

# Foundations of Data and Knowledge Systems

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### 2. First-Order Predicate Logic

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# Outline

## 2. Predicate Logic

- 2.1 Query Languages and Logic
- 2.2 Syntax of First-Order Predicate Logic
- 2.3 Semantics of First-Order Predicate Logic
- 2.4 Equality
- 2.5 Undecidability
- 2.6 Model Cardinalities

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### 2.1 Query Languages and Logic

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# Query Languages and Logic

## Motivation

- Foundations of query languages mostly stem from logic (and complexity theory)
- New query languages with new syntactical constructs and concepts depart from classical logic but keep a logical flavour.
- Typical strengths of this logical flavour are:
  - compound queries using connectives such as “and” and “or”
  - rules expressed as implications
  - declarative semantics reminiscent of Tarski’s model semantics
  - query optimisation based on equivalences of logical formulas
  - query evaluators based on methods and heuristics similar to, even though in some cases simpler than, those of theorem provers.

# What are Query Languages?

## Tentative Definitions

- 1 What are ... their purposes of use?  
selecting and retrieving data from “information systems”
- 2 What are ... their programming paradigms?  
declarative, hence related to logic
- 3 What are ... their major representatives?  
SQL (relational data), OQL (object-oriented data),  
XPath, XQuery (HTML and XML data),  
RQL, RDQL, SPARQL (RDF data, OWL ontologies)
- 4 What are ... their research issues?  
query paradigms, declarative semantics, complexity and expressive power,  
procedural semantics, implementations, optimisation, etc.

# Logic vs. Logics

## The development of logic(s)

- starting in antiquity: logic as an activity of philosophy aimed at **analysing rational reasoning**.
- late 19th century: parts of logic were **mathematically formalised**.
- early 20th century: logic used as a tool in a (not fully successful) attempt to overcome a foundational crisis of mathematics.
- logic in computer science: Today, logic provides the **foundations in many areas of computer science**, such as knowledge representation, database theory, programming languages, and query languages.
- Key features of logic: the use of **formal languages** for representing statements (which may be true or false) and the quest for **computable reasoning** about those statements.
- Logic vs. logics: “**Logic**” is the name of the scientific discipline investigating such formal languages for statements, but any of those languages is also called “**a logic**” – **logic investigates logics**.

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# Preliminaries

## Definition (Enumerable)

A set  $S$  is called **enumerable** if there is a surjection  $\mathbb{N} \rightarrow S$ .

A set  $S$  is called **computably enumerable** (or **recursively enumerable**), if it is enumerable with a surjection that is computable by some algorithm.

## Example

- Any finite set is enumerable and computably enumerable.
- The infinite set of all syntactically correct C programs is computably enumerable and thus enumerable.
- Its subset consisting of those syntactically correct C programs that do not terminate for each input is enumerable, but not computably enumerable.



# Symbols

## Symbols in First-Order Predicate Logic

First-order predicate logic is not just a single formal language, because some of its symbols may depend on the intended application.

- The symbols common to all languages of first-order predicate logic are called **logical symbols**.
- The symbols that are specified in order to determine a specific language are called the **signature** (or **vocabulary**) of that language.

## Definition (Signature)

A **signature** or **vocabulary** for first-order predicate logic is a pair  $\mathcal{L} = (\{Fun_{\mathcal{L}}^n\}_{n \in \mathbb{N}}, \{Rel_{\mathcal{L}}^n\}_{n \in \mathbb{N}})$  of two families of computably enumerable symbol sets, called  $n$ -ary **function symbols** of  $\mathcal{L}$  and  $n$ -ary **relation symbols** or **predicate symbols** of  $\mathcal{L}$ .

The 0-ary function symbols are called **constants** of  $\mathcal{L}$ . The 0-ary relation symbols are called **propositional relation symbols** of  $\mathcal{L}$ .

# Logical Symbols

## Definition (Logical Symbols)

The **logical symbols** of first-order predicate logic are:

| symbol class        |       | symbols                                       | name                   |
|---------------------|-------|---|------------------------|
| punctuation symbols |       | , ) (   |                        |
| connectives         | 0-ary | $\perp$                                       | falsity symbol         |
|                     |       | $\top$  | truth symbol           |
|                     | 1-ary | $\neg$  | negation symbol        |
|                     | 2-ary | $\wedge$                                      | conjunction symbol     |
| $\vee$              |       | disjunction symbol                            |                        |
| $\Rightarrow$       |       | implication symbol                            |                        |
| quantifiers         |       | $\forall$                                     | universal quantifier   |
|                     |       | $\exists$                                     | existential quantifier |
| variables           |       | $u v w x y z \dots$<br>(possibly subscripted) |                        |

The set of variables is infinite and computably enumerable.

# Terms and Atoms

## Definition ( $\mathcal{L}$ -term)

Let  $\mathcal{L}$  be a signature. **Terms** are defined inductively:

- 1 Each variable  $x$  is an  $\mathcal{L}$ -term.
- 2 Each constant  $c$  of  $\mathcal{L}$  is an  $\mathcal{L}$ -term.
- 3 For each  $n \geq 1$ , if  $f$  is an  $n$ -ary function symbol of  $\mathcal{L}$  and  $t_1, \dots, t_n$  are  $\mathcal{L}$ -terms, then  $f(t_1, \dots, t_n)$  is an  $\mathcal{L}$ -term.

## Definition ( $\mathcal{L}$ -atom)

Let  $\mathcal{L}$  be a signature.

For  $n \in \mathbb{N}$ , if  $p$  is an  $n$ -ary relation symbol of  $\mathcal{L}$  and  $t_1, \dots, t_n$  are  $\mathcal{L}$ -terms, then  $p(t_1, \dots, t_n)$  is an  $\mathcal{L}$ -atom or **atomic  $\mathcal{L}$ -formula**.

For  $n = 0$ , the atom may be written  $p()$  or  $p$  and is called a **propositional  $\mathcal{L}$ -atom**.

# Formulas

## Definition ( $\mathcal{L}$ -formula)

Let  $\mathcal{L}$  be a signature. **Formulas** are defined inductively:

- 1 Each  $\mathcal{L}$ -atom is an  $\mathcal{L}$ -formula. (atoms)
- 2  $\perp$  and  $\top$  are  $\mathcal{L}$ -formulas. (0-ary connectives)
- 3 If  $\varphi$  is an  $\mathcal{L}$ -formula, then  $\neg\varphi$  is an  $\mathcal{L}$ -formula. (1-ary connectives)
- 4 If  $\varphi$  and  $\psi$  are  $\mathcal{L}$ -formulas, then  $(\varphi \wedge \psi)$  and  $(\varphi \vee \psi)$  and  $(\varphi \Rightarrow \psi)$  are  $\mathcal{L}$ -formulas. (2-ary connectives)
- 5 If  $x$  is a variable and  $\varphi$  is an  $\mathcal{L}$ -formula, then  $\forall x\varphi$  and  $\exists x\varphi$  are  $\mathcal{L}$ -formulas. (quantifiers)

## Remark

In most cases the signature  $\mathcal{L}$  is clear from context, and we simply speak of terms, atoms, and formulas without the prefix “ $\mathcal{L}$ -”.

# Notational Conventions

## Symbols

In particular, if no signature is specified, one usually assumes the conventions:

- $p, q, r, \dots$  are relation symbols with appropriate arities.
- $f, g, h, \dots$  are function symbols with appropriate arities  $\neq 0$ .
- $a, b, c, \dots$  are constants, i.e., function symbols with arity 0.

## Use of Parentheses

**Unique Parsing of Terms and Formulas.** Since formulas constructed with a binary connective are enclosed by parentheses, any term or formula has an unambiguous syntactical structure.

**Precedence of Operators.** For the sake of readability this strict syntax definition can be relaxed by the convention that  $\wedge$  takes precedence over  $\vee$  and both of them take precedence over  $\Rightarrow$ .

**Example.**  $q(a) \vee q(b) \wedge r(b) \Rightarrow p(a, f(a, b))$  is a shorthand for the fully parenthesised form  $((q(a) \vee (q(b) \wedge r(b))) \Rightarrow p(a, f(a, b)))$ .

# Variables in Formulas

## Example (Bound/free variable)

Let  $\varphi$  be  $(\forall x[\exists xp(x) \wedge q(x)] \Rightarrow [r(x) \vee \forall xs(x)])$ . The  $x$  in  $p(x)$  is bound in  $\varphi$  by  $\exists x$ . The  $x$  in  $q(x)$  is bound in  $\varphi$  by the first  $\forall x$ . The  $x$  in  $r(x)$  is free in  $\varphi$ . The  $x$  in  $s(x)$  is bound in  $\varphi$  by the last  $\forall x$ .

Let  $\varphi'$  be  $\forall x([\exists xp(x) \wedge q(x)] \Rightarrow [r(x) \vee \forall xs(x)])$ . Here both the  $x$  in  $q(x)$  and the  $x$  in  $r(x)$  are bound in  $\varphi'$  by the first  $\forall x$ .

## Definition (Rectified formula)

A formula  $\varphi$  is **rectified**, if for each occurrence  $Qx$  of a quantifier for a variable  $x$ , there is neither any free occurrence of  $x$  in  $\varphi$  nor any other occurrence of a quantifier for the same variable  $x$ .

## Remark

Any formula can be rectified by consistently renaming its quantified variables. E.g., the above  $\varphi$  can be rectified to  $(\forall u[\exists vp(v) \wedge q(u)] \Rightarrow [r(x) \vee \forall ws(w)])$ .

## Ground and Propositional Case

### Definition (Ground term or formula, closed formula)

A **ground term** is a term containing no variable.

A **ground formula** is a formula containing no variable.

A **closed formula** or **sentence** is a formula containing no free variable.

### Definition (Propositional formula)

A **propositional formula** is a formula containing no quantifier and no relation symbol of arity  $> 0$ .

### Ground vs. Propositional

Obviously, each propositional formula is ground. Conversely, every ground formula can be regarded as propositional in a broader sense:

Let  $\mathcal{L}$  be an arbitrary signature and let  $\mathcal{L}'$  be a new signature defining each ground  $\mathcal{L}$ -atom as a 0-ary relation “symbol” of  $\mathcal{L}'$ . Then each ground  $\mathcal{L}$ -formula can be considered as a propositional  $\mathcal{L}'$ -formula.

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# Semantics of First-Order Predicate Logic

## Classical Tarski Model Theory

- Goal: attribution of meaning to terms and formulas
- Principle of a Tarski-style semantics: The interpretation of a compound term and the truth value of a compound formula are defined recursively over the structure of the term or formula.
- Advantage of this approach: recursive definition makes things simple; well-defined, finite, and restricted computation scope.
- Disadvantage of this approach: allowing for any kind of sets for interpreting terms makes it apparently incomputable.

# Semantics of First-Order Predicate Logic

## Definition (Variable assignment)

Let  $D$  be a nonempty set. A **variable assignment in  $D$**  is a function  $V$  mapping each variable to an element of  $D$ . We denote the image of  $x$  under  $V$  by  $x^V$ .

## Definition ( $\mathcal{L}$ -Interpretation)

Let  $\mathcal{L}$  be a signature. An  **$\mathcal{L}$ -interpretation** is a triple  $\mathcal{I} = (D, I, V)$  where

- $D$  is a nonempty set called the **domain** or **universe (of discourse)** of  $\mathcal{I}$ .

**Notation:**  $dom(\mathcal{I}) := D$ .

- $I$  is a function defined on the symbols of  $\mathcal{L}$  mapping
  - each  $n$ -ary function symbol  $f$  to an  $n$ -ary function  $f^I : D^n \rightarrow D$ .  
For  $n = 0$  this means  $f^I \in D$ .
  - each  $n$ -ary relation symbol  $p$  to an  $n$ -ary relation  $p^I \subseteq D^n$ .  
For  $n = 0$  this means either  $p^I = \emptyset$  or  $p^I = \{()\}$ .

**Notation:**  $f^{\mathcal{I}} := f^I$  and  $p^{\mathcal{I}} := p^I$ .

- $V$  is a variable assignment in  $D$ . **Notation:**  $x^{\mathcal{I}} := x^V$ .

# Value of Terms

## Definition

The value of a term  $t$  in an interpretation  $\mathcal{I}$ , denoted  $t^{\mathcal{I}}$ , is an element of  $\text{dom}(\mathcal{I})$  and inductively defined:

- 1 If  $t$  is a variable or a constant, then  $t^{\mathcal{I}}$  is defined as above.
- 2 If  $t$  is a compound term  $f(t_1, \dots, t_n)$ , then  $t^{\mathcal{I}}$  is defined as  $f^{\mathcal{I}}(t_1^{\mathcal{I}}, \dots, t_n^{\mathcal{I}})$

## Notation

Let  $V$  be a variable assignment in  $D$ ,  $x \in V$ , and  $d \in D$ . Then  $V[x \mapsto d]$  is the variable assignment which, for every variable  $z$ , is defined as follows:

$$z^{V[x \mapsto d]} = \begin{cases} d & \text{if } z = x \\ z^V & \text{if } z \neq x \end{cases}$$

Let  $\mathcal{I} = (D, I, V)$  be an interpretation. Then  $\mathcal{I}[x \mapsto d] := (D, I, V[x \mapsto d])$ .

# Value of Formulas

## Definition (Tarski, model relationship)

Let  $\mathcal{I}$  be an interpretation and  $\varphi$  a formula. The **relationship**  $\mathcal{I} \models \varphi$ , pronounced “ $\mathcal{I}$  is a model of  $\varphi$ ” or “ $\mathcal{I}$  satisfies  $\varphi$ ” or “ $\varphi$  is true in  $\mathcal{I}$ ”, and its negation  $\mathcal{I} \not\models \varphi$ , pronounced “ $\mathcal{I}$  falsifies  $\varphi$ ” or “ $\varphi$  is false in  $\mathcal{I}$ ”, are defined inductively:

$$\mathcal{I} \models p(t_1, \dots, t_n) \quad \text{iff} \quad (t_1^{\mathcal{I}}, \dots, t_n^{\mathcal{I}}) \in p^{\mathcal{I}} \quad (n\text{-ary } p, n \geq 1)$$

$$\mathcal{I} \models p \quad \text{iff} \quad () \in p^{\mathcal{I}} \quad (0\text{-ary } p)$$

$$\mathcal{I} \not\models \perp$$

$$\mathcal{I} \models \top$$

$$\mathcal{I} \models \neg\psi \quad \text{iff} \quad \mathcal{I} \not\models \psi$$

$$\mathcal{I} \models (\psi_1 \wedge \psi_2) \quad \text{iff} \quad \mathcal{I} \models \psi_1 \text{ and } \mathcal{I} \models \psi_2$$

$$\mathcal{I} \models (\psi_1 \vee \psi_2) \quad \text{iff} \quad \mathcal{I} \models \psi_1 \text{ or } \mathcal{I} \models \psi_2$$

$$\mathcal{I} \models (\psi_1 \Rightarrow \psi_2) \quad \text{iff} \quad \mathcal{I} \not\models \psi_1 \text{ or } \mathcal{I} \models \psi_2$$

$$\mathcal{I} \models \forall x \psi \quad \text{iff} \quad \mathcal{I}[x \mapsto d] \models \psi \text{ for each } d \in D$$

$$\mathcal{I} \models \exists x \psi \quad \text{iff} \quad \mathcal{I}[x \mapsto d] \models \psi \text{ for at least one } d \in D$$

For a set  $S$  of formulas,  $\mathcal{I} \models S$  iff  $\mathcal{I} \models \varphi$  for each  $\varphi \in S$ .

# Example

## Signature:

function symbols: 0-ary  $a, b$  1-ary  $f$

relation symbols: 1-ary  $p, q$  2-ary  $r$

## Interpretation $\mathcal{I}$ :

$$\text{dom}(\mathcal{I}) = \{ \text{red person}, \text{blue person}, \text{green person}, \text{red person} \}$$

$$a^{\mathcal{I}} = \text{red person} \quad b^{\mathcal{I}} = \text{red person} \quad f^{\mathcal{I}} = \{ \text{red person} \mapsto \text{blue person}, \text{blue person} \mapsto \text{green person}, \text{green person} \mapsto \text{blue person}, \text{red person} \mapsto \text{green person} \}$$

$$p^{\mathcal{I}} = \{ \text{red person}, \text{red person} \} \quad q^{\mathcal{I}} = \{ \text{red person}, \text{blue person} \}$$

$$r^{\mathcal{I}} = \{ (\text{red person}, \text{blue person}), (\text{red person}, \text{green person}), (\text{red person}, \text{red person}), (\text{red person}, \text{blue person}), (\text{red person}, \text{green person}) \}$$

## Model relationship " $\models$ ":

$$\mathcal{I} \models q(a) \wedge r(a, b) \wedge \neg r(f(a), b) \wedge \forall x (p(x) \Rightarrow r(x, f(x)))$$

# Semantic Properties, Entailment, Logical Equivalence

Semantic Properties. A formula is

**valid** iff it is satisfied in each interpretation

$$p \vee \neg p$$

**satisfiable** iff it is satisfied in at least one interpretation

$$p$$

**falsifiable** iff it is falsified in at least one interpretation

$$p$$

**unsatisfiable** iff it is falsified in each interpretation

$$p \wedge \neg p$$

Entailment, Logical Equivalence. For formulas  $\varphi$  and  $\psi$

$\varphi \models \psi$  iff for each interpretation  $\mathcal{I}$ :

if  $\mathcal{I} \models \varphi$  then  $\mathcal{I} \models \psi$

$$(p \wedge q) \models (p \vee q)$$

$\varphi \vDash \psi$  iff  $\varphi \models \psi$  and  $\psi \models \varphi$

$$(p \wedge q) \vDash (q \wedge p)$$

**Inter-translatability:** Being able to determine one of validity, unsatisfiability, or entailment, is sufficient to determine all of them:

$\varphi$  is valid iff  $\neg\varphi$  is unsatisfiable iff  $\top \models \varphi$ .

$\varphi$  is unsatisfiable iff  $\neg\varphi$  is valid iff  $\varphi \models \perp$ .

$\varphi \models \psi$  iff  $(\varphi \Rightarrow \psi)$  is valid iff  $(\varphi \wedge \neg\psi)$  is unsatisfiable.

# Calculi, Proof Systems

## Motivation

- Entailment  $\varphi \models \psi$  formalises the concept of **logical consequence**.
- A major concern in logic is the development of **calculi**, also called **proof systems**, which formalise the notion of **deductive inference**.

## Definition (Calculus)

- A **calculus** defines derivation rules, with which formulas can be derived from formulas by purely syntactic operations.
- The derivability relationship  $\varphi \vdash \psi$  for a calculus holds iff there is a finite sequence of applications of derivation rules of the calculus, which applied to  $\varphi$  result in  $\psi$ .
- Ideally, derivability should mirror entailment: a calculus is called **sound** iff whenever  $\varphi \vdash \psi$  then  $\varphi \models \psi$  and **complete** iff whenever  $\varphi \models \psi$  then  $\varphi \vdash \psi$ .

# Important Results about Tarski Model Theory

## Theorem (Gödel, completeness theorem)

*There exist calculi for first-order predicate logic such that  $S \vdash \varphi$  iff  $S \models \varphi$  for any set  $S$  of closed formulas and any closed formula  $\varphi$ .*

## Theorem (Church-Turing, undecidability theorem)

*For signatures with a non-propositional relation symbol and a relation or function symbol of arity  $\geq 2$ , satisfiability is undecidable.*

## Theorem (Gödel-Malcev, finiteness or compactness theorem)

*Let  $S$  be an infinite set of closed formulas. If every finite subset of  $S$  is satisfiable, then  $S$  is satisfiable.*

**Remark.** Proofs to be provided later.



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# Equality

## Motivation

- In many applications, we want to express equality: For this purpose, let the signature  $\mathcal{L}$  contain a special 2-ary relation symbol  $\doteq$  for equality.
- The relation symbol  $\doteq$  shall indeed be *interpreted* as equality: we consider **normal interpretations** (and thus treat equality as a built-in predicate).
- Alternatively, we can add **equality axioms** to the formula: this is fine for many purposes; but it does not exclude non-normal models!

## Definition (Normal interpretation)

An interpretation  $\mathcal{I}$  is **normal**, iff it interprets the relation symbol  $\doteq$  with the equality relation on its domain, i.e.,  $\doteq^{\mathcal{I}}$  is the relation  $\{(d, d) \mid d \in \text{dom}(\mathcal{I})\}$ .  
For formulas or sets of formulas  $\varphi$  and  $\psi$ , we write:

$\mathcal{I} \models_{=} \varphi$  iff  $\mathcal{I}$  is normal and  $\mathcal{I} \models \varphi$ .

$\varphi \models_{=} \psi$  iff for each normal interpretation  $\mathcal{I}$ : if  $\mathcal{I} \models_{=} \varphi$  then  $\mathcal{I} \models_{=} \psi$ .

# Equality Axioms

## Definition (Equality axioms)

Given a signature  $\mathcal{L}$  with 2-ary relation symbol  $\doteq$ , the set  $EQ_{\mathcal{L}}$  of **equality axioms** for  $\mathcal{L}$  consists of the formulas:

- $\forall x x \doteq x$  (reflexivity of  $\doteq$ )
- $\forall x \forall y (x \doteq y \Rightarrow y \doteq x)$  (symmetry of  $\doteq$ )
- $\forall x \forall y \forall z ((x \doteq y \wedge y \doteq z) \Rightarrow x \doteq z)$  (transitivity of  $\doteq$ )
- for each  $n$ -ary function symbol  $f$ ,  $n > 0$  (substitution axiom for  $f$ )  
 $\forall x_1 \dots x_n \forall x'_1 \dots x'_n ((x_1 \doteq x'_1 \wedge \dots \wedge x_n \doteq x'_n) \Rightarrow$   
 $f(x_1, \dots, x_n) \doteq f(x'_1, \dots, x'_n))$
- for each  $n$ -ary relation symbol  $p$ ,  $n > 0$  (substitution axiom for  $p$ )  
 $\forall x_1 \dots x_n \forall x'_1 \dots x'_n ((x_1 \doteq x'_1 \wedge \dots \wedge x_n \doteq x'_n \wedge p(x_1, \dots, x_n)) \Rightarrow$   
 $p(x'_1, \dots, x'_n))$

## Theorem (Equality axioms)

- For each interpretation  $\mathcal{I}$ , if  $\mathcal{I}$  is normal then  $\mathcal{I} \models EQ_{\mathcal{L}}$ .
- For each interpretation  $\mathcal{I}$  with  $\mathcal{I} \models EQ_{\mathcal{L}}$  there is a normal interpretation  $\mathcal{I}_=$  such that for each formula  $\varphi$ :  $\mathcal{I} \models \varphi$  iff  $\mathcal{I}_= \models \varphi$ .
- For each set  $S$  of formulas and formula  $\varphi$ :  $EQ_{\mathcal{L}} \cup S \models \varphi$  iff  $S \models \varphi$ .

## Corollary (Finiteness or compactness theorem with equality)

Let  $S$  be an infinite set of closed formulas with equality. If every finite subset of  $S$  has a normal model, then  $S$  has a normal model.

# Model Extension Theorem and Non-normal Models

## Theorem (Model extension theorem)

*For each interpretation  $\mathcal{I}$  and each set  $D' \supseteq \text{dom}(\mathcal{I})$  there is an interpretation  $\mathcal{I}'$  with  $\text{dom}(\mathcal{I}') = D'$  such that for each formula  $\varphi$ :  $\mathcal{I} \models \varphi$  iff  $\mathcal{I}' \models \varphi$ .*

## Proof (sketch)

Fix an arbitrary element  $d \in \text{dom}(\mathcal{I})$ . The idea is to let all “new” elements behave exactly like  $d$ . For this purpose, we define an auxiliary function  $\pi$  mapping each “new” element to  $d$  and each “old” element to itself:

$\pi : D' \rightarrow \text{dom}(\mathcal{I})$ ,  $\pi(d') := d$  if  $d' \notin \text{dom}(\mathcal{I})$ ,  $\pi(d') := d'$  if  $d' \in \text{dom}(\mathcal{I})$ .

Then we define  $f^{\mathcal{I}'} : D'^n \rightarrow D'$ ,  $f^{\mathcal{I}'}(d_1, \dots, d_n) := f^{\mathcal{I}}(\pi(d_1), \dots, \pi(d_n))$  and  $p^{\mathcal{I}'} \subseteq D'^n$ ,  $p^{\mathcal{I}'} := \{(d_1, \dots, d_n) \in D'^n \mid (\pi(d_1), \dots, \pi(d_n)) \in p^{\mathcal{I}}\}$  for all signature symbols and arities. □

# Model Extension Theorem and Non-normal Models

## Corollary (Existence of non-normal models)

*Every satisfiable set of formulas has non-normal models.*

## Proof (sketch)

By the construction in the above proof, if  $(d, d) \in \dot{=}^{\mathcal{I}}$  then  $(d, d') \in \dot{=}^{\mathcal{I}'}$  for each  $d' \in D'$  and the fixed element  $d \in \text{dom}(\mathcal{I})$ . Hence, if  $\mathcal{I}'$  is any proper extension of a normal interpretation  $\mathcal{I}$ , then  $\mathcal{I}'$  is not normal.  $\square$

## Remarks

- Every model of  $EQ_{\mathcal{L}}$  interprets  $\dot{=}$  by a congruence relation on the domain.
- The equality relation is the special case with singleton congruence classes.
- Because of the model extension theorem, there is no way to prevent models with larger congruence classes, unless equality is treated as built-in by making interpretations normal by definition.

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# Undecidability of First-Order Predicate Logic

## Motivation

- We inspect a proof of the undecidability of (the satisfiability or validity of) first-order predicate logic.
- The proof is folklore (communicated by Gernot Salzer)
  - It does not make use of equality at all.
  - The first-order formula is a conjunction of Horn clauses.

## Proof idea

We reduce (a variant of) the **Halting Problem** to the unsatisfiability problem:

*Given a deterministic Turing machine  $T$  with halting state  $h$ , it is undecidable if  $T$  when starting with the empty tape eventually reaches the halting state  $h$ .*



# Turing Machines

## Definition (Deterministic Turing machine)

A **deterministic Turing machine (DTM)** is defined as a quadruple  $(S, \Sigma, \delta, q_0)$  with the following meaning:  $S$  is a finite set of **states**,  $\Sigma$  is a finite alphabet of **symbols**,  $\delta$  is a **transition function**, and  $q_0 \in S$  is the **initial state**. The alphabet  $\Sigma$  contains a special symbol  $\sqcup$  called **blank**. The transition function  $\delta$  is a map

$$\delta: S \times \Sigma \rightarrow (S \cup \{\mathbf{h}\}) \times \Sigma \times \{-1, 0, +1\},$$

where  $\mathbf{h}$  denotes an additional state (the halting state) not occurring in  $S$ , and  $-1, 0, +1$  denote motion directions.

We may assume w.l.o.g., that the machine never moves off the left end of the tape, i.e.,  $d \neq -1$  whenever the cursor is on the leftmost cell; this can be easily ensured by a special symbol  $\triangleright$  which marks the left end of the tape.

# Computation of a Turing Machine

## Configurations

Let  $T$  be a DTM  $(\Sigma, S, \delta, q_0)$ . The tape of  $T$  is divided into cells containing symbols of  $\Sigma$ . There is cursor that may move along the tape. At every time instant, the current **configuration** of  $T$  is characterized by a tuple  $(q, w, \sigma, w')$ , where  $q$  denotes the **state**,  $w$  and  $w'$  denote the **tape contents** (written as string) to the left/right of the cursor and  $\sigma$  denotes the **currently scanned symbol**.

## Initial Configuration

On input string  $I$ , the TM  $T$  is initially in configuration  $(q_0, \varepsilon, \triangleright, I)$ , i.e.,  $T$  is in the initial state  $q_0$ , the tape contains the start symbol  $\triangleright$  followed by the input string  $I$ , and the cursor points to the leftmost cell of the tape.

## Notation

We denote a configuration  $(q, w, \sigma, w')$  in the format  $B\Sigma^*S\Sigma^*E$ , with the state symbol written in front of the currently scanned tape symbol.  $B$  and  $E$  are symbols marking the beginning and the end of the tape contents  $(\omega\sigma w')$ .

## Computation Step

The **transition relation** for  $T$ , denoted by  $\vdash_T$ , is defined as follows:

- 1  $Bw\alpha q\sigma w'E \vdash_T Bwq'a\sigma'w'E$ , if  $\delta(q, \sigma) = (q', \sigma', -1)$ .
- 2  $Bwq\sigma w'E \vdash_T Bwq'\sigma'w'E$ , if  $\delta(q, \sigma) = (q', \sigma', 0)$ .
- 3  $Bwq\sigma aw'E \vdash_T Bw\sigma'q'aw'E$  and  $Bwq\sigma E \vdash_T Bw\sigma'q' \sqcup E$ ,  
if  $\delta(q, \sigma) = (q', \sigma', +1)$ .

We write  $\vDash_T^*$  to denote the reflexive and transitive closure of  $\vdash_T$ .

## Halting

$T$  halts when it reaches the state  $\mathfrak{h}$ , i.e., there exist values  $w$ ,  $\sigma$ , and  $w'$ , s.t.  $T$  reaches the configuration  $(\mathfrak{h}, w, \sigma, w')$ .

That is,  $T$  halts on input  $I$  if  $Bq_0 \triangleright IE \vDash_T^* Bw\mathfrak{h}\sigma w'E$  for some  $w$ ,  $\sigma$ , and  $w'$ .

# Proof of the Undecidability of First-Order Predicate Logic

## Encoding of TM configurations as atoms

For every state  $q \in S$ , let  $\hat{q}$  be a constant symbol.

For every tape symbol  $a \in \Sigma$ , let  $\hat{a}$  be a unary function symbol.

The constant symbols  $\hat{B}$  and  $\hat{E}$  correspond to the end-of-tape markers  $B$  and  $E$ .

A configuration  $B\sigma_1 \dots \sigma_m q \sigma_{m+1} \dots \sigma_n E$  is represented by the atom

$$P(\widehat{\sigma_m}(\dots \widehat{\sigma_1}(\hat{B}) \dots), \hat{q}, \widehat{\sigma_{m+1}}(\dots \widehat{\sigma_n}(\hat{E}) \dots))$$

(The tape to the left of the current position is represented in reversed order.)

## Encoding of TM computations

Given an arbitrary TM  $T$ , we define the set  $\Phi_T$  of formulas as the smallest set containing the following formulas (i.e., Horn clauses):

- 1 If  $\delta(q, \sigma) = (q', \sigma', -1)$  then for all  $a \in \Sigma$ ,  
 $(\forall x)(\forall y)P(\widehat{a}(x), \widehat{q}, \widehat{\sigma}(y)) \Rightarrow P(x, \widehat{q}', \widehat{a}(\widehat{\sigma}'(y))) \in \Phi_T$
- 2 If  $\delta(q, \sigma) = (q', \sigma', 0)$  then  $(\forall x)(\forall y)P(x, \widehat{q}, \widehat{\sigma}(y)) \Rightarrow P(x, \widehat{q}', \widehat{\sigma}'(y)) \in \Phi_T$
- 3 If  $\delta(q, \sigma) = (q', \sigma', +1)$  then  $(\forall x)(\forall y)P(x, \widehat{q}, \widehat{\sigma}(y)) \Rightarrow P(\widehat{\sigma}'(x), \widehat{q}', y) \in \Phi_T$
- 4  $(\forall x)P(x, \widehat{q}, \widehat{E}) \Rightarrow P(x, \widehat{q}, \widehat{\sqcap}(\widehat{E})) \in \Phi_T.$

## Proposition

For any Turing machine  $T$ , every  $v, v', w, w' \in \Sigma^*$  and  $q, q' \in S$  with  $v = v_1, \dots, v_r$ ,  $v' = v'_1, \dots, v'_{r'}$ ,  $w = w_1, \dots, w_s$ , and  $w = w'_1, \dots, w'_{s'}$ :  
 $BvqwE \stackrel{*}{\vdash}_T Bv'q'w'E$  iff

$$\Phi_T \models P(\widehat{v}_r(..\widehat{v}_1(\widehat{B})..), \widehat{q}, \widehat{w}_1(..\widehat{w}_s(\widehat{E})..)) \Rightarrow P(\widehat{v}'_{r'}(..\widehat{v}'_1(\widehat{B})..), \widehat{q}', \widehat{w}'_1(..\widehat{w}'_{s'}(\widehat{E})..))$$

For any Turing machine  $T$ ,  $\Phi_T \cup \{P(\widehat{B}, \widehat{q}_0, \widehat{\triangleright}(\widehat{E})), (\forall x)(\forall y)\neg P(x, \widehat{h}, y)\}$  is **unsatisfiable** iff  $T$ , when starting with the empty tape, eventually halts.

# Outline

## 2. Predicate Logic

2.1 Query Languages and Logic

2.2 Syntax of First-Order Predicate Logic

2.3 Semantics of First-Order Predicate Logic

2.4 Equality

2.5 Undecidability

**2.6 Model Cardinalities**

# Model Cardinalities

## Motivation

We sometimes want to enforce that a formula only has models of a certain cardinality, e.g.:

- (only) infinite models
- (only) finite models
- (only) finite models with cardinality bounded by some constant
- etc.

Some of these properties cannot be expressed in first-order logic (possibly not even if we may use equality).

## Theorem

*Lower bounds of model cardinalities can be expressed in first-order predicate logic (even without equality).*

## Example

All models of the following satisfiable set of formulas have domains with cardinality  $\geq 3$ :

$$\left\{ \begin{array}{l} \exists x_1 ( p_1(x_1) \wedge \neg p_2(x_1) \wedge \neg p_3(x_1)), \\ \exists x_2 (\neg p_1(x_2) \wedge p_2(x_2) \wedge \neg p_3(x_2)), \\ \exists x_3 (\neg p_1(x_3) \wedge \neg p_2(x_3) \wedge p_3(x_3)) \end{array} \right\}$$

## Example

All models of the following satisfiable set of formulas have infinite domains:

$$\{ \forall x \neg(x < x), \quad \forall x \forall y \forall z (x < y \wedge y < z \Rightarrow x < z), \quad \forall x \exists y x < y \}.$$



# Inexpressibility without Equality

## Theorem

*Upper bounds of model cardinalities cannot be expressed in first-order predicate logic without equality.*

## Theorem

*Each satisfiable set of formulas without equality has models with infinite domain.*

## Corollary

*Finiteness cannot be expressed in first-order predicate logic without equality.*

## Proof (sketch)

All three results immediately follow from the model extension theorem. □

# Expressibility and Inexpressibility with Equality

## Theorem

*Bounded finiteness can be expressed in first-order predicate logic with equality. That is, for any given natural number  $k \geq 1$ , the upper bound  $k$  of model cardinalities can be expressed.*

## Example

All *normal* models of the following satisfiable formula have domains with cardinality  $\leq 3$ :

$$\exists x_1 \exists x_2 \exists x_3 \forall y (y \doteq x_1 \vee y \doteq x_2 \vee y \doteq x_3).$$

## Theorem

*If a set of formulas with equality has arbitrarily large finite normal models, then it has an infinite normal model.*

## Proof

Let  $S$  be such that for each  $k \in \mathbb{N}$  there is a normal model of  $S$  whose domain has finite cardinality  $> k$ . We show that  $S$  has an infinite normal model.

For each  $n \in \mathbb{N}$  let  $\varphi_n$  be the formula  $\forall x_0 \dots x_n \exists y (\neg(y \doteq x_0) \wedge \dots \wedge \neg(y \doteq x_n))$  expressing “more than  $n + 1$  elements”. Then every finite subset of  $S \cup \{\varphi_n \mid n \in \mathbb{N}\}$  has a normal model. By the finiteness/compactness theorem with equality,  $S \cup \{\varphi_n \mid n \in \mathbb{N}\}$  has a normal model  $\mathcal{I}$ .

Obviously,  $\mathcal{I}$  cannot be finite, but is also a normal model of  $S$ . □

## Corollary

*A satisfiable set of formulas with equality has either only finite normal models of a bounded cardinality, or infinite normal models.*

## Corollary

*Unbounded finiteness cannot be expressed in first-order predicate logic with equality.*

## Theorem (Löwenheim-Skolem)

*Every satisfiable enumerable set of closed formulas has a model with a finite or infinite enumerable domain.*