Foundations of Data and Knowledge Systems

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7. Complexity and Expressive Power

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Outline

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The Story So far

- Query languages with the form of logics
- Syntax, declarative and operational semantics

- How much resource (time, space) do we need for the computation of these semantics? ⇒ Complexity

- What kind of properties can a given query language express?
- Is Q_1 more expressive than Q_2? ⇒ Expressive power

A dream query language should have:

- lower complexity, and
- more expressive power

The Results Overview

Query	Data Complexity	Program Complexity
Conjunctive query	AC_0	NP-complete
FO	AC_0	PSPACE-complete
Prop. LP		P-complete
Datalog	P-complete	EXPTIME-complete
Stratified Datalog	P-complete	EXPTIME-complete
Datalog(WFM)	P-complete	EXPTIME-complete
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Disjun. Datalog	Π_2^p-complete	co-NEXPTIME$^{\mathsf{NP}}$-complete

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The Goal of this Lecture

- Basic concept of Turing machine, reduction, data complexity and program complexity
- How to prove completeness, Logspace reduction
- Get a taste of the hardness proofs of logic programming via nice encoding of a Turing machine
- Learn basics about expressive power

Decision Problems

- Problems where the answer is “yes” or “no”
- Formally,
  - A language L over some alphabet Σ.
  - An *instance* is given as a word $x \in \Sigma^*$.
  - Question: whether $x \in L$ holds
- The resources (i.e., either time or space) required in the worst case to find the correct answer for any instance x of a problem L is referred to as the *complexity* of the problem L

Complexities

Let P be a program with some query language, D_{in} input database and A a ground atom.

- **data complexity**
  Let P be *fixed*

  Instance. D_{in} and A.
  Question. Does $D_{in} \cup P \models A$ hold?
- **program complexity** (a.k.a. **expression complexity**)
  Let D_{in} be *fixed*.

  Instance. P and A.
  Question. Does $D_{in} \cup P \models A$ hold?
- **combined complexity**

  Instance. P, D_{in} and A.
  Question. Does $D_{in} \cup P \models A$ hold?

Complexity classes

$$\mathsf{L} \subseteq \mathsf{NL} \subseteq \mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{PSPACE} \subseteq \mathsf{EXPTIME} \subseteq \mathsf{NEXPTIME}$$

These are the classes of problems which can be solved in

- logarithmic space (L),
- non-deterministic logarithmic space (NL),
- polynomial time (P),
- non-deterministic polynomial time (NP),
- polynomial space (PSPACE),
- exponential time (EXPTIME), and
- non-deterministic exponential time (NEXPTIME).

we shall encounter in this course: P, NP, PSPACE, EXPTIME

Complexity classes – co Problems

- Any complexity class $\mathcal{C}$ has its complementary class denoted by co-$\mathcal{C}$.
- For every language $L \subseteq \Sigma^*$, let $\overline{L}$ denote its complement, i.e. the set $\Sigma^* \setminus L$. Then co-$\mathcal{C}$ is $\{\overline{L} \mid L \in \mathcal{C}\}$.
- Every deterministic complexity class is closed under complement, because one can simply add a last step to the algorithm which reverses the answer. (co-P?)

Complexity classes – Reductions

- **Logspace Reduction**
  - Let L_1 and L_2 be decision problems (languages over some alphabet Σ).
  - $R : \Sigma^* \to \Sigma^*$ be a function which can be computed in logarithmic space
  - The following property holds: for every $x \in \Sigma^*$, $x \in L_1$ iff $R(x) \in L_2$.
  - Then R is called a logarithmic-space reduction from L_1 to L_2 and we say that L_1 is reducible to L_2.
- **Hardness, Completeness**
  Let $\mathcal{C}$ be a set of languages. A language L is called $\mathcal{C}$-hard if any language L' in $\mathcal{C}$ is reducible to L. If L is $\mathcal{C}$-hard and $L \in \mathcal{C}$ then L is called *complete for* $\mathcal{C}$ or simply $\mathcal{C}$-complete.

Turing machines

A deterministic Turing machine (DTM) is defined as a quadruple

$$(S, \Sigma, \delta, s_0)$$

- S is a finite set of states,
- Σ is a finite alphabet of symbols, which contains a special symbol ␣ called the blank.
- δ is a transition function,
- and $s_0 \in S$ is the initial state.

The transition function δ is a map

$$\delta : \quad S \times \Sigma \quad \rightarrow \quad (S \cup \{\mathtt{yes}, \mathtt{no}\}) \times \Sigma \times \{\mathtt{-1,\ 0,\ +1}\},$$

where `yes`, and `no` denote two additional states not occurring in S, and `-1, 0, +1` denote motion directions.

Turing machines

DTM quadruple:

$$(\Sigma, S, \delta, s_0)$$

Transition function:

$$\delta(s, \sigma) = (s', \sigma', d).$$

The tape of the TM

$\triangleright$	a	b	$\cdots$	b	$\boxed{a}$	a	$\sqcup$	$\sqcup$	$\sqcup$	$\sqcup$	$\cdots$
					s_1						

Turing machines

DTM quadruple:

$$(\Sigma, S, \delta, s_0)$$

Transition function:

$$\delta(s, \sigma) = (s', \sigma', d).$$

The tape of the TM

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c}
\hline
\triangleright & \boxed{a} & b & \cdots & b & a & a & \sqcup & \sqcup & \sqcup & \sqcup & \cdots \\
\hline
\multicolumn{2}{c}{s_0} & & & & & & & & & & \\
\end{array}$$

Turing machines

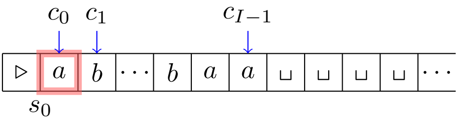
DTM quadruple:

$$(\Sigma, S, \delta, s_0)$$

Transition function:

$$\delta(s, \sigma) = (s', \sigma', d).$$

The tape of the TM



Turing machines

DTM quadruple:

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Transition function:

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The tape of the TM

$\triangleright$	a	b	$\cdots$	b	$\boxed{a}$	a	$\sqcup$	$\sqcup$	$\sqcup$	$\sqcup$	$\cdots$
					s						

Transition Function example:

$$\delta(s, a) = (s', b, -1)$$

Turing machines

DTM quadruple:

$$(\Sigma, S, \delta, s_0)$$

Transition function:

$$\delta(s, \sigma) = (s', \sigma', d).$$

The tape of the TM

$\triangleright$	a	b	$\cdots$	b	$\boxed{b}$	a	$\sqcup$	$\sqcup$	$\sqcup$	$\sqcup$	$\cdots$

s

Transition Function example:

$$\delta(s, a) = (s', b, -1)$$

Turing machines

DTM quadruple:

$$(\Sigma, S, \delta, s_0)$$

Transition function:

$$\delta(s, \sigma) = (s', \sigma', d).$$

The tape of the TM

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c}
\hline
\triangleright & a & b & \cdots & \boxed{b} & b & a & \sqcup & \sqcup & \sqcup & \sqcup & \cdots \\
\hline
\multicolumn{4}{c}{} & s' & \multicolumn{7}{c}{}
\end{array}$$

Transition Function example:

$$\delta(s, a) = (s', b, -1)$$

Turing machines

DTM quadruple:

$$(\Sigma, S, \delta, s_0)$$

Transition function:

$$\delta(s, \sigma) = (s', \sigma', d).$$

The tape of the TM

$\triangleright$	a	$\boxed{b}$	$\cdots$	b	b	a	a	a	b	b	$\cdots$

$\texttt{yes}$

Accept!

T halts, when any of the states $\texttt{yes}$ or $\texttt{no}$ is reached

Turing machines

DTM quadruple:

$$(\Sigma, S, \delta, s_0)$$

Transition function:

$$\delta(s,\sigma) = (s', \sigma', d).$$

The tape of the TM

$\triangleright$	a	$\boxed{b}$	$\cdots$	b	b	a	a	a	b	b	$\cdots$

no

Reject!

T halts, when any of the states yes or no is reached

NDTM

A non-deterministic Turing machine (NDTM) is defined as a quadruple

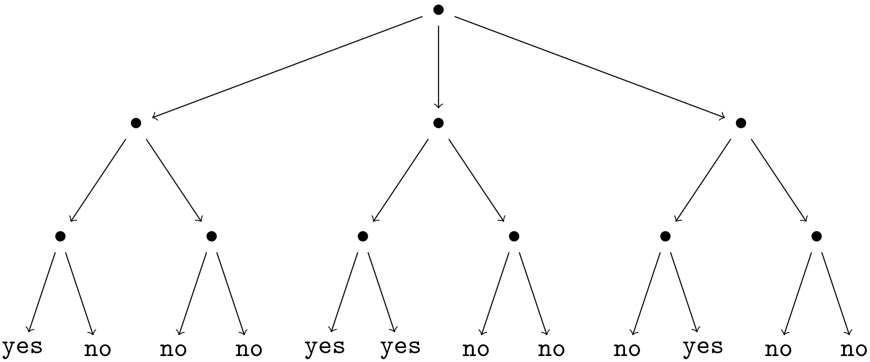
$$(S, \Sigma, \Delta, s_0)$$

- S, Σ, s_0 are the same as DTM
- Δ is no longer a function, but a relation:

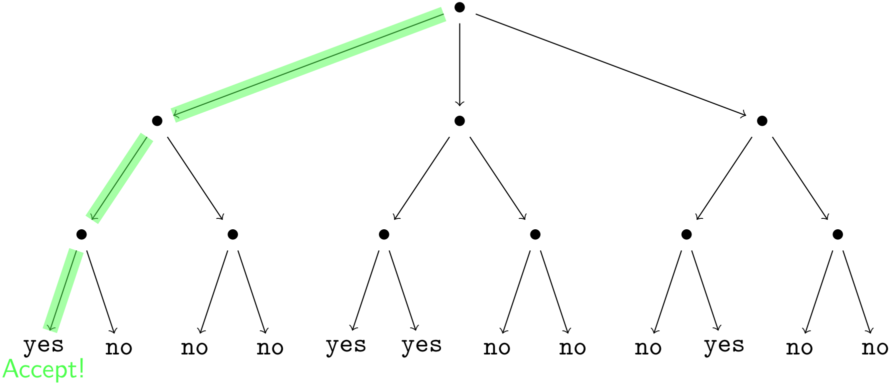
$$\Delta \ \subseteq \ (S \times \Sigma) \times (S \cup \{\texttt{yes}, \texttt{no}\}) \times \Sigma \times \{\texttt{-1, 0, +1}\}.$$

- A tuple with s and σ. If the number of such tuples is greater than one, the NDTM non-deterministically chooses any of them and operates accordingly.
- Unlike the case of a DTM, the definition of acceptance and rejection by a NDTM is asymmetric.

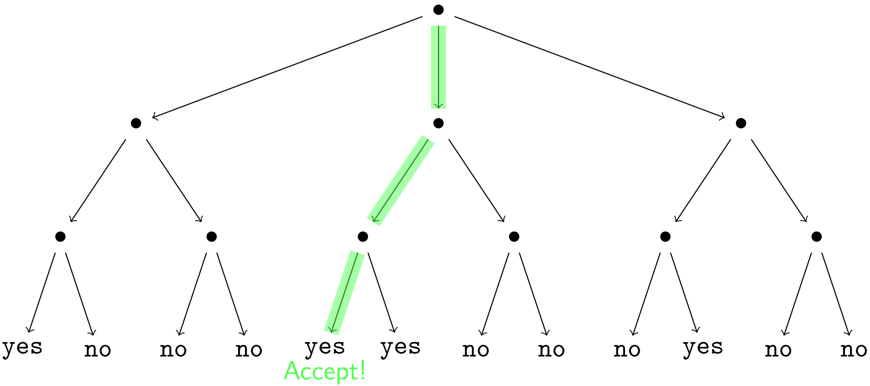
Nondeterministic Computation (Accept)



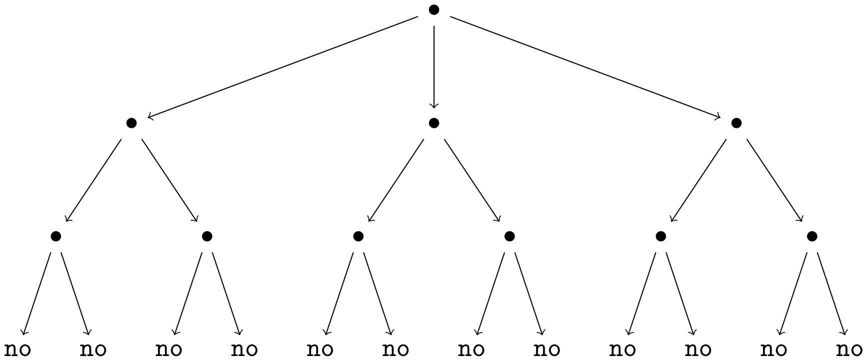
Nondeterministic Computation (Accept)



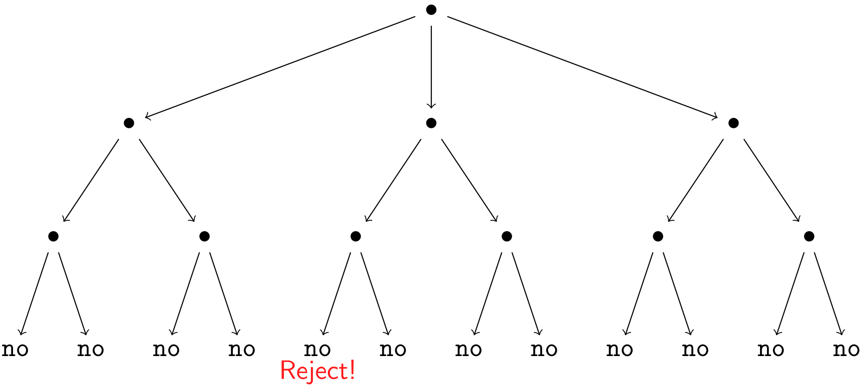
Nondeterministic Computation (Accept)



Nondeterministic Computation (Rejection)



Nondeterministic Computation (Rejection)



Outline

7.2 Propositional Logic Programming

Today we shall concentrate on

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Conjunctive query	AC_0	NP-complete
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Propositional LP

Theorem

Propositional logic programming is P*-complete.*

Proof: (Membership)

- The semantics of a given program P can be defined as the least fixpoint of the immediate consequence operator $\mathbf{T}_P$
- This least fixpoint $\mathit{lfp}(\mathbf{T}_P)$ can be computed in polynomial time even if the "naive" evaluation algorithm is applied.
- The number of iterations (i.e. applications of $\mathbf{T}_P$) is bounded by the number of rules plus 1.
- Each iteration step is clearly feasible in polynomial time.

Propositional LP P-hardness Proof

Proof: (Hardness)

- Encoding of a a deterministic Turing machine (DTM) T. Given a DTM T, an input string I and a number of steps N, where N is a polynomial of $|I|$, construct in logspace a program $P = P(T, I, N)$. An atom A such as $P \models A$ iff T accepts I in N steps.
- The transition function δ of a DTM with a single tape can be represented by a table whose rows are tuples $t = \langle s, \sigma, s', \sigma', d\rangle$. Such a tuple t expresses the following if-then-rule:

  if at some time instant τ the DTM is in state s, the cursor points to cell number π, and this cell contains symbol σ
  then at instant $\tau + 1$ the DTM is in state s', cell number π contains symbol σ', and the cursor points to cell number $\pi + d$.

Propositional LP P-hardness: the atoms

The propositional atoms in $P(T, I, N)$.
(there are many, but only polynomially many...)

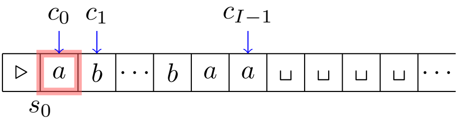
- *symbol*$_\alpha[\tau, \pi]$ for $0 \leq \tau \leq N$, $0 \leq \pi \leq N$ and $\alpha \in \Sigma$. Intuitive meaning: at instant τ of the computation, cell number π contains symbol α.
- *cursor*$[\tau, \pi]$ for $0 \leq \tau \leq N$ and $0 \leq \pi \leq N$. Intuitive meaning: at instant τ, the cursor points to cell number π.
- *state*$_s[\tau]$ for $0 \leq \tau \leq N$ and $s \in S$. Intuitive meaning: at instant τ, the DTM T is in state s.
- *accept* Intuitive meaning: T has reached state yes.

Propositional LP P-hardness: the rules

initialization facts: in $P(T, I, N)$:

$$
\begin{array}{rcl}
\mathit{symbol}_\sigma[0,\pi] & \leftarrow & \text{for } 0 \leq \pi < |I|, \text{ where } I_\pi = \sigma \\
\mathit{symbol}_\sqcup[0,\pi] & \leftarrow & \text{for } |I| \leq \pi \leq N \\
\mathit{cursor}[0,0] & \leftarrow & \\
\mathit{state}_{s_0}[0] & \leftarrow &
\end{array}
$$

The tape of the TM



Propositional LP P-hardness: the rules

- transition rules: for each entry $\langle s, \sigma, s', \sigma', d \rangle$, $0 \leq \tau < N$, $0 \leq \pi < N$, and $0 \leq \pi + d$.

$$\begin{array}{rcl} \mathit{symbol}_{\sigma'}[\tau+1,\pi] & \leftarrow & \mathit{state}_s[\tau], \mathit{symbol}_\sigma[\tau,\pi], \mathit{cursor}[\tau,\pi] \\ \mathit{cursor}[\tau+1,\pi+d] & \leftarrow & \mathit{state}_s[\tau], \mathit{symbol}_\sigma[\tau,\pi], \mathit{cursor}[\tau,\pi] \\ \mathit{state}_{s'}[\tau+1] & \leftarrow & \mathit{state}_s[\tau], \mathit{symbol}_\sigma[\tau,\pi], \mathit{cursor}[\tau,\pi] \end{array}$$

- inertia rules: where $0 \leq \tau < N$, $0 \leq \pi < \pi' \leq N$

$$\begin{array}{rcl} \mathit{symbol}_{\sigma}[\tau+1,\pi] & \leftarrow & \mathit{symbol}_\sigma[\tau,\pi], \mathit{cursor}[\tau,\pi'] \\ \mathit{symbol}_{\sigma}[\tau+1,\pi'] & \leftarrow & \mathit{symbol}_\sigma[\tau,\pi'], \mathit{cursor}[\tau,\pi] \end{array}$$

- accept rules: for $0 \leq \tau \leq N$

$$\mathit{accept} \leftarrow \mathit{state}_{\mathsf{yes}}[\tau]$$

Propositional LP P-hardness

- The encoding precisely simulates the behaviour machine T on input I up to N steps. (This can be formally shown by induction on the time steps.)
- $P(T, I, N) \models$ *accept* iff the DTM T accepts the input string I within N steps.
- The construction is feasible in Logspace

Horn clause inference is P-complete

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7.3 Datalog Complexity

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Complexity of Datalog Programs – Data complexity

Theorem

Datalog is data complete for P*.*

Proof: (Membership)
Effective reduction to Propositional Logic Programming is possible. Given P, D, A:

- Generate $ground(P, D)$
- Decide whether $ground(P, D) \models A$

Grounding of Datalog Rules

- Let U_D be the universe of D (usually the active universe (domain), i.e., the set of all domain elements present in D).
- The grounding of a rule r, denoted $ground(r, D)$, is the set of all rules obtained from r by all possible uniform substitutions of elements of U_D for the variables in r.

For any datalog program P and database D,

$$ground(P, D) = \bigcup_{r \in P} ground(r, D).$$

Grounding example

P and D:

$parent(X,Y) \leftarrow father(X,Y) \quad parent(X,Y) \leftarrow mother(X,Y).$
$ancestor(X,Y) \leftarrow parent(X,Y).$
$ancestor(X,Y) \leftarrow parent(X,Z), ancestor(Z,Y).$

$father(john, mary).\ father(joe, kurt).$
$mother(mary, joe).mother(tina, kurt).$

$ground(P,D)$:

$parent(john, john) \leftarrow father(john, john)$
$parent(john, john) \leftarrow father(john, marry)$
$\ldots$
$parent(john, john) \leftarrow mother(john, john)$
$parent(john, marry) \leftarrow mother(john, marry)$
$\ldots$
$ancestor(john, john) \leftarrow parent(john, john)$
$\ldots$

Grounding complexity

Given P, D, the number of rules in $ground(P, D)$ is bounded by

$$|P| * \#consts(D)^{vmax}$$

- $vmax (\geq 1)$ is the maximum number of different variables in any rule $r \in P$
- $\#consts(D) = |U_D|$ is the number of constants in D (ass.: $|U_D| > 0$).
- $ground(P \cup D)$ can be exponential in the size of P.
- $ground(P \cup D)$ is polynomial in the size of D.

Hence, the complexity of propositional logic programming is an upper bound for the data complexity.

Datalog data complexity: hardness

Proof: Hardness The P-hardness can be shown by writing a simple datalog meta-interpreter for propositional $\mathrm{LP}(k)$, where k is a constant.

- Represent rules $A_0 \leftarrow A_1, \ldots, A_i$, where $0 \leq i \leq k$, by tuples $\langle A_0, \ldots, A_i \rangle$ in an $(i+1)$-ary relation R_i on the propositional atoms.
- Then, a program P in $\mathrm{LP}(k)$ which is stored this way in a database $D(P)$ can be evaluated by a fixed datalog program $P_{MI}(k)$ which contains for each relation R_i, $0 \leq i \leq k$, a rule

$$T(X_0) \leftarrow T(X_1), \ldots, T(X_i), R_i(X_0, \ldots, X_i).$$

- $T(x)$ intuitively means that atom x is true.
  Then, $P \models A$ just if $P_{MI} \cup P(D) \models T(A)$. P-hardness of the data complexity of datalog is then immediately obtained.

Program Complexity Datalog

Theorem

Datalog is program complete for EXPTIME.

- **Membership**. Grounding P on D leads to a propositional program $grounding(P, D)$ whose size is exponential in the size of the fixed input database D.
  Hence, the program complexity is in EXPTIME.
- **Hardness**.
  - Adapt the propositional program $P(T, I, N)$ deciding acceptance of input I for T within N steps, where $N = 2^m$, $m = n^k (n = |I|)$ to a datalog program $P_{dat}(T, I, N)$
  - Note: We can not simply generate $P(T, I, N)$, since this program is exponentially large (and thus the reduction would not be polynomial!)

Datalog Program Complexity: Hardness

Main ideas for lifting $P(T, I, N)$ to $P_{dat}(T, I, N)$:

- use the predicates *symbol*$_\sigma(X, Y)$, *cursor*(X, Y) and *state*$_s(X)$ instead of the propositional letters *symbol*$_\sigma[X, Y]$, *cursor*$[X, Y]$ and *state*$_s[X]$ respectively.
- The time points τ and tape positions π from 0 to $N - 1$ are encoded in binary, i.e. by m-ary tuples $t_\tau = \langle c_1, \ldots, c_m\rangle$, $c_i \in \{0, 1\}$, $i = 1, \ldots, m$, such that $0 = \langle 0, \ldots, 0\rangle$, $1 = \langle 0, \ldots, 1\rangle$, $N - 1 = \langle 1, \ldots, 1\rangle$.
- The functions $\tau + 1$ and $\pi + d$ are realized by means of the successor *Succ*^{m} from a linear order $\leq^m$ on U^m.

Datalog Program Complexity: Hardness

The predicates Succ^m, First^m, and Last^m are provided.

- The initialization facts $\mathit{symbol}_\sigma[0,\pi]$ are readily translated into the datalog rules

$$\mathit{symbol}_\sigma(\mathbf{X},\mathbf{t}) \leftarrow \mathit{First}^m(\mathbf{X}),$$

  where $\mathbf{t}$ represents the position π,
- Similarly the facts $\mathit{cursor}[0,0]$ and $\mathit{state}_{s_0}[0]$.
- Initialization facts $\mathit{symbol}_\sqcup[0,\pi]$, where $|I| \leq \pi \leq N$, are translated to the rule

$$\mathit{symbol}_\sqcup(\mathbf{X},\mathbf{Y}) \leftarrow \mathit{First}^m(\mathbf{X}),\ \leq^m(\mathbf{t},\mathbf{Y})$$

  where $\mathbf{t}$ represents the number $|I|$.

Datalog Program Complexity: Hardness

- Transition and inertia rules: for realizing $\tau + 1$ and $\pi + d$, use in the body atoms $\mathit{Succ}^m(\mathbf{X}, \mathbf{X}')$. For example, the clause

  $$\mathit{symbol}_{\sigma'}[\tau + 1, \pi] \leftarrow \mathit{state}_s[\tau], \mathit{symbol}_\sigma[\tau, \pi], \mathit{cursor}[\tau, \pi]$$

  is translated into

  $$\mathit{symbol}_{\sigma'}(\mathbf{X}', \mathbf{Y}) \leftarrow \mathit{state}_s(\mathbf{X}), \mathit{symbol}_\sigma(\mathbf{X}, \mathbf{Y}), \mathit{cursor}(\mathbf{X}, \mathbf{Y}), \mathit{Succ}^m(\mathbf{X}, \mathbf{X}').$$

- The translation of the accept rules is straightforward.

Defining $\mathit{Succ}^m(\mathbf{X}, \mathbf{X}')$ and $\leq^m$

- The ground facts $\mathit{Succ}^1(0,1)$, $\mathit{First}^1(0)$, and $\mathit{Last}^1(1)$ are provided.
- For an inductive definition, suppose $\mathit{Succ}^i(\mathbf{X}, \mathbf{Y})$, $\mathit{First}^i(\mathbf{X})$, and $\mathit{Last}^i(\mathbf{X})$ tell the successor, the first, and the last element from a linear order $\leq^i$ on U^i, where $\mathbf{X}$ and $\mathbf{Y}$ have arity i. Then, use rules

$$\begin{array}{rcl}
\mathit{Succ}^{i+1}(Z, \mathbf{X}, Z, \mathbf{Y}) & \leftarrow & \mathit{Succ}^i(\mathbf{X}, \mathbf{Y}) \\
\mathit{Succ}^{i+1}(Z, \mathbf{X}, Z', \mathbf{Y}) & \leftarrow & \mathit{Succ}^1(Z, Z'), \mathit{Last}^i(\mathbf{X}), \mathit{First}^i(\mathbf{Y}) \\
\mathit{First}^{i+1}(Z, \mathbf{X}) & \leftarrow & \mathit{First}^1(Z), \mathit{First}^i(\mathbf{X}) \\
\mathit{Last}^{i+1}(Z, \mathbf{X}) & \leftarrow & \mathit{Last}^1(Z), \mathit{Last}^i(\mathbf{X})
\end{array}$$

Defining $Succ^m(\mathbf{X}, \mathbf{X}')$ and $\leq^m$

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$$
\begin{array}{rcl}
Succ^{i+1}(0, \mathbf{X}, 0, \mathbf{Y}) & \leftarrow & Succ^i(\mathbf{X}, \mathbf{Y}) \\
Succ^{i+1}(1, \mathbf{X}, 1, \mathbf{Y}) & \leftarrow & Succ^i(\mathbf{X}, \mathbf{Y}) \\
Succ^{i+1}(0, \mathbf{X}, 1, \mathbf{Y}) & \leftarrow & Last^i(\mathbf{X}), First^i(\mathbf{Y}) \\
First^{i+1}(0, \mathbf{X}) & \leftarrow & First^i(\mathbf{X}) \\
Last^{i+1}(1, \mathbf{X}) & \leftarrow & Last^i(\mathbf{X})
\end{array}
$$

Defining $\mathit{Succ}^m(\mathbf{X}, \mathbf{X}')$ and $\leq^m$

- The ground facts $\mathit{Succ}^1(0, 1)$, $\mathit{First}^1(0)$, and $\mathit{Last}^1(1)$ are provided.
- For an inductive definition, suppose $\mathit{Succ}^i(\mathbf{X}, \mathbf{Y})$, $\mathit{First}^i(\mathbf{X})$, and $\mathit{Last}^i(\mathbf{X})$ tell the successor, the first, and the last element from a linear order $\leq^i$ on U^i, where $\mathbf{X}$ and $\mathbf{Y}$ have arity i. Then, use rules

$$
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\mathit{Succ}^{i+1}(1, \mathbf{X}, 1, \mathbf{Y}) & \leftarrow & \mathit{Succ}^i(\mathbf{X}, \mathbf{Y}) \\
\mathit{Succ}^{i+1}(0, \mathbf{X}, 1, \mathbf{Y}) & \leftarrow & \mathit{Last}^i(\mathbf{X}), \mathit{First}^i(\mathbf{Y}) \\
\mathit{First}^{i+1}(0, \mathbf{X}) & \leftarrow & \mathit{First}^i(\mathbf{X}) \\
\mathit{Last}^{i+1}(1, \mathbf{X}) & \leftarrow & \mathit{Last}^i(\mathbf{X})
\end{array}
$$

- The order $\leq^m$ is easily defined from Succ^m by two clauses

$$
\begin{array}{rcl}
\leq^m(\mathbf{X}, \mathbf{X}) & \leftarrow & \\
\leq^m(\mathbf{X}, \mathbf{Y}) & \leftarrow & \mathit{Succ}^m(\mathbf{X}, \mathbf{Z}),\ \leq^m(\mathbf{Z}, \mathbf{Y})
\end{array}
$$

Datalog Program Complexity Conclusion

- Let $P_{dat}(T, I, N)$ denote the datalog program with empty edb described for T, I, and $N = 2^m$, $m = n^k$ (where $n = |I|$)
- $P_{dat}(T, I, N)$ is constructible from T and I in polynomial time (in fact, careful analysis shows feasibility in logarithmic space).
- $P_{dat}(T, I, N)$ has *accept* in its least model $\Leftrightarrow$ T accepts input I within N steps.
- Thus, the decision problem for any language in EXPTIME is reducible to deciding $P \models A$ for datalog program P and fact A.
- Consequently, deciding $P \models A$ for a given datalog program P and fact A is EXPTIME-hard.

Complexity of Datalog with Stratified Negation

Theorem

Stratified propositional logic programming with negation is P*-complete. Stratified datalog with negation is data complete for* P *and program complete for* EXPTIME*.*

- stratified P can be partitioned into disjoint sets $S_1, \ldots, S_n$ s.t. the semantics of P is computed by successively computing fixpoints of the immediate consequence operators $\mathbf{T}_{S_1}, \ldots, \mathbf{T}_{S_n}$.
- Let $\mathbf{I}_0$ be the initial instance over the extensional predicate symbols of P and let $\mathbf{I}_i$ (with $1 \leq i \leq n$) be defined as follows:

$$\mathbf{I}_1 := \mathbf{T}^{\omega}_{S_1}(\mathbf{I}_0),\ \ \mathbf{I}_2 := \mathbf{T}^{\omega}_{S_2}(\mathbf{I}_1),\ \ \ldots,\ \ \mathbf{I}_n := \mathbf{T}^{\omega}_{S_n}(\mathbf{I}_{n-1})$$

  Then the semantics of program P is given through the set $\mathbf{I}_n$.
- In the propositional case, $\mathbf{I}_n$ is clearly polynomially computable. Hence, stratified negation does not increase the complexity.

Outline

7.4 Complexity Stable Model

Today we shall concentrate on

Query	Data Complexity	Program Complexity
Conjunctive query	AC_0	NP-complete
FO	AC_0	PSPACE-complete
Prop. LP		P-complete
Datalog	P-complete	EXPTIME-complete
Stratified Datalog	P-complete	EXPTIME-complete
Datalog(WFM)	P-complete	EXPTIME-complete
Datalog(INF)	P-complete	EXPTIME-complete
Datalog(Stable Model)	co-NP-complete	co-NEXPTIME-complete
Disjun. Datalog	Π_2^p-complete	co-NEXPTIME$^{\mathrm{NP}}$-complete

Recall Stable Model Semantics

Let S be a (possibly infinite) set of ground normal clauses, i.e., of formulas of the form $A \leftarrow L_1 \wedge \ldots \wedge L_n$ where $n \geq 0$ and A is a ground atom and the L_i for $1 \leq i \leq n$ are ground literals.

Gelfond-Lifschitz Transformation

Let $B \subseteq HB$. The Gelfond-Lifschitz transform $GL_B(S)$ of S with respect to B is obtained from S as follows:

1. remove each clause whose antecedent contains a literal $\neg A$ with $A \in B$.
2. remove from the antecedents of the remaining clauses all negative literals.

Stable Model

An Herbrand interpretation $HI(B)$ is a stable model of S iff it is the unique minimal Herbrand model of $GL_B(S)$.

Complexity Prop. LP Stable model

Theorem

Given a propositional normal logic program P, deciding whether P has a stable model is NP*-complete.*

Membership. Clearly, P^I is polynomial time computable from P and I. Hence, a stable model M of P can be guessed and checked in polynomial time.

Stable Model Prop. LP - Hardness

Proof hardness

- Encoding of a non-deterministic Turing machine (NDTM) T.
  - Given a NDTM T, an input string I and a number of steps N, where N is a polynomial of $|I|$, construct in logspace a program $P = P(T, I, N)$.
  - P has a stable model iff T accepts I in non-deterministically N steps.
- Much similar to the encoding of DTM with propositional LP. Modification on deterministic property.

Stable Model Prop. LP - Hardness

Example: $\langle s, \sigma, s_1, \sigma'_1, d_1\rangle$, $\langle s, \sigma, s_2, \sigma'_2, d_2\rangle$
Transition rules $0 \leq \tau < N$, $0 \leq \pi < N$, and $0 \leq \pi + d$.

$$\begin{array}{rcl}
\mathit{symbol}_{\sigma'_1}[\tau+1,\pi] & \leftarrow & \mathit{state}_s[\tau], \mathit{symbol}_\sigma[\tau,\pi], \mathit{cursor}[\tau,\pi] \\
\mathit{cursor}[\tau+1,\pi+d_1] & \leftarrow & \mathit{state}_s[\tau], \mathit{symbol}_\sigma[\tau,\pi], \mathit{cursor}[\tau,\pi] \\
\mathit{state}_{s_1}[\tau+1] & \leftarrow & \mathit{state}_s[\tau], \mathit{symbol}_\sigma[\tau,\pi], \mathit{cursor}[\tau,\pi]
\end{array}$$

$$\begin{array}{rcl}
\mathit{symbol}_{\sigma'_2}[\tau+1,\pi] & \leftarrow & \mathit{state}_s[\tau], \mathit{symbol}_\sigma[\tau,\pi], \mathit{cursor}[\tau,\pi] \\
\mathit{cursor}[\tau+1,\pi+d_2] & \leftarrow & \mathit{state}_s[\tau], \mathit{symbol}_\sigma[\tau,\pi], \mathit{cursor}[\tau,\pi] \\
\mathit{state}_{s_2}[\tau+1] & \leftarrow & \mathit{state}_s[\tau], \mathit{symbol}_\sigma[\tau,\pi], \mathit{cursor}[\tau,\pi]
\end{array}$$

What is wrong here?
Enforcement violated:
At any time instance τ, there is exactly one cursor; each cell of the tape contains exactly one element; in exactly one state.

Stable Model Prop. LP - Hardness

- For each state s and symbol σ, introduce atoms $B_{s,\sigma,1}[\tau]$,..., $B_{s,\sigma,k}[\tau]$ for all $1 \leq \tau < N$ and for all transitions $\langle s, \sigma, s_i, \sigma'_i, d_i \rangle$, where $1 \leq i \leq k$.
- Add $B_{s,\sigma,i}[\tau]$ in the bodies of the transition rules for $\langle s, \sigma, s_i, \sigma'_i, d_i \rangle$.
- Add the rule

$$B_{s,\sigma,i}[\tau] \quad \leftarrow \quad \neg B_{s,\sigma,1}[\tau], \ldots, \neg B_{s,\sigma,i-1}[\tau], \neg B_{s,\sigma,i+1}[\tau], \ldots, \neg B_{s,\sigma,k}[\tau].$$

  Intuitively, these rules non-deterministically select precisely one of the possible transitions for s and σ at time instant τ, whose transition rules are enabled via $B_{s,\sigma,i}[\tau]$.
- Finally, add a rule

$$\mathit{accept} \leftarrow \neg \mathit{accept}.$$

  It ensures that *accept* is true in every stable model.

Stable Model Prop. LP - Hardness

Example: $\langle s, \sigma, s_1, \sigma'_1, d_1\rangle$, $\langle s, \sigma, s_2, \sigma'_2, d_2\rangle$

$$\begin{array}{rcl}
\mathit{symbol}_{\sigma'_1}[\tau+1,\pi] & \leftarrow & \mathit{state}_s[\tau], \mathit{symbol}_\sigma[\tau,\pi], \mathit{cursor}[\tau,\pi], {\color{red}B_{s,\sigma,1}[\tau]} \\
\mathit{cursor}[\tau+1,\pi+d_1] & \leftarrow & \mathit{state}_s[\tau], \mathit{symbol}_\sigma[\tau,\pi], \mathit{cursor}[\tau,\pi], {\color{red}B_{s,\sigma,1}[\tau]} \\
\mathit{state}_{s_1}[\tau+1] & \leftarrow & \mathit{state}_s[\tau], \mathit{symbol}_\sigma[\tau,\pi], \mathit{cursor}[\tau,\pi], {\color{red}B_{s,\sigma,1}[\tau]} \\
\mathit{symbol}_{\sigma'_2}[\tau+1,\pi] & \leftarrow & \mathit{state}_s[\tau], \mathit{symbol}_\sigma[\tau,\pi], \mathit{cursor}[\tau,\pi], {\color{red}B_{s,\sigma,2}[\tau]} \\
\mathit{cursor}[\tau+1,\pi+d_2] & \leftarrow & \mathit{state}_s[\tau], \mathit{symbol}_\sigma[\tau,\pi], \mathit{cursor}[\tau,\pi], {\color{red}B_{s,\sigma,2}[\tau]} \\
\mathit{state}_{s_2}[\tau+1] & \leftarrow & \mathit{state}_s[\tau], \mathit{symbol}_\sigma[\tau,\pi], \mathit{cursor}[\tau,\pi], {\color{red}B_{s,\sigma,2}[\tau]}
\end{array}$$

$$\begin{array}{rcl}
B_{s,\sigma,1}[\tau] & \leftarrow & \neg B_{s,\sigma,2}[\tau] \\
B_{s,\sigma,2}[\tau] & \leftarrow & \neg B_{s,\sigma,1}[\tau]
\end{array}$$

One and only one atom from $B_{s,\sigma,1}[\tau]$ and $B_{s,\sigma,2}[\tau]$ is true. Which one?
Non-deterministic

Stable Model Prop. LP - Hardness

Proof.

- Assume there is a sequence of choices leading to the state yes.
  Let I be the set of the propositional atoms along the computation path reaching the state *accept*.
  Then *accept* $\in I$ due to the rule:

  $$\mathit{accept} \quad \leftarrow \quad \mathit{state}_{\mathrm{yes}}[\tau]$$

  Clearly I is a stable model of P.
- Assume there exists no sequence of choices leading to the state yes in the computation tree. Suppose I is a stable model of P and *accept* $\in I$.
  By minimality of I for P^I, it follows that $\mathit{state}_{\mathrm{yes}}[\tau] \in I$ for some τ;
  moreover, this means that a sequence of choices leads to yes.
  Contradiction.

Further Complexity Results

Theorem

Propositional logic programming with negation under well-founded semantics is P*-complete. Datalog with negation under well-founded semantics is data complete for* P *and program complete for* EXPTIME*.*

Theorem

Propositional logic programming with negation under inflationary semantics is P*-complete. Datalog with negation under inflationary semantics is data complete for* P *and program complete for* EXPTIME*.*

Further Complexity Results

Theorem

Propositional logic programming with negation under stable model semantics is co-NP*-complete. Datalog with negation under stable model semantics is data complete for* co-NP *and program complete for* co-NEXPTIME*.*

Note that the decision problem here is whether an atom is true in all stable models.

Further Complexity Results

Theorem

The program complexity of conjunctive queries is NP*-complete.*

Theorem

First-order queries are program-complete for PSPACE*. Their data complexity is in the class* AC^0*, which contains the languages recognized by unbounded fan-in circuits of polynomial size and constant depth.*

Further Complexity Results

Theorem

Logic programming is r.e.-complete.

Theorem

Nonrecursive logic programming is NEXPTIME*-complete.*

Outline

7.5 Expressive Power

- A query q defines a mapping $\mathcal{M}_q$ that assigns to each suitable input database D_{in} (over a fixed input schema) a result database $D_{out} = \mathcal{M}_q(D_{in})$ (over a fixed output schema)
- Formally, the expressive power of a query language Q is the set of mappings $\mathcal{M}_q$ for all queries q expressible in the language Q by some query expression (program) E
- Research tasks concerning expressive power:
  - Comparing two query languages Q_1 and Q_2 in their relative expressive power (e.g. FO vs. SQL vs. Datalog).
    This is important for designing and analysing a query language.
  - determining the absolute expressive power of a query language, e.g. proving that a given query language Q is able to express exactly all queries whose evaluation complexity is in a complexity class $\mathcal{C}$.
    We say Q captures $\mathcal{C}$ and write simply $Q = \mathcal{C}$.

Expressive Power

There is a substantial difference between showing that the query evaluation problem for a certain query language Q is $\mathcal{C}$-complete and showing that Q captures $\mathcal{C}$.

- If the evaluation problem for Q is $\mathcal{C}$-complete, then at least one $\mathcal{C}$-hard query is expressible in Q.
- If Q captures $\mathcal{C}$, then Q expresses all queries evaluable in $\mathcal{C}$ (including, of course, all $\mathcal{C}$-hard queries).
- Example: Evaluating Datalog is P hard (data complexity), but positive Datalog can only express monotone properties, however, there are of course problems in P which are non-monotonic.

Expressive Power: Ordered Structures

- To prove that a query language Q captures a machine-based complexity class $\mathcal{C}$, one usually shows that each $\mathcal{C}$-machine with (encodings of) finite structures as inputs that computes a generic query can be represented by an expression in language Q.
- A Turing machine works on a string encoding of the input database D. Such an encoding provides an implicit *linear order* on D, in particular, on all elements of the universe U_D
- Therefore, one often assumes that a linear ordering of the universe elements is predefined
- We consider here ordered databases whose schemas contain special relation symbols *Succ*, *First*, and *Last*

Expressive Power: Datalog

Theorem

datalog$^+$ $\subsetneq$ P.

Show that there exists no datalog$^+$ program P that can tell whether the universe U of the input database has an even number of elements.

Theorem

On ordered databases, datalog$^+$ *captures* P.

Expressive Power: More Results

Theorem

Nonrecursive range-restricted datalog with negation
= relational algebra
= domain-independent relational calculus
= first-order logic (without function symbols).

Theorem

On ordered databases, the following query languages capture P*:*

- *stratified datalog,*
- *datalog under well-founded semantics,*
- *datalog under inflationary semantics.*

Theorem

Datalog under stable model semantics captures co-NP*.*