unique Σ-full set Λ such that $\Delta = SE_\Sigma(\Lambda)$. Now assume that α_Λ is a tautology. Note that for $L\psi \in \Lambda$, $\Lambda \models (L\psi \leftrightarrow \top)$, and for $\neg L\psi \in \Lambda$, $\Lambda \models L\psi \leftrightarrow \perp)$. Hence, under the assumption that Λ is valid, by the substitution theorem of propositional logic, we may replace in α_Λ the occurrences of $\top$ ad $\perp$ by appropriate literals from $\Lambda^{\pm L}$ getting α. Thus, $\Lambda \models \alpha$. Since $\Lambda \subset \Delta$, it follows that $\Delta \models \alpha$, hence $\alpha \in \Delta$ and thus $L\alpha \in L\Delta$. Thus $\Delta = cons(\Sigma \cup L\Delta \cup \neg L\overline{\Delta}) = cons(\Sigma \cup \{L\alpha\} \cup L\Delta \cup \neg L\overline{\Delta})$, therefore $\Delta \in \mathbf{CEXP}(\Sigma \cup \{L\alpha\})$. $\square$

We are now ready for defining the "grounding condition" $G(\Sigma)$. At a first glance, this condition appears to be a rather long and inintelligible formula. However, the meaning of the formula will soon become clear. In particular, in the proof of the subsequent Theorem 8.2, it will become clear that $G(\Sigma)$ is drastically simplified in the context of a stable expansion of Σ. An immediate understanding of this formula is therefore not necessary.

Definition 8.1 (Grounding Condition) *Let $\Sigma \subset \mathcal{L}_L$ be a finite set of premises. Assume that z is a propositional variable not occurring in Σ. Then $G(\Sigma)$ is the formula*

$$\left(\left([\Sigma^z] \wedge \bigwedge_{L\psi \in \Sigma^L} (\neg L\psi \rightarrow \neg[L\psi^z]) \wedge \bigwedge_{L\psi \in \Sigma^L} \big(u_\psi \rightarrow [L\psi^z]\big) \right) \rightarrow \bigwedge_{L\psi \in \Sigma^L} (L\psi \rightarrow [L\psi^z]) \right) \vee$$
$$\bigvee_{L\gamma \in \Sigma^L} \left(\neg u_\gamma \wedge \left(\left([\Sigma^\gamma] \wedge \bigwedge_{L\psi \in \Sigma^L} (\neg L\psi \rightarrow \neg[L\psi^\gamma]) \wedge \bigwedge_{L\psi \in \Sigma^L} \big(u_\psi \rightarrow [L\psi^\gamma]\big) \right) \rightarrow [\gamma^\gamma] \right) \right)$$

As we will show in the proof of the next theorem, in the context of each particular AEL-expansion of Σ with kernel Λ, the grounding condition $G(\Sigma)$ becomes equivalent to the formula $F(\Sigma \cup \Lambda^-, \Lambda^+)$. Thus, by Theorem 7.4, $G(\Sigma)$ is valid iff $\Sigma \cup \Lambda^- \models_{\overline{\mathbf{N}}} \Lambda^+$, i.e., iff the expansion corresponding to Λ is an **N**-expansion. The Expansion Cutting Theorem (Thm. 8.1) then guarantees that the theory $\Sigma \cup LG(\Sigma)$ has as standard AEL-expansions exactly the **N**-expansions of Σ.

Theorem 8.2 *Let $\Sigma \subset \mathcal{L}_L$ be a finite set of premises. Then Δ is a consistent* **N**-*expansion of Σ iff Δ is a consistent stable AE-expansion of $\Sigma \cup \{LG(\Sigma)\}$.*

PROOF. Let $\Sigma_* = \Sigma \cup \{LG(\Sigma)\}$. Note that conditions 1) and 2) of Theorem 8.1 both apply, hence $\mathbf{CEXP}(\Sigma_*)$ consists exactly of those consistent stable AE-expansions of Σ which have a kernel Λ such that $G(\Sigma)_\Lambda$ is a propositional tautology. Now recall that $G(\Sigma)_\Lambda$ is obtained by replacing in $G(\Sigma)$ each modal atom $L\psi$ by $\top$ if $L\psi \in \Lambda^+$ and by $\bot$ if $L\psi \in \Lambda^-$. Thus after these replacements the term

$$\bigwedge_{L\psi \in \Sigma^L} (\neg L\psi \rightarrow \neg[L\psi^z])$$

of $G(\Sigma)$ becomes equivalent to

$$\bigwedge_{\neg L\psi \in \Lambda} (\neg[L\psi^z])$$

which in turn is equivalent to $[(\Lambda^-)^z]$.
Similarly, the term

$$\bigwedge_{L\psi \in \Sigma^L} (L\psi \rightarrow [L\psi^z])$$

becomes equivalent to $[(\Lambda^+)^z]$, and each term

$$\bigwedge_{L\psi \in \Sigma^L} (\neg L\psi \rightarrow \neg[L\psi^\gamma])$$

becomes equivalent to $[(\Lambda^-)^\gamma]$.
Thus $G(\Sigma)_\Lambda$ is equivalent to

$$\left(\left([\Sigma^z] \wedge [(\Lambda^-)^z] \ \wedge \bigwedge_{L\psi \in \Sigma^L} (u_\psi \rightarrow [L\psi^z]) \right) \rightarrow [(\Lambda^+)^z] \right) \ \vee$$
$$\bigvee_{L\gamma \in \Sigma^L} \left(\neg u_\gamma \ \wedge \left(\left([\Sigma^\gamma] \wedge [(\Lambda^-)^\gamma] \ \wedge \bigwedge_{L\psi \in \Sigma^L} (u_\psi \rightarrow [L\psi^\gamma]) \right) \rightarrow [\gamma^\gamma] \right) \right).$$

Now note that $\Sigma^L = (\Sigma \cup \Lambda^-)^L \cup (\Lambda^+)^L$ and $[\Sigma^z] \wedge [(\Lambda^-)^z] = [(\Sigma \cup \Lambda^-)^z]$. Thus $G(\Sigma)_\Lambda$ is equivalent to

$$\left(\left([(\Sigma \cup \Lambda^-)^z] \ \wedge \bigwedge_{L\psi \in (\Sigma \cup \Lambda^-)^L \cup (\Lambda^+)^L} (u_\psi \rightarrow [L\psi^z]) \right) \rightarrow [(\Lambda^+)^z] \right) \ \vee$$

$$\bigvee_{\gamma\in(\Sigma\cup\Lambda^-)^L\cup(\Lambda^+)^L}\left(\neg u_\gamma \wedge \left(\left([(\Sigma\cup\Lambda^-)^\gamma] \wedge \bigwedge_{L\psi\in(\Sigma\cup\Lambda^-)^L\cup(\Lambda^+)^L}(u_\psi \to [L\psi^\gamma])\right) \to [\gamma^\gamma]\right)\right).$$

But the last formula is precisely $F(\Sigma\cup\Lambda^-,\Lambda^+)$. Hence, by Theorem 7.4 $G(\Sigma)_\Lambda$ is a tautology iff $\Sigma\cup\Lambda^- \models_{\overline{\mathbf{N}}} \Lambda^+$. It thus holds that **CEXP**(Σ_*) consists exactly of all those consistent stable AE-expansions of Σ whose kernel Λ satisfies $\Sigma\cup\Lambda^- \models_{\overline{\mathbf{N}}} \Lambda^+$. By Propositions 6.2 and 6.3 these are exactly the consistent **N**-expansions of Σ. $\Box$

We have thus established a polynomial-time translation from Nonmonotonic **N** to Standard AEL. Let us conclude this Section with an example.

Example 8.1 *Let $\Sigma = \{Lp \to p\}$. Then $G(\Sigma) =$*

$$\left(\left(([Lp^z]\to p^z)\wedge(\neg Lp\to\neg[Lp^z])\wedge(u_p\to[Lp^z])\right)\to(Lp\to[Lp^z])\right)\vee$$
$$\left(\neg u_p \wedge \left(\left(([Lp^p]\to p^p)\wedge(\neg Lp\to\neg[Lp^p])\wedge(u_p\to[Lp^p])\right)\to p^p\right)\right).$$

As we have noted earlier in this paper, Σ has two stable AE-expansions: $\Delta_1 = E(\emptyset)$ and $\Delta_2 = E(\{p\})$. The kernels of these expansions are respectively: $\Lambda_1 = \{\neg Lp\}$ and $\Lambda_2 = \{Lp\}$. It is easy to see that $G(\Sigma)_{\Lambda_1}$ is a tautology since after replacing Lp by $\bot$ and $\neg Lp$ by $\top$, the term $(Lp \to [Lp^z])$ becomes true and so does the first disjunct, and hence the entire formula evaluates to true. *On the other hand, $G(\Sigma)_{\Lambda_2}$ is not a tautology. To see this, replace Lp by $\top$ and $\neg Lp$ by $\bot$ as required and assign* false *to u_p as well as to $[Lp^z]$, $[Lp^p]$, p^z, and p^p. The formula evaluates to false.* $\Box$

9 Translating Default Logic into standard AEL

Marek and Truszczyński [15, 34] have shown that default logic can be translated into nonmonotonic **N**.

The translation function t defined in [34][6] mapping a default theory $\langle D, W\rangle$ to an autoepistemic theory $t(\langle D, W\rangle)$ is as follows.

Definition 9.1 ([34]) *Let $\langle D, W\rangle$ be a default theory. For each default in $d \in D$ of the form*

$$\frac{\alpha\ :\ M\beta_1,\ M\beta_2,\ldots,M\beta_n}{\omega},$$

[6] The translation function is called *emb* in that paper

$t(d)$ denotes the autoepistemic formula

$$(L\alpha \wedge L\neg L\neg\beta_1 \wedge \ldots \wedge L\neg L\neg\beta_n) \rightarrow \omega.$$

The translation $t(\langle D, W\rangle)$ is defined by

$$t(\langle D, W\rangle) = W \cup \bigcup_{d\in D} \{t(d)\}.$$

Proposition 9.1 ([34]) *The extensions of a default theory $\langle D, W\rangle$ are given by the sets $\Delta \cap \mathcal{L}$, i.e., the objective parts of Δ, where Δ is an* **N***-expansion of $t(\langle D, W\rangle)$.*

By this result, default logic can be polynomially embedded into nonmonotonic logic **N**. By the composition of this embedding and our translation from nonmonotonic **N** to standard AEL, we obtain a polynomial translation mapping each default theory $\langle D, W\rangle$ to an autoepistemic theory

$$emb(\langle D, W\rangle) = t(\langle D, W\rangle) \cup \{LG(t(\langle D, W\rangle))\}.$$

By Proposition 9.1 and Theorem 8.2 it is immediate that the consistent extensions of $\langle D, W\rangle$ exactly correspond to the objective parts of the consistent stable AE-expansions of $emb(\langle D, W\rangle)$. However, as the following example shows, there exists still a subtle problem with *inconsistent* AE-expansions.

Example 9.1 *Consider a default theory $\langle D, W\rangle$, where $D = \{\frac{p:}{p}, \frac{\neg p:}{\neg p}\}$ and $W = \emptyset$. Note that the default rules in D are trivial inference rules (with empty justifications). The unique extension of $\langle D, W\rangle$ is the set $cons(\emptyset)$ of all propositional tautologies. By applying Truszczyński's transformation t, we get the theory $t(\langle D, W\rangle) = \{Lp \rightarrow p,\ L\neg p \rightarrow \neg p\}$. It is easy to see (and formally guaranteed by the results of [34]) that this theory has a unique* **N***-expansion equal to $E(\emptyset)$. Now consider our translation*

$$emb(\langle D, W\rangle) = \{Lp \rightarrow p,\ L\neg p \rightarrow \neg p,\ LG(t(\langle D, W\rangle))\}.$$

This theory has two stable AE-expansions in standard AEL, namely $E(\emptyset)$ and the inconsistent expansion $E(\{\perp\}) = \mathcal{L}_L$. $\Box$

This problem can be settled by applying minor "cosmetics" to Truszczyński's transformation t.

Definition 9.2 *Let $\langle D, W\rangle$ be a default theory. Let v be a propositional variable from $\mathcal{L}$ which does not occur in $\langle D, W\rangle$. For each default in $d \in D$ of the form*

$$\frac{\alpha \ : \ M\beta_1,\, M\beta_2, \ldots, M\beta_n}{\omega},$$

let $t_v(d)$ denote the autoepistemic formula

$$(\neg Lv \wedge L\alpha \wedge L\neg L\neg\beta_1 \wedge \ldots \wedge L\neg L\neg\beta_n) \rightarrow \omega.$$

Define the translation $t_v(\langle D, W\rangle)$ by

$$t_v(\langle D, W\rangle) = W \cup \bigcup_{d\in D} \{t_v(d)\}.$$

We are now able to define the definitive translation function e between default logic and standard AEL.

Definition 9.3 *If $\langle D, W\rangle$ is a default theory then*

$$e(\langle D, W\rangle) = t_v(\langle D, W\rangle) \cup \{LG(t_v(\langle D, W\rangle))\}.$$

Theorem 9.1 (Faithful Translation) *The extensions of a default theory $\langle D, W\rangle$ are given by the sets $\Delta\cap\mathcal{L}$, i.e., the objective parts of Δ, where Δ is a stable AE-expansion of $e(\langle D, W\rangle)$.*

PROOF. We first prove that $t_v(\langle D, W\rangle)$ and $t(\langle D, W\rangle)$ have exactly the same consistent $\mathbf{N}$-expansions. Later we will deal with inconsistent expanions.

Let Δ be a consistent $\mathbf{N}$-expansion of $t(\langle D, W\rangle)$. Since v does not occur as an atom in $t(\langle D, W\rangle)$, nor in $\neg L\overline{\Delta}$, we have $v \notin \Delta = cons_{\mathbf{N}}(t(\langle D, W\rangle)\cup\neg L\overline{\Delta})$. Thus $v \in \overline{\Delta}$, and $\neg Lv \in \neg L\overline{\Delta}$. Hence, for each $d \in D$, $t(\langle D, W\rangle)\cup\neg L\overline{\Delta} \models t_v(d)$. Thus $t(\langle D, W\rangle)\cup\neg L\overline{\Delta} \models t_v(\langle D, W\rangle)$. In a similar way, it follows that $t_v(\langle D, W\rangle) \cup \neg L\overline{\Delta} \models t(\langle D, W\rangle)$. Thus the two sets $t(\langle D, W\rangle) \cup \neg L\overline{\Delta}$ and $t_v(\langle D, W\rangle) \cup \neg L\overline{\Delta}$ are logically equvalent and hence $\Delta = cons_{\mathbf{N}}(t(\langle D, W\rangle) \cup \neg L\overline{\Delta}) = cons_{\mathbf{N}}(t_v(\langle D, W\rangle) \cup \neg L\overline{\Delta})$ from where it follows that Δ is an $\mathbf{N}$-expansion of $t_v(\langle D, W\rangle)$.

Conversely, assume that Δ is a consistent $\mathbf{N}$-expansion of $t_v(\langle D, W\rangle)$. Since v does not occur *as an atom* in $t_v(\langle D, W\rangle)\cup\neg L\overline{\Delta}$, we have, as before, $v \notin \Delta = cons_{\mathbf{N}}(t_v(\langle D, W\rangle)\cup \neg L\overline{\Delta})$. Thus $v \in \overline{\Delta}$, and $\neg Lv \in \neg L\overline{\Delta}$. Hence, by exactly the same proof argument as before, it follows that the two sets $t(\langle D, W\rangle)\cup\neg L\overline{\Delta}$ and $t_v(\langle D, W\rangle)\cup\neg L\overline{\Delta}$ are logically equivalent and thus Δ is a consistent $\mathbf{N}$-expansion of $t(\langle D, W\rangle)$.

Thus the consistent **N**-expansions of $t(\langle D, W\rangle)$ and $t_v(\langle D, W\rangle)$ coincide and hence, by Proposition 9.1 and Theorem 8.2, the consistent extensions of $\langle D, W\rangle$ coincide with the objective parts of the consistent AE-expansions of $e(\langle D, W\rangle)$. It thus remains to prove that $\langle D, W\rangle$ has an inconsistent extension if $e(\langle D, W\rangle)$ has an inconsistent AE-expansion.

Assume that $\langle D, W\rangle$ has an inconsistent extension. It is easy to see (and well-known) that this is only the case if W is propositionally inconsistent. Therefore, $e(\langle D, W\rangle)$ is propositionally inconsistent and thus admits the inconsistent stable set $E(\{\perp\}) = \mathcal{L}_L$ as stable AE-expansion.

Conversely, assume that $e(\langle D, W\rangle)$ admits the inconsistent stable AE-expansion $E(\{\perp\}) = \mathcal{L}_L$. Then, since $\overline{\mathcal{L}_L} = \emptyset$, by Def. 2.3 we have $\mathcal{L}_L = cons(e(\langle D, W\rangle) \cup L\mathcal{L}_L)$, thus $e(\langle D, W\rangle) \cup L\mathcal{L}_L$ is inconsistent. Now assume W is consisent. Consider any truth assignment σ that satisfies W and assigns *true* to each modal atom, i.e., to each element of $L\mathcal{L}_L$. It is easy to see that σ satisfies $e(\langle D, W\rangle)$. Indeed, σ satisfies W by definition; furthermore, σ satisfies each formula $t_v(d)$ for $d \in D$ because $t_v(d)$ is an implication having $\neg Lv$ in its antecedent, thus, σ falsifies the antecedent and hence satisfies the implication; finally, σ satisfies $LG(t_v(\langle D, W\rangle))$ because this is just a modal atom from $L\mathcal{L}_L$. It follows that σ is a satisfying truth value assignment for the (inconsistent) set of formulas $e(\langle D, W\rangle) \cup L\mathcal{L}_L$. Contradiction. Therefore, W is inconsistent, and thus $\langle D, W\rangle$ admits the inconsistent extension $\mathcal{L}$. $\square$

Theorem 9.1 establishes a faithful polynomial translation from default logic into standard AEL. It is easy to see that a faithful translation in the opposite direction is not possible. Just consider the autoepistemic theory $\Sigma = \{Lp \rightarrow p\}$. As we have noted in Section 5, Σ has two stable AE-expansions, $E(\emptyset)$ and $E(\{p\})$. The objective parts of these extensions are $T_1 = cons(\emptyset)$ and $T_2 = cons(\{p\})$. Obviously, $T_1 \subset T_2$. However, it is well known (and follows easily from the definition of default-extension [26]) that no extension of a default theory $\langle D, W\rangle$ can be a subset of any other extension of $\langle D, W\rangle$. Thus, the stable AE-expansions of Σ do not correspond to the extensions of any default theory. If the *expressive power* of a nonmonotonic logic stands for its capacity of expressing sets of propositional theories (i.e., sets of alternative epistemic states) through sets of premises, then the expressive power of standard AEL is strictly higher than that of default logic.

10 Conclusion and Philosophical Remarks

Our investigation was originally motivated by the theoretical interest of comparing the expressive power of two well-known knowledge representation formalisms, default logic and autoepistemic logic. However, we believe that our results are also useful to practical aspects of knowledge engineering.

By showing that there is no modular translation from default logic to AEL, we have provided evidence, that these formalisms are rather different. In more practical terms, for a knowledge engineer who wants to represent default knowledge in the sense of Reiter's default logic, this means that AEL is not a suitable language: it is impossible to represent each default by a separate formula. Thus, as long as the knowledge engineer is interested to represent pure default knowledge, he or she is advised to use default logic or some modal logic close to default logic (such as nonmonotonic **N**) as a knowledge representation language.

On the other hand, we have shown that default logic and nonmonotonic **N** can be translated by a rather complicated translation into AEL. Roughly speaking, this means that the nonmonotonicity of AEL can be restricted *on demand* to nonmonotonicity arising merely from negative introspection. This is advantageous in case one needs a KR language integrating both types of reasoning. Of course, writing the formula $LG(\Sigma)$ is not really a practical approach for a human knowledge engineer, since $LG(\Sigma)$ may be rather intricate. However, $LG(\Sigma)$ can be computed easily by a machine, because the translation is polynomial. So one may imagine an autoepistemic reasoning tool with an enhanced syntax, where, say, $\Sigma!$ stands for $\Sigma \cup LG(\Sigma)$. If the user types $\Sigma!$, the formula is automatically expanded by the system, and the system reasons like a nonmonotonic **N** system, i.e., similar to a default reasoning engine.

The formula $G(\Sigma)$ can also be used by an AEL reasoning tool for recognizing whether or not a computed AEL expansion E is appropriately grounded. A relatively simple membership-check testing whether $G(\Sigma)$ belongs to E suffices. In contexts where appropriately grounded expansions are considered to be more important or more reliable than just weakly grounded expansions, this may turn out to be useful. For example, the system may consider appropriately grounded expansions first, or warn the user in case an inappropriately grounded expansion was used in the reasoning process.

Let us also remark that by our results it is possible to represent mixed knowledge, i.e, knowledge consisting of one part Σ_A that should be dealt-with in a more carefully grounded (default logic like) manner and another part Σ_B that may be used in a more liberal (AEL like) fashion. The overall theory could then be represented within AEL as

$\Sigma = \Sigma_A \cup LG(\Sigma_A) \cup \Sigma_B$, or, more shortly, as $\Sigma_A! \cup \Sigma_B$. This will certainly work fine in case Σ_A and Σ_B have no common atoms. The set of expansions of T is then a kind of direct product of the set of appropriately grounded expansions of Σ_A and of the set of standard AEL expansions of Σ_B. A mathematical and semantical characterization of the expansions of T in case Σ_A and Σ_B have common popositional variables is an interesting research problem that we have not yet looked at.

Note also that our translatability result admit a somewhat speculative, but nevertheless very appealing and intriguing philosophical interpretation.

Recall that standard AEL is a logic for expressing beliefs, and that a set of premises Σ is conceived as the set of initial beliefs of an ideally rational agent. Hence, the addition of the formula $LG(\Sigma)$ to a theory Σ can be interpreted as the belief of the agent that its own beliefs (w.r.t. Σ) must be sufficiently grounded in Σ. We have shown in this paper that the addition of this formula effectively enforces all beliefs based on Σ to be sufficiently grounded in Σ. Hence, we delivered a mathematical proof for the fact that (polynomially many) beliefs in groundedness-principles about a theory may control the *way* of reasoning on the basis of that theory. In particular, if we agree that appropriately grounded expansions are "better" (i.e., more rational or more scientific) than just weakly grounded expansions, then our result may be paraphrased as follows: *an autoepistemic agent can improve its way of reasoning by believing in better principles of reasoning.*

A yet different interpretation or use of our results concerns the problem of *communicating knowledge* to an autoepistemic agent. If we tell a story Σ to such an agent, we must be prepared to the possibility that the agent bases insufficiently grounded beliefs on Σ and acts according to such beliefs. In order to restrict the agent's phantasy and put its feet back on the ground, it is sufficient to add the formula $LG(\Sigma)$ to the intended information Σ.

References

[1] Ph. Besnard. *Default Logic*, Springer Verlag, Berlin Heidelberg New-York, (Symbolic Computation Series), 1989.

[2] M. Cadoli and M. Schaerf. A Survey of Complexity Results for Non-Monotonic Logics *Journal of Logic Programming*, vol. 17, pp.127-160, 1993.

[3] J. Delgrande and W. Jackson. Default Logic Revisited. In *Proc. KR-91*, pp. 118–127, 1991.

[4] Th. Eiter and G. Gottlob Reasoning with Parsimonious and Moderately Grounded Expansions. *Fundamenta Informaticae*, 17:31-53, 1992.

[5] M.C. Fitting, W. Marek, and M. Truszczyński. The Pure Logic of Necessitation. *The Journal of Logic and Computation*, 2(3):349–373, 1992.

[6] M. Garey and D. S. Johnson. *Computers and Intractability – A Guide to the Theory of NP-Completeness*. W. H. Freeman, New York, 1979.

[7] G. Gottlob. Complexity Results for Nonmonotonic Logics. *The Journal of Logic and Computation*, 2:3:397–425, June 1992.

[8] G. Gottlob. The Power of Beliefs: Translating Default Logic Into Standard Autoepistemic Logic. *Proceedings of the Thirteenth International Joint Conference on Artificial Intelligence (IJCAI-93), Chambéry, France, August 28–Sept. 3, 1993*, vol.1, pp. 570–575. *Note: This paper was also presented at the Knowledge Representation Workshop at ECAI-92, Vienna Austria, and was published in the respective workshop notes.*

[9] T. Imielinski. Results on Translating Defaults to Circumscription. In *Proc. 9th Intl. Joint Conf. on Artificial Intelligence, IJCAI-1985, UCLA, Los Angeles, CA, Aug. 1985*, vol I, Morgan Kaufmann, Los Altos, CA, pp. 114–120, 1985.

[10] M. Kaminski. Embedding a Default System into Nonmonotonic Logic. *Fundamenta Informaticae*, 14:345–354, 1991.

[11] K. Konolige. On the Relationship between Default and Autoepistemic Logic. *AIJ*, 35:343–382, 1988. (+ Errata, vol 41(1989/90), p.115,1989).

[12] F. Lin and Y. Shoham. Epistemic Semantics for Fixed-Point Non-Monotonic Logics. In *Proc. TARK-90*, pp. 111–120. 1990.

[13] W. Marek, G.F. Schwartz, and M. Truszczyński. Modal Nonmonotonic Logics: Ranges, Characterization, Computation. Proc. KR-91.

[14] W. Marek and M. Truszczyński. Relating Autoepistemic and Default Logics. In *Proc. KR-89.*

[15] W. Marek and M. Truszczyński. Modal logic for default reasoning. *Annals of Mathematics and Artificial Intelligence*, 1:275–302, 1990.

[16] W. Marek and M. Truszczyński. Autoepistemic Logic. *Journal of the ACM*, 38:3:588–619, 1991.

[17] W. Marek and M. Truszczyński. *Nonmonotonic Logic.* Springer Verlag, Berlin Heidelberg New York, 1993, 417 pp.

[18] D. McDermott. Non-Monotonic Logic II: Nonmonotonic Modal Theories. *JACM*, 29:33–57, 1982.

[19] D. McDermott and J. Doyle. Non-Monotonic Logic I. *Artificial Intelligence*, 13:41–72, 1980.

[20] R.C. Moore. Semantical Considerations on Nonmonotonic Logic. *AIJ*, 25:75–94, 1985.

[21] P.H. Morris. Autoepistemic Stable Closures and Contradiction Resolution. In *Proc. 2nd Int. Workshop on Nonmonotonic Reasoning (Grassau, FRG, June 13-15, 1988)*, Lecture Notes in Computer Science (subseries LNAI) 346, pages 60–73, Berlin, 1991. Springer Verlag.

[22] I. Niemelä. Decision Procedures for Autoepistemic Logic. In *Proceedings of the 9th International Conference on Automated Deduction (Argonne, May 23-26,1988)*, Lecture Notes in Computer Science 310, pages 676–684, Berlin, 1988. Springer Verlag.

[23] I. Niemelä. Constructive Tightly Grounded Autoepistemic Reasoning. In *Proc. IJCAI-1991*, 1991.

[24] I. Niemelä. Towards Automatic Autoepistemic Reasoning. In *Logics in AI (Proc. Europ. Workshop on Logics in AI, Amsterdam, September 1990)*, Lecture Notes in Computer Science (subseries LNAI) 478, Berlin, 1991. Springer Verlag.

[25] Ch. Papadimitriou and M. Sideri. On Finding Extensions of Default Theories. In *Proceedings of the 4th International Conference on Database Theory (ICDT'92), Berlin, Germany, October 1992*, Springer LNCS No.646, pages 276–281, 1992.

[26] R. Reiter. A Logic for Default Reasoning. *AIJ*, 13:81–132, 1980.

[27] G. Shvarts. Autoepistemic Modal Logics. In *Proc. TARK-90*, pp. 97–109. 1990.

[28] G. Schwarz and M. Truszczyński. Nonmonotonic Reasoning is Sometimes Easier. In *Proc. 3rd Kurt Goedel Colloquium, Brno, Czech Republic, August 1993*, Springer Verlag, LNCS Nr. 713, pages 313–324, 1993.

[29] P. Siegel. A Modal Language for Nonmonotonic Logic. In *Proc. Workshop DRUMS/'CEE, Marseille, France, 1990*, 1990.

[30] R. Staerk. On the Existence of Fixpoints in Moore's Autoepistemic Logic and the Non-Monotonic Logic of McDermott and Doyle. In *Proc. Computer Science Logic 1990, Springer LNCS 533*, 1990.

[31] R.C. Stalnaker. A Note on Non-Monotonic Modal Logic. 1980. Manuscript, Dept of Philosophy, Cornell Univ. Reprinted in *Artificial Intelligence*, 64:2, pp.183–196, 1993.

[32] J. Stillman. The Complexity of Propositional Default Logic. In *Proceedings AAAI-92*, 1992.

[33] M. Truszczyński. Modal Interpretations of Default Logic. In *Proc. IJCAI-1991*, 1991.

[34] M. Truszczyński. Embedding Default Logic into Modal Nonmonotonic Logics. *Proc. Workshop on Nonmon. Reas. and LP, Washington, July 1991.*