A Minimal Model Semantics for Rational Closure

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Abstract

This paper provides a general semantic framework for nonmonotonic reasoning, based on a minimal models semantics on the top of KLM systems for nonmonotonic reasoning. This general framework can be instantiated in order to provide a semantic reconstruction within modal logic of the notion of rational closure, introduced by Lehmann and Magidor. We give two characterizations of rational closure: the first one in terms of minimal models where propositional interpretations associated to worlds are fixed along minimization, the second one where they are allowed to vary. In both cases a knowledge base must be expanded with a suitable set of consistency assumptions, represented by negated conditionals. The correspondence between rational closure and minimal model semantics suggests the possibility of defining variants of rational closure by changing either the underlying modal logic or the comparison relation on models.

Introduction

In a seminal work Kraus Lehmann and Magidor (Kraus, Lehmann, and Magidor 1990) (henceforth KLM) proposed an axiomatic approach to nonmonotonic reasoning. Plausible inferences are represented by nonmonotonic conditionals of the form $A \mid\sim B$, to be read as "typically or normally A entails B": for instance $monday \mid\sim go\_work$, "normally on Monday I go to work". The conditional is nonmonotonic since from $A \mid\sim B$ one cannot derive $A \wedge C \mid\sim B$, in our example, one cannot derive $monday \ \wedge \ ill \mid\sim go\_work$. KLM proposed a hierarchy of four systems, from the weakest to the strongest: cumulative logic **C**, loop-cumulative logic **CL**, preferential logic **P** and rational logic **R**. Each system is characterized by a set of postulates expressing natural properties of nonmonotonic inference. We present below an axiomatization of the two stronger logics **P** and **R** (**C** and **CL** being too weak to be taken as an axiomatic base for nonmonotonic reasoning). But before presenting the axiomatization, let us clarify one point: in the original presentation of KLM systems, (Kraus, Lehmann, and Magidor 1990) a conditional $A \mid\sim B$ is considered as a consequence relation between a pair of formulas A and B, so that their systems provide a set of "postulates" (or closure conditions) that the intended consequence relations must satisfy. Alternatively, these postulates may be seen as *rules* to derive new conditionals from given ones. We take a slightly different viewpoint, shared among others by Halpern and Friedman (Friedman and Halpern 2001) (see Section 8) and Boutilier (Boutilier 1994) who proposed a modal interpretation of KLM systems **P** and **R**: in our understanding these systems are ordinary logical systems in which a conditional $A \mid\sim B$ is a propositional formula belonging to the object language. Whenever we restrict our consideration, as done by Kraus Lehmann and Magidor, to the entailment of a conditional from a set of conditionals, the two viewpoints *coincide*: a conditional is a logical consequence in logic **P**/**R** of a set of conditionals if and only if it belongs to all preferential/rational consequence relations extending that set of conditionals, or (in semantic terms), it is valid in all preferential/rational models (as defined by KLM) of that set.

Here is the axiomatization of logics **P** and **R**, in our presentation KLM postulates/rules are just *axioms*. We use $\vdash_{PC}$ (resp. $\models_{PC}$) to denote provability (resp. validity) in the propositional calculus.

All axioms and rules of propositional logic	
$A \mid\sim A$	(REF)
if $\vdash_{PC} A \leftrightarrow B$ then $(A \mid\sim C) \rightarrow (B \mid\sim C)$,	(LLE)
if $\vdash_{PC} A \rightarrow B$ then $(C \mid\sim A) \rightarrow (C \mid\sim B)$	(RW)
$((A \mid\sim B) \wedge (A \mid\sim C)) \rightarrow (A \wedge B \mid\sim C)$	(CM)
$((A \mid\sim B) \wedge (A \mid\sim C)) \rightarrow (A \mid\sim B \wedge C)$	(AND)
$((A \mid\sim C) \wedge (B \mid\sim C)) \rightarrow (A \vee B \mid\sim C)$	(OR)
$((A \mid\sim B) \wedge \neg(A \mid\sim \neg C)) \rightarrow (A \wedge C) \mid\sim B)$	(RM)

The axiom (CM) is called cumulative monotony and it is characteristic of all KLM logics, axiom (RM) is called rational monotony and it characterizes the logic of rational entailment **R**. The weaker logic of preferential entailment **P** contains all axioms, but (RM). **P** and **R** seem to capture the core properties of nonmonotonic reasoning, as shown in (Friedman and Halpern 2001) they are quite ubiquitous being characterized by different semantics (all of them being instances of so-called plausibility structures).

Logics **P** and **R** enjoy a very simple modal semantics, actually it turns out that they are the flat fragment of some well-known conditional logics. For **P** the modal semantics is defined by considering a set of worlds W equipped by an accessibility (or preference) relation $<$ assumed to be transitive, irreflexive, and satisfying the so-called Smoothness Condition. For the stronger **R** $<$ is further assumed to be modular. Intuitively the meaning of $x < y$ is that x is more typical/more normal/less exceptional than y. We say that $A \mid\sim B$ is true in a model if B holds in all most normal worlds where A is true, i.e. in all $<$-minimal worlds satisfying A.

KLM systems formalize desired properties of nonmonotonic inference. However, they are too weak to perform useful nonmonotonic inferences. For instance KLM systems cannot handle irrelevant information in conditionals: from $monday \mid\sim go\_work$, there is no way of concluding $monday \wedge shines \mid\sim go\_work$ in any one of KLM systems. Partially motivated by the weakness of the axiomatic approach, Lehmann and Magidor have proposed a true nonmonotonic mechanism on the top of logic **R** called *rational closure*. Rational clsure on the one hand preserves the properties of **R**, on the other hand allows one to perform some truthful nonmonotonic inferences, like the one just mentioned ($monday \wedge shines \mid\sim go\_work$).[1] The authors has given a syntactic procedure to calculate the set of conditionals entailed by the rational closure as well as a quite complex semantic construction. It is worth noticing that a strongly related construction has been proposed by Pearl (Pearl 1990) with his notion of 1-entailment, motivated by a probabilistic interpretation of conditionals.

In this work we tackle the problem of giving a purely semantic characterization of rational closure, stemming directly from the modal semantics of logic **R**. Notice that we restrict our attention to finite knowledge bases. More precisely, we try to answer to the following question: given the fact that logic **R** is characterized by a specific class of Kripke models, how can we characterize the Kripke models of the rational closure of a set of positive conditionals?

The characterization we propose may be seen as an instance of a general recipe for defining nonmonotonic inference: (i) fix an underlying modal semantics for conditionals (such as the one of **P** or **R**), (ii) obtain nonmonotonic inference by restricting semantic consequence to a class of "minimal" models according to some preference relation on models. The preference relation in itself is defined independently from the *language* and from the *set of conditionals* (knowledge base) whose nonmonotonic consequences we want to determine. In this respect our approach is similar in spirit to "minimal models" approaches to nonmonotonic reasoning, such as circumscription.

The general recipe for defining nonmonotonic inference we have sketched may have a wider interest than that of capturing Lehmnan and Magidor's rational closure. First of all, we may think of studying variants of rational closure based on other modal logics and/or on other comparison relations on models. Secondly, being a purely semantic approach (the preference relation is independent from the language), our semantics can cope with a larger language than the one considered in KLM framework. To this regard, already in this paper, we consider a richer language allowing boolean combinations of conditionals[2]. In the future, we may think of applying our semantics to Nonmonotonic Description Logics, where an extension of rational closure has been recently considered (Casini and Straccia 2010).

In any case, the quest of a modal characterization of rational closure turns out to be harder than expected. Our semantic account is based on the minimization of the *height* of worlds in models, where the height of a world is defined in terms of length of the $<$-chains starting from the world. Intuitively, the lower the height of a world, the more normal (or less exceptional) is the world and our minimization corresponds intuitively to the idea of minimizing less-normal or less-plausible worlds (or maximizing most plausible ones).

We begin by considering the nonmonotonic inference relation determined by restricting considerations to models which minimize the *height of worlds*. In this first characterization we keep fixed the propositional interpretation associated to worlds. The consequence relation makes sense in its own, but as we show it is *strictly weaker* than rational closure. We can obtain nonetheless a first characterization of rational closure if we further restrict attention to minimal *canonical models* that is to say, to models that contain all propositional interpretations compatible with the knowledge base K (i.e. all propositional interpretations except those that satisfy some formulas inconsistent with the knowledge base K). Restricting attention to canonical models amounts to expanding K by all formulas $\neg(A \mid\sim \bot)$ (read as "A is possible", as it encodes S5 $\Diamond A$) for all formulas A such that $K \not\models_R A \mid\sim \bot$. We thus obtain a very simple and neat characterization of rational closure, but at the price of an *exponential* increase of the K.

We then propose a second characterization that does not entail this exponential blow up. In analogy with circumscription, we consider a stronger form of minimization where we minimize the height of worlds, but *we allow to vary* the propositional interpretation associated to worlds. Again the resulting minimal consequence relation makes sense in its own, but as we show it still does not correspond to rational closure. In order to capture rational closure, we must basically add the assumption that there are "enough" worlds to satisfy all conditionals consistent with the knowledge base K. This amounts to adding a *small* set of consistency assumptions (represented by negative conditionals). In this way we capture exactly rational closure, without an exponential increase of the knowledge base.

[1] Actually the main motivation of Lehmann and Magidor leading to the definition of rational closure was *technical*: it turns out that the intersection of all rational consequence relations satisfying a set of conditionals coincides with the least *preferential* consequence relation satisfying that set, so that (i) the axiom/rule (RM) does not add anything and (ii) such relation in itself *fails* to satisfy (RM). Their notion of rational closure provides a solution to both problems and can be seen as the "minimal" (in some sense) rational consequence completing a set of conditionals.

[2] An extension of rational closure to knowledge bases comprising both positive and negative conditionals has been proposed in (Booth and Paris 1998).

General Semantics

In KLM framework the language of both logics **P** and **R** consists only of conditionals $A \mid\sim B$. We consider here a richer language allowing boolean combinations of conditionals (and propositional formulas). Our language $\mathcal{L}$ is defined from a set of propositional variables ATM. We use $A, B, C, \ldots$ to denote propositional formulas (not containing $\mid\sim$), and $F, G, \ldots$ to denote arbitrary formulas. More precisely, the formulas of $\mathcal{L}$ are defined as follows: if A is a propositional formula, $A \in \mathcal{L}$; if A and B are propositional formulas, $A \mid\sim B \in \mathcal{L}$; if F is a boolean combination of formulas of $\mathcal{L}$, $F \in \mathcal{L}$. A knowledge base K is any set of formulas: as already mentioned in this work we restrict our attention to finite knowledge bases.

The semantics of **P** and **R** is defined respectively in terms of preferential and rational[3] models, that are possible world structures equipped with a preference relation $<$, intuitively $x < y$ means that the world/individual x is *more normal*/ *more typical* than the world/individual y. The intuitive idea is that $A \mid\sim B$ holds in a model if the most typical/normal worlds/individuals satisfying A satisfy also B. Preferential models presented in (Kraus, Lehmann, and Magidor 1990) characterize the system **P**, whereas the more restricted class of rational models characterizes the system **R** (Lehmann and Magidor 1992).

Definition 1 *A* preferential *model is a triple*

$$\mathcal{M} = \langle \mathcal{W}, <, V \rangle$$

where:

- *$\mathcal{W}$ is a non-empty set of items*
- *$<$ is an irreflexive, transitive relation on $\mathcal{W}$ satisfying the Smoothness relation defined below;*
- *V is a function $V : \mathcal{W} \longmapsto 2^{ATM}$, which assigns to every world w the set of atoms holding in that world.*

If F is a boolean combination of formulas, its truth conditions ($\mathcal{M}, w \models F$) are defined as for propositional logic. Let A be a propositional formula; we define $Min^{\mathcal{M}}_{<}(A) = \{w \in \mathcal{W} \mid \mathcal{M}, w \models A$ and $\forall w'$, $w' < w$ implies $\mathcal{M}, w' \not\models A\}$. We also define $\mathcal{M}, w \models A \mid\sim B$ if for all w', if $w' \in Min^{\mathcal{M}}_{<}(A)$ then $\mathcal{M}, w' \models B$. Last we define the Smoothness Condition*: if $\mathcal{M}, w \models A$, then $w \in Min^{\mathcal{M}}_{<}(A)$ or there is $w' \in Min^{\mathcal{M}}_{<}(A)$ such that $w' < w$. Validity and satisfiability of a formula are defined as usual. Given a set of formulas K of $\mathcal{L}$ and a model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$, we say that $\mathcal{M}$ is a model of K, written $\mathcal{M} \models K$, if, for every $F \in K$, and every $w \in \mathcal{W}$, we have that $\mathcal{M}, w \models F$. K preferentially entails a formula F, written $K \models_P F$ if F is valid in all preferential models of K.*

Since we limit our attention to finite knowledge bases, we can restrict our attention to finite models, as the logic enjoys the finite model property (observe that in this case the smoothness condition is ensured trivially by the irreflexivity of the preference relation). From Definition 1, we have that the truth condition of $A \mid\sim B$ is "global" in a model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$: given a world w, we have that $\mathcal{M}, w \models A \mid\sim B$ if, for all w', if $w' \in Min^{\mathcal{M}}_{<}(A)$ then $\mathcal{M}, w' \models B$. It immediately follows that $A \mid\sim B$ holds in w if only if $A \mid\sim B$ is valid in a model, i.e. it holds that $\mathcal{M}, w' \models A \mid\sim B$ for all w' in $\mathcal{W}$; for this reason we will often write $\mathcal{M} \models A \mid\sim B$. Moreover, when the reference to the model $\mathcal{M}$ is unambiguous, we will simply write $Min_{<}(A)$ instead of $Min^{\mathcal{M}}_{<}(A)$.

Definition 2 *A* rational *model is a preferential model in which $<$ is further assumed to be* modular*: for all $x, y, z \in \mathcal{W}$, if $x < y$ then either $x < z$ or $z < y$. K rationally entails a formula F, written $K \models_R F$ if F is valid in all rational models of K.*

When the logic is clear from the context we shall write $K \models F$ instead of $K \models_P F$ or $K \models_R F$.

From now on, we restrict our attention to *rational models*.

Definition 3 *The height $k_{\mathcal{M}}$ of a world w in $\mathcal{M}$ is the length of any chain $w_0 < \ldots < w$ from w to a w_0 such that for no w' it holds that $w' < w_0$* [4].

Notice that in a rational model $\langle \mathcal{W}, V, < \rangle$, $k_{\mathcal{M}}$ is uniquely determined. Moreover, finite Rational models can be equivalently defined by postulating the existence of a function $k : \mathcal{W} \to \mathbb{N}$, and then letting $x < y$ iff $k(x) < k(y)$.

Definition 4 *The height $k_{\mathcal{M}}(F)$ of a formula F is $i = min\{k_{\mathcal{M}}(w) : \mathcal{M}, w \models F\}$. If there is no $w : \mathcal{M}, w \models F$, F has no height.*

It is immediate to verify that:

Proposition 1 *For any $\mathcal{M} = \langle \mathcal{W}, V, < \rangle$ and any $w \in \mathcal{W}$, we have $\mathcal{M} \models A \mid\sim B$ iff $k_{\mathcal{M}}(A \wedge B) < k_{\mathcal{M}}(A \wedge \neg B)$.*

As already mentioned, although the operator $\mid\sim$ j is *non-monotonic*, the notion of logical entailment just defined is itself *monotonic*: if $K \models_P F$ and $K \subseteq K'$ then also $K' \models_P F$ (the same holds for $\models_R$). In order to define a non-monotonic entailment we introduce our second ingredient of minimal models. The underlying idea is to restrict attention to models that minimize *the height of worlds*. Informally, given two models of K, one in which a given x has height 2 (because for instance $z < y < x$) , and another in which it has height 1 (because only $y < x$), we would prefer the latter, as in this model x is "more normal" than in the former.

In analogy with circumscription, there are mainly two ways of comparing models with the same domain: 1) by keeping the valuation function fixed (only comparing $\mathcal{M}$ and $\mathcal{M}'$ if V and V' in the two models respectively coincide); 2) by also comparing $\mathcal{M}$ and $\mathcal{M}'$ in case $V \neq V'$. We consider the two possible semantics resulting from these alternatives. The first semantics is a *fixed interpretations minimal semantics*, for short $FIMS$.

[3] We use the expression "rational model" rather than "ranked model" which is also used in the literature in order to avoid any confusion with the notion of rank used in rational closure.

[4] In the literature the function $k_{\mathcal{M}}$ is usually called *ranking*, but we call it *height* in order to avoid any confusion with the different notion of *ranking* as defined by Lehmann and Magidor and used in this paper as well. Our notion of ranking is similar to the one originally introduced by Spohn (Spohn 1988) and to the one introduced by Pearl (Pearl 1990). The definition of height can be adapted to preferential models by considering the *longest* chain rather than any chain in the definition.

Definition 5 (*FIMS*) *Given $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ and $\mathcal{M}' = \langle \mathcal{W}', <', V' \rangle$ we say that $\mathcal{M}$ is preferred to $\mathcal{M}'$ with respect to the fixed interpretations minimal semantics ($\mathcal{M} <_{FIMS} \mathcal{M}'$) if $\mathcal{W} = \mathcal{W}'$, $V = V'$, and for all x, $k_{\mathcal{M}}(x) \leq k_{\mathcal{M}'}(x)$ whereas there exists $x' : k_{\mathcal{M}}(x') < k_{\mathcal{M}'}(x')$. We say that $\mathcal{M}$ is minimal with respect to $<_{FIMS}$ in case there is no $\mathcal{M}'$ such that $\mathcal{M}' <_{FIMS} \mathcal{M}$. We say that K minimally entails a formula F with respect to $FIMS$, and we write $K \models_{FIMS} F$, if F is valid in all models of K which are minimal with respect to $<_{FIMS}$.*

The following theorem shows that we can characterize minimal models with fixed interpretations in terms of conditionals that are falsified by a world. Intuitively minimal models are those where the worlds of height 0 satisfy all conditionals, and the height (> 0) of a world x is determined by the height $k_{\mathcal{M}}(C)$ of the antecedents C of conditionals *falsified* by x. Given a model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ and $x \in \mathcal{W}$, we define $S_x = \{C \mathrel{|\!\sim} D \in K \mid \mathcal{M}, x \models C \wedge \neg D\}$.

Proposition 2 *Let K be a knowledge base and $\mathcal{M}$ a model, then $\mathcal{M} \models K$ if and only if $\mathcal{M}$ satisfies the following, for every $x \in \mathcal{W}$:*

1. *if $k_{\mathcal{M}}(x) = 0$ then $S_x = \emptyset$*
2. *if $S_x \neq \emptyset$, then $k_{\mathcal{M}}(x) > k_{\mathcal{M}}(C)$ for every $C \mathrel{|\!\sim} D \in S_x$.*

Proof. (*Only if part*) We prove condition 2. Let $C \mathrel{|\!\sim} D \in S_x$, suppose, we have $\mathcal{M}, x \models C \wedge \neg D$, since $\mathcal{M} \models C \mathrel{|\!\sim} D$ we obtain that $x \notin Min_<(C)$, which entails that $k_{\mathcal{M}}(x) > k_{\mathcal{M}}(C)$. Condition 1 is a consequence of condition 2, since by 2 if $S_x \neq \emptyset$ then trivially $k_{\mathcal{M}}(x) > 0$.

(*If part*) Let $A \mathrel{|\!\sim} B \in K$, suppose that $\mathcal{M}$ satisfies the two conditions above, we show that $\mathcal{M} \models A \mathrel{|\!\sim} B$. Let $x \in Min_<(A)$, if $k_{\mathcal{M}}(x) = 0$, then $S_x = \emptyset$, thus we get that $\mathcal{M}, x \models A \rightarrow B$, whence $\mathcal{M}, x \models B$. Suppose now that $k_{\mathcal{M}}(x) > 0$, if $\mathcal{M}, x \models A \wedge \neg B$, then $A \mathrel{|\!\sim} B \in S_x$, but then by hypothesis we get $k_{\mathcal{M}}(x) > k_{\mathcal{M}}(A)$ against the fact that $x \in Min_<(A)$. ■

In the proof of Proposition 2, we have observed that condition 1 is a consequence of condition 2; we have explicitly mentioned it for clarity (see the subsequent proposition and theorem).

Proposition 3 *Let K be a knowledge base and let $\mathcal{M}$ be a* minimal *model of K with respect to $FIMS$; then $\mathcal{M}$ satisfies for every $x \in \mathcal{W}$:*

1. *if $S_x = \emptyset$ then $k_{\mathcal{M}}(x) = 0$.*
2. *if $S_x \neq \emptyset$, then $k_{\mathcal{M}}(x) = 1 + max\{k_{\mathcal{M}}(C) \mid C \mathrel{|\!\sim} D \in S_x\}$.*

Proof. Let $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$. Suppose that $S_x = \emptyset$, but $k_{\mathcal{M}}(x) > 0$, define a model $\mathcal{M}' = \langle \mathcal{W}, <', V \rangle$ by letting $k_{\mathcal{M}'}(x) = 0$ and $k_{\mathcal{M}'}(y) = k_{\mathcal{M}}(y)$ for $y \neq x$. We show that $\mathcal{M}' \models K$, obtaining a contradiction with the hypothesis that $\mathcal{M}$ is minimal. Let $A \mathrel{|\!\sim} B \in K$, suppose that $w \in Min_<^{\mathcal{M}'}(A)$. If $w = x$, since $S_x = \emptyset$, we have that $\mathcal{M}', x \models B$ (the evaluation function of $\mathcal{M}'$ is the same as the one in $\mathcal{M}$). If $w \neq x$ and $w \in Min_<^{\mathcal{M}'}(A)$ we must have that $w \in Min_<^{\mathcal{M}}(A)$, otherwise there would be a world y with $\mathcal{M}, y \models A$ and with $k_{\mathcal{M}'}(y) \leq k_{\mathcal{M}}(y) < k_{\mathcal{M}}(w) = k_{\mathcal{M}'}(w)$, against the fact that $w \in Min_<^{\mathcal{M}'}(A)$; we then conclude by the fact that $\mathcal{M} \models A \mathrel{|\!\sim} B$ so that $\mathcal{M}, w \models B$, whence $\mathcal{M}', w \models B$.

Suppose now that $S_x \neq \emptyset$, but $k_{\mathcal{M}}(x) \neq 1 + max\{k_{\mathcal{M}}(C) \mid C \mathrel{|\!\sim} D \in S_x\}$. By Proposition 2, it must be $k_{\mathcal{M}}(x) > 1 + max\{k_{\mathcal{M}}(C) \mid C \mathrel{|\!\sim} D \in S_x\}$. In this case, we define a model $\mathcal{M}' = (\mathcal{W}, <', V)$, by stipulating $k_{\mathcal{M}}(x) = 1 + max\{k_{\mathcal{M}}(C) \mid C \mathrel{|\!\sim} D \in S_x\}$ and $k_{\mathcal{M}'}(y) = k_{\mathcal{M}}(y)$ for $y \neq x$. We show that $\mathcal{M}' \models K$, obtaining a contradiction with the hypothesis that $\mathcal{M}$ is minimal. Let $A \mathrel{|\!\sim} B \in K$ and let $w \in Min_<^{\mathcal{M}'}(A)$. If $w \neq x$ we get as before that $w \in Min_<^{\mathcal{M}}(A)$ and we conclude by the fact that $\mathcal{M} \models A \mathrel{|\!\sim} B$. Let now $w = x$, if $A \mathrel{|\!\sim} B \notin S_x$, we are done as $\mathcal{M}', x \models A \rightarrow B$. If $A \mathrel{|\!\sim} B \in S_x$, then it must be $x \notin Min_<^{\mathcal{M}}(A)$, thus there is y s.t. $\mathcal{M}, y \models A$, with $k_{\mathcal{M}}(y) = k_{\mathcal{M}}(A)$ and $k_{\mathcal{M}}(y) < k_{\mathcal{M}}(x)$. Since $k_{\mathcal{M}'}(y) = k_{\mathcal{M}}(y)$ (and $k_{\mathcal{M}'}(A) = k_{\mathcal{M}}(A)$) we get that $k_{\mathcal{M}'}(x) > k_{\mathcal{M}'}(A)$, against the hypothesis that $x \in Min_<^{\mathcal{M}'}(A)$. ■

Theorem 1 *Let K be a knowledge base and let $\mathcal{M}$ be any model, then $\mathcal{M}$ is a $FIMS$ minimal model of K if and only if $\mathcal{M}$ satisfies for every $x \in \mathcal{W}$:*

1. *$S_x = \emptyset$ iff $k_{\mathcal{M}}(x) = 0$.*
2. *if $S_x \neq \emptyset$, then $k_{\mathcal{M}}(x) = 1 + max\{k_{\mathcal{M}}(C) \mid C \mathrel{|\!\sim} D \in S_x\}$.*

Proof. The *only if* direction immediately follows from Proposition 3. For the *if* direction, let $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ be a model with associated $k_{\mathcal{M}}$, if $\mathcal{M}$ satisfies the two conditions by Proposition 2 it follows that $\mathcal{M} \models K$. Let $\mathcal{M}' \models K$ with $\mathcal{M}' = \langle \mathcal{W}, <', V \rangle$, and associated $k_{\mathcal{M}'}$, then $\mathcal{M}'$ satisfies the conditions of Proposition 2. By induction on $k_{\mathcal{M}'}(x)$ we show that $k_{\mathcal{M}}(x) \leq k_{\mathcal{M}'}(x)$. If $k_{\mathcal{M}'}(x) = 0$ then $S_x = \emptyset$ so that by Lemma 3 $k_{\mathcal{M}}(x) = 0$. Let $k_{\mathcal{M}'}(x) > 0$: if $S_x = \emptyset$ then $k_{\mathcal{M}}(x) = 0 < k'(x)$. If $S_x \neq \emptyset$, then: (i) $k_{\mathcal{M}'}(x) > k_{\mathcal{M}'}(C)$ for every $C \mathrel{|\!\sim} D \in S_x$ and (ii) $k_{\mathcal{M}}(x) = 1 + max\{k_{\mathcal{M}}(C) \mid C \mathrel{|\!\sim} D \in S_x\}$. By (i) and induction hypothesis it follows $k_{\mathcal{M}}(C) \leq k_{\mathcal{M}'}(C)$, thus: $k(x) = 1 + max\{k_{\mathcal{M}}(C) \mid C \mathrel{|\!\sim} D \in S_x\} \leq 1 + max\{k_{\mathcal{M}'}(C) \mid C \mathrel{|\!\sim} D \in S_x\} \leq k_{\mathcal{M}'}(x)$. We have shown that for all $x \in \mathcal{W}$, $k_{\mathcal{M}}(x) \leq k_{\mathcal{M}'}(x)$, hence $\mathcal{M}' \not<_{FIMS} \mathcal{M}$, and $\mathcal{M}$ is minimal. ■

In our second semantics, we let the interpretations vary. The semantics is called variable interpretations minimal semantics, for short $VIMS$.

Definition 6 (*VIMS*) *Given $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ and $\mathcal{M}' = \langle \mathcal{W}', <', V' \rangle$ we say that $\mathcal{M}$ is preferred to $\mathcal{M}'$ with respect to the variable interpretations minimal semantics, and write $\mathcal{M} <_{VIMS} \mathcal{M}'$, if $\mathcal{W} = \mathcal{W}'$, and for all x, $k_{\mathcal{M}}(x) \leq k_{\mathcal{M}'}(x)$ whereas there exists $x' : k_{\mathcal{M}}(x') < k_{\mathcal{M}'}(x')$. We say that $\mathcal{M}$ is minimal with respect to $<_{VIMS}$ in case there is no $\mathcal{M}'$ such that $\mathcal{M}' <_{VIMS} \mathcal{M}$. We say that K minimally entails (with respect to $VIMS$) F, and write $K \models_{VIMS} F$, if F is valid in all models of K which are minimal with respect to $<_{VIMS}$.*

It is easy to realize that the two semantics, $FIMS$ and $VIMS$, define different sets of minimal models. This is illustrated by the following example.

Example 1 Let

$$K = \{penguin \mid\!\sim bird, penguin \mid\!\sim \neg fly, bird \mid\!\sim fly\}.$$

We derive that

$$K \not\models_{FIMS} penguin \wedge black \mid\!\sim \neg fly.$$

Indeed in $FIMS$ there can be a model $\mathcal{M}$ in which $\mathcal{W} = \{x, y, z\}$, $V(x) = \{penguin, bird, fly, black\}$, $V(y) = \{penguin, bird\}$, $V(z) = \{bird, fly\}$, and $z < y < x$. $\mathcal{M}$ is a model of K, and it is minimal with respect to $FIMS$ (indeed once fixed $V(x), V(y), V(z)$ as above, it is not possible to lower the height of x nor of y nor of z unless we falsify K). Furthermore, in $\mathcal{M}$ x is a typical black penguin (since there is no other black penguin preferred to it) that flies. Therefore, $K \not\models_{FIMS} penguin \wedge black \mid\!\sim \neg fly$.

On the other hand, $\mathcal{M}$ is not minimal with respect to $VIMS$. Indeed, consider the model $\mathcal{M}' = \langle \mathcal{W}, <', V' \rangle$ obtained from $\mathcal{M}$ by letting $V'(x) = \{penguin, bird, black\}$, $V'(y) = V(y)$, $V'(z) = V(z)$ and by defining $<'$ as: $z <' y$ and $z <' x$. Clearly $\mathcal{M}' \models K$, and $\mathcal{M}' <_{VIMS} \mathcal{M}$, since $k_{\mathcal{M}'}(x) < k_{\mathcal{M}}(x)$, while $k_{\mathcal{M}'} = k_{\mathcal{M}}$ for all other worlds.

The example above shows that $FIMS$ and $VIMS$ lead to different sets of minimal models for a given K. Notice however that the model $\mathcal{M}'$ we have used to illustrate this fact is not a minimal model for K in $VIMS$. A minimal model in $VIMS$ for K that can be defined on the domain $\mathcal{W}$ is given by $V(x) = V(y) = V(z) = \{bird, fly\}$, and the empty relation $<$. This is quite a degenerate model of K in which there are no penguins. This illustrates the strength of $VIMS$: in case of knowledge bases that only contain positive conditionals, logical entailment in $VIMS$ collapses into classical logic entailment. This feature corresponds to a similar feature of the non-monotonic logic $\mathbf{P}_{min}$ in (Giordano et al. 2010), and can be proven in the same way.

Proposition 4 *Let K be a set of positive conditionals. Let us replace all formulas of the form $A \mid\!\sim B$ in K with $A \rightarrow B$, and call K' the resulting set of formulas. We have that $K \models_{VIMS} A \mid\!\sim B$ if and only if $K' \models_{PC} A \rightarrow B$.*

As for $\mathbf{P}_{min}$ this strong feature of $VIMS$ can be avoided when considering knowledge bases that include existence assertions: these are negated conditionals, in the example for instance we could add $\neg(penguin \mid\!\sim \bot)$ to force us to consider non trivial models in which penguins exist. In the next section, we will use $VIMS$ in this kind of way, by always considering knowledge bases that include existence assertions (expressed by negated conditionals).

A Semantical Reconstruction of Rational Closure

We provide a semantic characterization of the well known rational closure, described in (Lehmann and Magidor 1992) within the two semantics described in the previous section. More precisely, we can give two semantic characterizations of rational closure, the first based on $FIMS$, the second based on $VIMS$. Since in rational closure no boolean combinations of conditionals are allowed, in the following, the knowledge base K is just a finite set of positive conditional assertions. We recall the notion rational closure, giving its syntactical definition in terms of *rank* of a formula.

Definition 7 *Let K be a knowledge base (i.e. a finite set of positive conditional assertions) and A a propositional formula. A is said to be* exceptional *for K iff $K \models_R \top \mid\!\sim \neg A$*[5].

A conditional formula $A \mid\!\sim B$ is exceptional for K if its antecedent A is exceptional for K. The set of conditional formulas which are exceptional for K will be denoted as $E(K)$. It is possible to define a non-decreasing sequence of subsets of K $C_0 \supseteq C_1, \ldots$ by letting $C_0 = K$ and, for $i > 0$, $C_i = E(C_{i-1})$. Observe that, being K finite, there is a $n \geq 0$ such that for all $m > n, C_m = C_n$ or $C_m = \emptyset$.

Definition 8 *A propositional formula A has* rank *i for K iff i is the least natural number for which A is not exceptional for C_i. If A is exceptional for all C_i then A has no rank.*

The notion of rank of a formula allows to define the rational closure of a knowledge base K.

Definition 9 *Let K be a conditional knowledge base. The rational closure $\underline{K}$ of K is the set of all $A \mid\!\sim B$ such that either (1) the rank of A is strictly less than the rank of $A \wedge \neg B$ (this includes the case A has a rank and $A \wedge \neg B$ has none), or (2) A has no rank.*

The rational closure of a knowledge base K seemingly contains all conditional assertions that, in the analysis of nonmonotonic reasoning provided in (Lehmann and Magidor 1992), one rationally wants to derive from K. For a full discussion, see (Lehmann and Magidor 1992).

Can we capture rational closure within our semantics? A first conjecture might be that the $FIMS$ of Definition 5 could capture rational closure. However, we are soon forced to recognize that this is not the case. For instance, Example 1 above illustrates that $\{penguin \mid\!\sim bird, penguin \mid\!\sim \neg fly, bird \mid\!\sim fly\} \not\models_{FIMS} penguin \wedge black \mid\!\sim \neg fly$. On the contrary, it can be easily verified that $penguin \wedge black \mid\!\sim \neg fly$ is in the rational closure of $\{penguin \mid\!\sim bird, penguin \mid\!\sim \neg fly, bird \mid\!\sim fly\}$. Therefore, $FIMS$ as it is does not allow us to define a semantics corresponding to rational closure. Things change if we consider $FIMS$ applied to models that contain *all possible valuations "compatible" with a given knowledge base K*. We call these models *canonical models*.

Example 2 Consider Example 1 above. If we restrict our attention to models that also contain a w with $V(w) = \{penguin, bird, black\}$ which is a black penguin that does not fly and can therefore be assumed to be a typical penguin, we are able to conclude that typically black penguins do not fly, as in rational closure. Indeed, in all minimal models of K that also contain w with $V(w) = \{penguin, bird, black\}$, it holds that $penguin \wedge black \mid\!\sim \neg fly$.

[5] In (Lehmann and Magidor 1992), $\models_P$ is used instead of $\models_R$. However when K contains only positive conditionals the two notions coincide (see footnote 1) and we prefer to use $\models_R$ here since we consider rational models.

We are led to the conjecture that $FIMS$ restricted to canonical models could be the right semantics for rational closure. Fix a propositional language $\mathcal{L}_{Prop}$ comprising a finite set of propositional variables ATM, a propositional interpretation $v : ATM \longrightarrow \{true, false\}$ is *compatible* with K, if there is no formula $A \in \mathcal{L}_{Prop}$ such that $v(A) = true$ and $K \models_R A \mathrel{|\!\sim} \bot$.

Definition 10 *A model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ satisfying a knowledge base K is said to be* canonical *if it contains (at least) a world associated to each propositional interpretation compatible with K, that is to say: if v is compatible with K, then there exists a world w in $\mathcal{W}$, such that for all propositional formulas B $\mathcal{M}, w \models B$ iff $v(B) = true$.*

It can be easily shown that:

Theorem 2 *For a given domain $\mathcal{W}$, there exists a unique canonical model $\mathcal{M}$ for K over $\mathcal{W}$ such that, for all other canonical models $\mathcal{M}'$ over $\mathcal{W}$, we have $\mathcal{M} <_{FIMS} \mathcal{M}'$.*

In the following, we show that the canonical models that are minimal with respect to $FIMS$ are an adequate semantic counterpart of rational closure.

To prove the correspondence between the rational closure of a knowledge base K and the fixed interpretation minimal model semantics of K, we need to prove some propositions. The next one is a restatement for rational models of Lemma 5.18 in (Lehmann and Magidor 1992), and it can be proved in a similar way. Note that, as a difference, point 2 in Lemma 5.18 is an “if and only if” rather than an “if” statement.

Proposition 5 *Let $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ be a rational model of K. Let $\mathcal{M}_0 = \mathcal{M}$ and, for all i, let $\mathcal{M}_i = \langle \mathcal{W}_i, <_i, V_i \rangle$ be the rational model obtained from $\mathcal{M}$ by removing all the worlds w with $k_{\mathcal{M}}(w) < i$, i.e., $\mathcal{W}_i = \{w \in \mathcal{W} : k_{\mathcal{M}}(w) \geq i\}$. For any propositional formula A, if $rank(A) \geq i$, then: (1) $k_{\mathcal{M}}(A) \geq i$; (2) If $C_i \models_R A \mathrel{|\!\sim} B$ then $\mathcal{M}_i \models A \mathrel{|\!\sim} B$.*

Proof. The proof is by induction on i. For $i = 0$, item (1) holds trivially. For item (2), observe that $\mathcal{M}_0 = \mathcal{M}$ and $C_0 = K$. From the hypothesis, $\mathcal{M}$ is a rational model of K, and hence a preferential model of K. Thus, if $K \vdash_R A \mathrel{|\!\sim} B$, then $\mathcal{M}$ satisfies $A \mathrel{|\!\sim} B$.

For $i > 0$, let us prove item (1). As $rank(A) \geq i$, then, for all $r < i$, $C_r \vdash_R \top \mathrel{|\!\sim} \neg A$. By inductive hypothesis (item 2), $\mathcal{M}_r$ satisfies $\top \mathrel{|\!\sim} \neg B$. Hence, there is no A world w with $k_{\mathcal{M}}(w) = r < i$. Therefore, $k_{\mathcal{M}}(A) \geq i$.

To prove item (2), observe that, as $rank(A) \geq i$, then, by Lemma 2.21 in (Lehmann and Magidor 1992), $C_0 \vdash_R A \mathrel{|\!\sim} B$ if and only if $C_i \vdash_R A \mathrel{|\!\sim} B$. Suppose $C_i \vdash_R A \mathrel{|\!\sim} B$, then $C_0 \vdash_R A \mathrel{|\!\sim} B$ and hence, by inductive hypothesis (case (2), i=0) $\mathcal{M}$ satisfies $A \mathrel{|\!\sim} B$. As we have shown in the proof of item (1) that in $\mathcal{M}$ there is no A world w with $k_{\mathcal{M}}(w) < i$, then it must be that $\mathcal{M}_i$ satisfies $A \mathrel{|\!\sim} B$. ■

We need the next fact in order to prove the following proposition.

Fact 1 *If $\{A1 \rightarrow B1, \ldots, An \rightarrow Bn\} \models_{PC} \neg C$ then $\{A1 \mathrel{|\!\sim} B1, \ldots, An \mathrel{|\!\sim} Bn\} \vdash_R \top \mathrel{|\!\sim} \neg C$.*

Proposition 6 *Let $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ be a canonical model of K, minimal with respect to $<_{FIMS}$. For all $w \in \mathcal{W}$ it holds that: if $\mathcal{M}, w \models A \rightarrow B$ for all $A \mathrel{|\!\sim} B$ in C_i, then $k_{\mathcal{M}}(w) \leq i$.*

Proof. The proof is by induction on i. If $i = 0$, it immediately follows by Lemma 3 (1).

For $i > 0$, let us consider $w \in \mathcal{W}$ such that for all $A \mathrel{|\!\sim} B$ in C_i, $\mathcal{M}, w \models A \rightarrow B$ but $k_{\mathcal{M}}(w) > i$. Let $\mathcal{M}'$ be a model obtained from $\mathcal{M}$ by changing $<$ in order to have $k_{\mathcal{M}'}(w) = i$. $\mathcal{M}'$ is preferred to $\mathcal{M}$ and it is a model of K, as it satisfies all the conditionals in K. Let $A \mathrel{|\!\sim} B \in K$. It is clear that, for all the worlds $w' \in \mathcal{W}$ with $w' \neq w$, w' satisfies $A \mathrel{|\!\sim} B$ in $\mathcal{M}'$, as it satisfies it in $\mathcal{M}$. To show that w satisfies $A \mathrel{|\!\sim} B$, let $w \in Min_<^{\mathcal{M}'}(A)$. If $A \mathrel{|\!\sim} B$ in C_i, we know from the hypothesis that w satisfies $A \rightarrow B$, and hence, w satisfies B. If $A \mathrel{|\!\sim} B$ in $K - C_i$, there is a $j < i$ such that $A \mathrel{|\!\sim} B \in C_j$, $C_j \not\vdash_R \top \mathrel{|\!\sim} \neg A$ while $C_{j-1} \vdash_R \top \mathrel{|\!\sim} \neg A$. Form $C_j \not\vdash_R \top \mathrel{|\!\sim} \neg A$, by Fact 1, we have that $\{A' \rightarrow B' : A' \mathrel{|\!\sim} B' \text{ in } C_{i-1}\} \not\models \neg A$. Hence, there is a world $w' \in \mathcal{W}$ such that w' satisfies all the implications $A' \rightarrow B'$ s.t. $A' \mathrel{|\!\sim} B'$ in C_{i-1} and w' satisfies A. By inductive hypothesis, $k_{\mathcal{M}}(w') < i$, and therefore $k_{\mathcal{M}}(A) < i$. By construction of $\mathcal{M}'$, $k_{\mathcal{M}'}(w') < i$, and therefore $k_{\mathcal{M}'}(A) < i$ which contradicts the hypothesis that $w \in Min_<^{\mathcal{M}'}(A)$. Hence, $\mathcal{M}'$ satisfies all the conditionals in K. The fact that $k_{\mathcal{M}}(w) > i$ and $k_{\mathcal{M}'}(w) = i$ contradicts the minimality of $\mathcal{M}$. hence, it must be $k_{\mathcal{M}}(w) \leq i$, and the proof is over. ■

Proposition 7 *Let $\mathcal{M}$ be a canonical model of K minimal with respect to $<_{FIMS}$. Then, $rank(A) = i$ iff $k_{\mathcal{M}}(A) = i$.*

Proof. (Only if part) Let us assume that $rank(A) = i$. We know that $C_i \not\vdash_R \top \mathrel{|\!\sim} \neg A$. Hence, by Fact 1, $\{A \rightarrow B : A \mathrel{|\!\sim} B \in C_i\} \not\models_{PC} \neg A$ Then, there is a world $w \in W$ such that, for all $A \mathrel{|\!\sim} B \in C_i$, w satisfies $A \rightarrow B$ and w satisfies A. By Proposition 6, $k_{\mathcal{M}}(w) \leq i$. Thus, $k_{\mathcal{M}}(A) \leq i$. As by Proposition 5 we know that $k_{\mathcal{M}}(A) \geq i$, we can conclude that $k_{\mathcal{M}}(A) = i$. *(If part)* This direction is obvious: if $k_{\mathcal{M}}(A) = i$ then $rank(A) = i$. Indeed if $rank(A) = j \neq i$, $k_{\mathcal{M}}(A) = j \neq i$, against the hypothesis. ■

We can now prove the following theorem:

Theorem 3 *Let K be a knowledge base and $\mathcal{M}$ be a canonical model of K minimal with respect to $<_{FIMS}$. For all conditionals $A \mathrel{|\!\sim} B$ we have:*

$$\mathcal{M} \models A \mathrel{|\!\sim} B \text{ if and only if } A \mathrel{|\!\sim} B \in \overline{K},$$

where $\overline{K}$ is the rational closure of K.

Proof. (Only if part) Let us assume that $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ satisfies $A \mathrel{|\!\sim} B$. Then, for each world $w \in Min_<(A)$, w satisfies B. If $Min_<(A) = \emptyset$, then there is no w s.t. $\mathcal{M}, w \models A$, hence A has no height in $\mathcal{M}$ and by Proposition 7 A has no rank. In this case by Definition 9 $A \mathrel{|\!\sim} B \in \overline{K}$. Let us assume that $k_{\mathcal{M}}(A) = i$. As $k_{\mathcal{M}}(A \wedge B) < k_{\mathcal{M}}(A \wedge \neg B)$, then $k_{\mathcal{M}}(A \wedge \neg B) > i$. By Proposition 7, $rank(A) = i$ and $rank(A \wedge \neg B) > i$. Hence, by Definition 9, $A \mathrel{|\!\sim} B \in \overline{K}$, the rational closure of K.

(If part) $A \mathrel{|\!\sim} B$ belong to $\overline{K}$, the rational closure of K, then, by Definition 9, either (a) $rank(A) < rank(A \wedge \neg B)$,

or (b) A has no rank. In the first case (a), let $rank(A \wedge \neg B) = i$, and $rank(A) < i$. Suppose for a contradiction that $\mathcal{M}$ does not satisfy $A \mid\!\sim B$, i.e., that $k_{\mathcal{M}}(A \wedge \neg B) \leq k_{\mathcal{M}}(A \wedge B)$. Hence, $k_{\mathcal{M}}(A \wedge \neg B) = i$ and $k_{\mathcal{M}}(A \wedge B) > i$, which contradicts the fact that $rank(A) < i$.

In case (b), Suppose for a contradiction that $\mathcal{M}$ does not satisfy $A \mid\!\sim B$, i.e., that $k_{\mathcal{M}}(A \wedge \neg B) \leq k_{\mathcal{M}}(A \wedge B)$. Let $k_{\mathcal{M}}(A \wedge \neg B) = i$. Then, it must be that $k_{\mathcal{M}}(A) = i$, which contradicts the fact that A has no rank. ■

In Theorem 3 we have shown a correspondence between rational closure and minimal models with fixed interpretations, *on the proviso that* we restrict our attention to minimal *canonical* models. We can obtain the same effect by extending K into K' by adding negated conditionals: $K' = K \cup \{\neg(C \mid\!\sim \bot) \mid K \not\models_R (C \mid\!\sim \bot)\}$. Indeed it can be easily verified that all models of K' are canonical, hence restricting $FIMS$ to canonical models on the one hand and considering the extension of K as K' on the other hand amounts to the same effect. We can therefore restate Theorem 3 above as follows:

Theorem 4 *Let K be a knowledge base and let $K' = K \cup \{\neg(C \mid\!\sim \bot) \mid K \not\models_R (C \mid\!\sim \bot)\}$. For all conditionals $A \mid\!\sim B$ we have:*

$$K' \models_{FIMS} A \mid\!\sim B \text{ if and only if } A \mid\!\sim B \in \overline{K}$$

where $\overline{K}$ is the rational closure of K.

Notice that the size of K' is exponential in that of K.

Can we lift the restriction to canonical models by adopting a semantics based on variable valuations? In the general case, the answer is negative. We have already mentioned that if we consider knowledge bases consisting only positive conditionals logical entailment in $VIMS$ collapses into classical logic entailment. To avoid this collapse, we can require that, when we are checking for entailment of a conditional $A \mid\!\sim B$ from a K, at least an $A \wedge B$ world and an $A \wedge \neg B$ world are present in K. This can be obtained by adding to K the conditionals $\neg(A \wedge B \mid\!\sim \bot)$ and $\neg(A \wedge \neg B \mid\!\sim \bot)$. Also in this case, however, we cannot give a positive answer to the above question. In fact, it is possible to build a model of K, minimal with respect to $VIMS$, which falsifies a conditional $A \mid\!\sim B$ which on the contrary is satisfied in all the canonical minimal models of K under $FIMS$. This is shown by the following example.

Example 3 Let K be the following:

$\{T \mid\!\sim S$,
$S \mid\!\sim \neg D$,
$L \mid\!\sim P$,
$R \mid\!\sim Q$,
$E \mid\!\sim F$,
$H \mid\!\sim G$,
$D \mid\!\sim \neg P \wedge \neg Q \wedge \neg F \wedge \neg G$,
$S \mid\!\sim \neg(L \wedge R)$,
$S \mid\!\sim \neg(L \wedge E)$,
$S \mid\!\sim \neg(L \wedge H)$,
$S \mid\!\sim \neg(R \wedge E)$,
$S \mid\!\sim \neg(R \wedge H)$,
$S \mid\!\sim \neg(E \wedge H)\}$.

Let

$A = D \wedge S \wedge R \wedge L \wedge E \wedge H$,
$B = \neg Q \wedge \neg P \wedge \neg F \wedge \neg G$

and let

$$K' = K \cup \{\neg(A \wedge B \mid\!\sim \bot),\ \neg(A \wedge \neg B \mid\!\sim \bot)\}.$$

We define a model $\mathcal{M} = (\mathcal{W}, <, V)$ of K', which is minimal with respect to $VIMS$, as follows: $\mathcal{W} = \{x, w, y_1.y_2, y_3\}$, where:

$V(y_1) = \{S, \neg D, \neg R, \neg L, \neg E, \neg H, P, Q, F, G\}$
$V(y_2) = \{\neg S, \neg D, R, L, E, H, P, Q, F, G\}$
$V(y_3) = \{\neg S, \neg D, R, L, E, H, P, Q, F, G\}$
$V(x) = \{D, S, R, L, E, H, \neg Q, \neg P, \neg F, \neg G\}$
$V(w) = \{D, S, R, L, E, H, Q, \neg P, \neg F, \neg G\}$

with $k_{\mathcal{M}}(y_1) = 0$, $k_{\mathcal{M}}(y_2) = 1$, $k_{\mathcal{M}}(y_3) = 1$, $k_{\mathcal{M}}(x) = 2$ and $k_{\mathcal{M}}(w) = 2$. Observe that: x is an $A \wedge B$ minimal world; w is an $A \wedge \neg B$ minimal world; y_1 is an S minimal world; y_2 is a minimal world for R, L, E and H; and y_3 is a D minimal world.

$\mathcal{M}$ is a model of K which is minimal with respect to $VIMS$. Also, $A \mid\!\sim B$ is falsified in $\mathcal{M}$, while, on the contrary, $A \mid\!\sim B$ holds in all the canonical models minimal with respect to $FIMS$. Indeed, in all such models the height of $k(A \wedge B) = 2$ while $k(A \wedge \neg B) = 3$. However, it is not possible to construct a model $\mathcal{M}'$ with 5 worlds so that $\mathcal{M}' <_{VIMS} \mathcal{M}$. In particular, assigning to x or w height 1 would require the introduction of minimal worlds for R, L, E and H with height 0. But world y_2 cannot be given height 0, as it does not satisfy the conditionals with antecedent S. In canonical models there are distinct minimal R worlds, L worlds, E worlds and H worlds height 0 (which are also minimal S worlds).

As suggested by this example, in order to characterize rational closure in terms of $VIMS$, we should restrict our consideration to models which contain "enough" worlds. In the following, as in Theorem 4, we enrich K with negated conditionals but, as a difference with K' of Theorem 4, we only need to add to K a polynomial number of negated conditionals (instead of an exponential number). The purpose of the addition is that of restricting our attention to models minimal with respect to $<_{VIMS}$ that have a domain large enough to have, in principle, a distinct most-preferred world for each antecedent of conditional in K. For this reason, we add for each antecedent C of K a new corresponding atom ϕ_C. If the problem to be addressed is that of knowing whether $A \mid\!\sim B$ is logically entailed by K, we also introduce $\phi_{A \wedge B}$ and $\phi_{A \wedge \neg B}$, and we define K' as follows.

Definition 11 *We define:*

- $A_{K,A \mid\sim B} = \{C \mid$ *either for some* D, $C \mid\!\sim D \in K$ *or* $C = A \wedge B$ *or* $C = A \wedge \neg B$, *and* $K \not\vdash_R C \mid\!\sim \bot\}$;
- $K' = K \cup \{\neg(C \wedge \phi_C \mid\!\sim \bot) \mid C \in A_{K,A \mid\sim B}\} \cup \{(\phi_{C_i} \wedge \phi_{C_j} \mid\!\sim \bot) \mid C_i, C_j \in A_{K,A \mid\sim B}\}$.

We here establish a correspondence between $FIMS$ and $VIMS$. By virtue of Theorem 3, this allows us to establish a correspondence between rational closure and $VIMS$, as stated by Theorem 6.

Theorem 5 *Let $\mathcal{M}$ be a canonical model of K, minimal with respect to $FIMS$, and let K' be the extension of K defined as in Definition 11. We have that:*

$$\mathcal{M} \models A \mid\!\sim B \textit{ iff } K' \models_{VIMS} A \mid\!\sim B.$$

Proof. We show the contrapositive of the two directions. First suppose $K' \not\models_{VIMS} A \mid\!\sim B$. Let $\mathcal{M}' = \langle \mathcal{W}', <', V' \rangle$ be a model of K' minimal with respect to $<_{VIMS}$ that does not satisfy $A \mid\!\sim B$, i.e., such that $k_{\mathcal{M}'}(A \wedge \neg B) \leq k_{\mathcal{M}'}(A \wedge B)$. We want to show that also $\mathcal{M} \not\models A \mid\!\sim B$, i.e., $k_{\mathcal{M}}(A \wedge \neg B) \leq k_{\mathcal{M}}(A \wedge B)$. For a contradiction, suppose in the canonical $\mathcal{M}$, $k_{\mathcal{M}}(A \wedge \neg B) = j > k_{\mathcal{M}}(A \wedge B) = i$. By Propositions 7 and 5, $k_{\mathcal{M}'}(A \wedge \neg B) \geq j$ and $k_{\mathcal{M}'}(A \wedge B) \geq i$, and since by hypothesis $k_{\mathcal{M}'}(A \wedge \neg B) \leq k_{\mathcal{M}'}(A \wedge B)$, it follows that $k_{\mathcal{M}'}(A \wedge B) \geq j > i$. We show that this goes against the minimality of $\mathcal{M}'$. Consider $\mathcal{M}^* = \langle \mathcal{W}^*, <^*, V^* \rangle$ built from $\mathcal{M}$ by cutting out a portion containing: x in $Min^{\mathcal{M}}_{<}(A \wedge B)$, $x' \in Min^{\mathcal{M}}_{<}(A \wedge \neg B)$ and an element $y \in Min^{\mathcal{M}}_{<}(C)$ for each antecedent C of conditional in K s.t. $K \not\vdash_R A \mid\!\sim \bot$. For these worlds, we define $V^* = V$ and $k_{\mathcal{M}^*} = k_{\mathcal{M}}$. If the same element y is associated to two different formulas it must be duplicated into y and y' (and $V^*(y') = V^*(y)$ and $k_{\mathcal{M}^*}(y') = k_{\mathcal{M}^*}(y)$). Furthermore, for each world y associated to a formula C, $V^*(y)$ is extended in order to include ϕ_C. $<^*$ is straightly defined from $k_{\mathcal{M}^*}$ in the obvious way. The construction is almost finished. Notice that up to this point we have introduced in $\mathcal{W}^*$ no more elements than those in $\mathcal{W}'$. To conclude we have to rename the elements of $\mathcal{W}^*$ with the names as the elements of $\mathcal{W}'$ that satisfy the same ϕ_C, and we have to add to $\mathcal{W}^*$ the elements of $\mathcal{W}'$ that are eventually missing (we let for these cases $V^* = V'$ and $k_{\mathcal{M}^*} = k_{\mathcal{M}'}$).

It can be shown that $\mathcal{M}^*$ is a model of K', and $\mathcal{M}^* <_{VIMS} \mathcal{M}'$, against the minimality of $\mathcal{M}'$. First of all, we show that $\mathcal{M}^*$ is a model of K'. Indeed, by construction we have introduced a new element of $\mathcal{M}$ for each C antecedent of conditional in K or equal to $A \wedge B$ or $A \wedge \neg B$, and this element is still in $Min^{\mathcal{M}^*}_{<}(C)$ (otherwise, $k_{\mathcal{M}^*}(C) < k_{\mathcal{M}^*}(y) = k_{\mathcal{M}}(y) = k_{\mathcal{M}}(C)$, against Propositions 7 and 5). Hence, $\mathcal{M}^*$ satisfies all negated conditionals in K'. Consider now the positive conditionals $C \mid\!\sim D$ in K'. Consider any y that inserted in $\mathcal{M}^*$ from $\mathcal{M}$. Let $y \in Min^{\mathcal{M}^*}_{<}(C)$. Then also $y \in Min^{\mathcal{M}}_{<}(C)$ (otherwise another $y' \in Min^{\mathcal{M}}_{<}(C)$ would have been taken in the construction with $\mathcal{M}^*, y' \models C$ and $k_{\mathcal{M}^*}(y') < k_{\mathcal{M}^*}(y)$, against $y \in Min^{\mathcal{M}^*}_{<}(C)$). Since $\mathcal{M}$ is a model of K, and $C \mid\!\sim D \in K$, $\mathcal{M}, y \models D$ hence also $\mathcal{M}^*, y \models D$. Consider now y introduced in $\mathcal{M}^*$ from $\mathcal{M}'$. If $y \in Min^{\mathcal{M}^*}_{<}(C)$, then we reason as follows to show that $y \in Min^{\mathcal{M}'}_{<}(C)$. First of all, we know that $k_{\mathcal{M}^*}(y) = k_{\mathcal{M}}(C)$. Indeed in $\mathcal{M}^*$ we have inserted a y' that was in $Min^{\mathcal{M}}_{<}(C)$. As shown above, $y' \in Min^{\mathcal{M}^*}_{<}(C)$. Hence $k_{\mathcal{M}^*}(y) = k_{\mathcal{M}^*}(y')$ (otherwise $y \notin Min^{\mathcal{M}^*}_{<}(C)$), and $k_{\mathcal{M}^*}(y) = k_{\mathcal{M}}(C)$. But by construction $k_{\mathcal{M}^*}(y) = k_{\mathcal{M}'}(y)$ and if $y \notin Min^{\mathcal{M}'}_{<}(C)$, there would be a y' s.t. $\mathcal{M}', y' \models C$ and $k_{\mathcal{M}'}(y') < k_{\mathcal{M}'}(y)$, hence $k_{\mathcal{M}'}(C) < k_{\mathcal{M}}(C)$, against Propositions 7 and 5. Hence, since $C \mid\!\sim D$ holds in $\mathcal{M}'$, $\mathcal{M}', y \models D$ and by construction $\mathcal{M}, y \models D$.

For the conditionals with form $\phi_{C_i} \wedge \phi_{C_j} \mid\!\sim \bot$: they hold in $\mathcal{M}^*$ since we have suitably extended V^* in order to include at most one ϕ_C at a time.

Last, it holds that $\mathcal{M}^* <_{VIMS} \mathcal{M}'$. Indeed the domain of the two models coincide, and for all y taken from $\mathcal{M}'$, $k_{\mathcal{M}^*}(y) = k_{\mathcal{M}'}(y)$, and for all y taken from $\mathcal{M}$, they were introduced as representatives of a given C antecedent of conditional or equal to $A \wedge B$, $A \wedge \neg B$. For all these formulas by Propositions 7 and 5, it holds that $k_{\mathcal{M}^*}(C) = k_{\mathcal{M}}(C) \leq k_{\mathcal{M}'}(C)$, hence $k_{\mathcal{M}^*}(y) \leq k_{\mathcal{M}'}(C)$. Furthermore, for $A \wedge B$ we have shown above that $k_{\mathcal{M}^*}(A \wedge B) = k_{\mathcal{M}}(A \wedge B) = i < k_{\mathcal{M}'}(A \wedge B)$, hence $\mathcal{M}^* <_{VIMS} \mathcal{M}'$, which contradicts the minimality of $\mathcal{M}'$. We conclude that if $K' \not\models_{VIMS} A \mid\!\sim B$, then also $K \not\models_{FIMS} A \mid\!\sim B$.

For the other direction, suppose $\mathcal{M} \not\models A \mid\!\sim B$, i.e. in a minimal canonical model of K, $\mathcal{M}$, $k_{\mathcal{M}}(A \wedge \neg B) \leq k_{\mathcal{M}}(A \wedge B)$. Let $k_{\mathcal{M}}(A \wedge \neg B) = i$ and $k_{\mathcal{M}}(A \wedge B) = j$. Consider the model $\mathcal{M}^*$ built as in the first part of the construction used above. More precisely $\mathcal{M}^* = \langle W^*, <^*, V^* \rangle$ is built from $\mathcal{M}$ by cutting out its portion containing: x in $Min^{\mathcal{M}}_{<}(A \wedge B)$, $x' \in Min^{\mathcal{M}}_{<}(A \wedge \neg B)$ and an element $y \in Min^{\mathcal{M}}_{<}(C)$ for each antecedent C of conditional in K (i.e. we introduce an element y for each element of $A_{K,A \mid\sim B}$). $V^* = V$ and $k_{\mathcal{M}^*} = k_{\mathcal{M}}$. If the same element y is associated to two different formulas, it must be duplicated into y and y' (and $V^*(y') = V^*(y)$ and $k_{\mathcal{M}^*}(y') = k_{\mathcal{M}^*}(y)$). Furthermore, for each world y associated to a formula C, $V^*(y)$ is extended in order to include ϕ_C. Last, $<^*$ is obviously defined from $k_{\mathcal{M}^*}$. By reasoning similarly to what we have done above, we can show that $\mathcal{M}^*$ is a model of K'. Furthermore, there cannot be a $\mathcal{M}^{*'} <_{VIMS} \mathcal{M}^*$. Indeed, any model of K' must have a distinct element x satisfying $C \wedge \phi_C$ for each C in $A_{K,A \mid\sim B}$. Now suppose there was a model $\mathcal{M}^{*'}$ of K' with $\mathcal{M}^{*'} <_{VIMS} \mathcal{M}^*$. Suppose the same elements of the domains of $\mathcal{M}$ and $\mathcal{M}^{*'}$ satisfy the same $C \wedge \phi_C$ for C in $A_{K,A \mid\sim B}$ (hence $\mathcal{M}^*, x \models C \wedge \phi_C$ iff $\mathcal{M}^{*'}, x \models C \wedge \phi_C$, otherwise consider the model equivalent to $\mathcal{M}^{*'}$ that respects this constraint). If $\mathcal{M}^{*'} <_{VIMS} \mathcal{M}^*$, then for some x, $k_{\mathcal{M}^{*'}}(x) < k_{\mathcal{M}^*}(x)$. Suppose in $\mathcal{M}^*, x \models C \wedge \phi_C$ (and hence also $\mathcal{M}^{*'}, x \models C \wedge \phi_C$). By construction of $\mathcal{M}^*$, $k_{\mathcal{M}^*}(x) = k_{\mathcal{M}}(C)$. If $k_{\mathcal{M}^{*'}}(x) < k_{\mathcal{M}^*}(x)$, then $k_{\mathcal{M}^{*'}}(C) < k_{\mathcal{M}^*}(C)$, against Propositions 5 and 7. Then, it cannot be $\mathcal{M}^{*'} <_{VIMS} \mathcal{M}^*$, hence $\mathcal{M}^*$ is a minimal model of K'. Furthermore by construction $k_{\mathcal{M}^*}(A \wedge \neg B) \leq k_{\mathcal{M}^*}(A \wedge B)$. We conclude that $K' \not\models_{VIMS} A \mid\!\sim B$. ■

From Theorem 3 and Theorem 5 just shown, it follows that:

Theorem 6 *$A \mid\!\sim B \in \overline{K}$ iff $K' \models_{VIMS} A \mid\!\sim B$ for K' of Definition 11.*

Relation with P_{min} and Pearl's System Z

In (Giordano et al. 2010) an alternative nonmonotonic extension of preferential logic **P** called $\mathbf{P}_{min}$ is proposed.

Similarly to the semantics presented in this work, $\mathbf{P}_{min}$ is based on a minimal modal semantics. However the preference relation among models is defined in a different way. Intuitively, in $\mathbf{P}_{min}$ the fact that a world x is a minimal A-world is expressed by the fact that x satisfies $A \wedge \Box\neg A$, where $\Box$ is defined with respect to the inverse of the preference relation (i.e. with respect to the accessibility relation given by Ruv iff $v < u$). The idea is that preferred models are those that minimize the set of worlds where $\neg\Box\neg A$ holds, that is A-worlds which are not minimal. As a difference from the approach presented in this work, the semantics of $\mathbf{P}_{min}$ is defined starting from preferential models of Definition 1, in which the relation $<$ is irreflexive and transitive (thus, no longer modular). $\mathbf{P}_{min}$ is a nonmonotonic logic considering only **P** models that, intuitively, minimize the non-typical worlds. More precisely, given a set of formulas K, a model $\mathcal{M} = \langle \mathcal{W}_{\mathcal{M}}, <_{\mathcal{M}}, V_{\mathcal{M}} \rangle$ of K and a model $\mathcal{N} = \langle \mathcal{W}_{\mathcal{N}}, <_{\mathcal{N}}, V_{\mathcal{N}} \rangle$ of K, we say that $\mathcal{M}$ is preferred to $\mathcal{N}$ if $\mathcal{W}_{\mathcal{M}} = \mathcal{W}_{\mathcal{N}}$, and the set of pairs $(w, \neg\Box\neg A)$ such that $\mathcal{M}, w \models \neg\Box\neg A$ is strictly included in the corresponding set for $\mathcal{N}$. A model $\mathcal{M}$ is a *minimal model* for K if it is a model of K and there is no a model $\mathcal{M}'$ of K which is preferred to $\mathcal{M}$. Entailment in $\mathbf{P}_{min}$ is restricted to minimal models of a given set of formulas K. In Section 3 of (Giordano et al. 2010) it is observed that the logic $\mathbf{P}_{min}$ turns out to be quite strong. In general, if we only consider knowledge bases containing only positive conditionals, we get the same trivialization result (part of Proposition 1 in (Giordano et al. 2010)) as the one contained in Proposition 4 for $VIMS$.

This does not hold for rational closure. This is the reason why we have introduced the additional assumptions of Definition 11 in order to obtain an equivalence with rational closure. Similarly, in order to tackle this trivialization in $\mathbf{P}_{min}$, Section 3 in (Giordano et al. 2010) is focused on the so called *well-behaved knowledge bases*, that explicitly include that A is possible ($\neg(A \mid\!\sim \bot)$) for all conditional assertions $A \mid\!\sim B$ in the knowledge base.

We can now wonder whether $\mathbf{P}_{min}$ is equivalent to $VIMS$, which is the semantics to which it resembles the most. Or whether $VIMS$ is equivalent to a stronger version of $\mathbf{P}_{min}$ obtained by replacing **P** with **R** as the underlying logic. We call $\mathbf{R}_{min}$ this stronger version of $\mathbf{P}_{min}$.

Example 4 Let $K = \{PhD \mid\!\sim \neg worker, PhD \mid\!\sim adult, adult \mid\!\sim worker, italian \mid\!\sim house\_owner, PhD \mid\!\sim \neg house\_owner\}$.

What do we derive in $\mathbf{P}_{min}$ and $\mathbf{R}_{min}$, and what in $VIMS$? By what said above, since K only contains positive conditionals, both in $\mathbf{P}_{min}$ and $\mathbf{R}_{min}$, on the one side, and in $VIMS$, on the other side, we derive that $italian \wedge PhD \mid\!\sim \bot$. So let's add to K the constraint that people who are italian and have a PhD do exist by introducing in K a conditional $\neg(italian \wedge PhD \mid\!\sim \bot)$, thus obtaining: $K' = \{PhD \mid\!\sim \neg worker, PhD \mid\!\sim adult, adult \mid\!\sim worker, italian \mid\!\sim house\_owner, PhD \mid\!\sim \neg house\_owner, \neg(italian \wedge PhD \mid\!\sim \bot)\}$.

Notice that since $\neg(italian \wedge PhD \mid\!\sim \bot)$ entails both that $\neg(italian \mid\!\sim \bot)$ and that $\neg(PhD \mid\!\sim \bot)$, and that this in turn entails $\neg(adult \mid\!\sim \bot)$, K' is also well-behaved.

It can be easily verified that the logical consequences of K' in $\mathbf{P}_{min}$, $\mathbf{R}_{min}$, and $VIMS$ differ. In both $\mathbf{P}_{min}$ and $\mathbf{R}_{min}$, for instance, we derive neither that $italian \wedge PhD \mid\!\sim house\_owner$ nor that $italian \wedge PhD \mid\!\sim \neg house\_owner$: the two alternatives are equivalent. On the other hand, in $VIMS$ we derive that $italian \wedge PhD \mid\!\sim \neg house\_owner$.
The previous example shows that in some cases $VIMS$ is stronger than both $\mathbf{P}_{min}$ and $\mathbf{R}_{min}$. The following one shows that the two approaches are incomparable, since there are also logical consequences that hold for both $\mathbf{P}_{min}$ and $\mathbf{R}_{min}$ but not for $VIMS$.

Example 5 Let $K = \{PhD \mid\!\sim adult, adult \mid\!\sim work, PhD \mid\!\sim \neg work, italian \mid\!\sim house\_owner\}$.

What do we derive about typical $italian \wedge PhD \wedge work$, for instance? Do they inherit the property of typical italians of being $house\_owner$? Again, in order to prevent the entailment of $italian \wedge PhD \wedge work \mid\!\sim \bot$ from K both in $VIMS$ and in $\mathbf{P}_{min}$ and $\mathbf{R}_{min}$, we add to K the constraint that italians with a PhD who work exist, henceforth they also have typical instances. Therefore we expand K into:

$$K' = \{PhD \mid\!\sim adult, adult \mid\!\sim work, PhD \mid\!\sim \neg work, italian \mid\!\sim house\_owner, \neg(italian \wedge PhD \wedge work \mid\!\sim \bot)\}.$$

By reasoning as in Example 4 we can show that K' is a well-behaved knowledge base. Now it can be shown that

$$italian \wedge PhD \wedge work \mid\!\sim house\_owner$$

is entailed in $\mathbf{P}_{min}$ and $\mathbf{R}_{min}$, whereas nothing is entailed in $VIMS$. This difference can be explained intuitively as follows. The set of properties for which an individual is atypical matters in $\mathbf{P}_{min}$ and $\mathbf{R}_{min}$ where one has to minimize the set of distinct $\neg\Box\neg C$: even if an $italian \wedge PhD \wedge work$ is an atypical PhD, $\mathbf{P}_{min}$ and $\mathbf{R}_{min}$ still maximize its typicality as an italian, and therefore entail that it is a house_owner, as all typical italians. As a difference, in $VIMS$, what matters is *the set of individuals which are more typical* than a given x, rather than *the set of properties* with respect to which they are more typical. As a consequence, since an x which is $italian \wedge PhD \wedge work$ is an atypical PhD, there is no need to maximize its typicality as an italian, as long as this does not increase the set of individuals more typical than x.

In (Pearl 1990) Pearl has introduced two notions of 0-entailment and 1-entailment to perform nonmonotonic reasoning. We recall here the semantic definition of both and then we remark upon their relation with our semantics and rational closure. A model $\mathcal{M}$ for a finite knowledge base K has the form $\mathcal{M} = (\{true, false\}^{ATM}, k_{\mathcal{M}})$ where $\{true, false\}^{ATM}$ is the set of propositional interpretations for, say, a fixed finite propositional language, and $k_{\mathcal{M}}$ is our height function mapping propositional interpretations to $\mathcal{N}$, the definition of height $k_{\mathcal{M}}(A)$ of a formula is the same as in our semantic. A conditional $A \mid\!\sim B$ is true in a model $\mathcal{M}$ if $k_{\mathcal{M}}(A \wedge B) < k_{\mathcal{M}}(A \wedge \neg B)$. Then the two entailments relations are defined as follows:

$K \models_{0-ent} A \mid\!\sim B$ if $A \mid\!\sim B$ is true in all models of K
$K \models_{1-ent} A \mid\!\sim B$ if $A \mid\!\sim B$ is true in the (unique) model $\mathcal{M}$ of K which is *minimal* with respect to $k_{\mathcal{M}}$.

(minimal with respect to $k_{\mathcal{M}}$ means that no other model $\mathcal{M}'$ assigns a lower value $k_{\mathcal{M}'}$ to any propositional interpretation). First, observe that Pearl's semantics (both 0 and 1 entailment) cannot cope with conditionals having an inconsistent antecedent. This limitation is deliberate and is motivated by a probabilistic interpretation of conditionals: in asserting $A \mid\sim B$, A must not be impossible, no matter how it is unlikely. For this reason, a knowledge base such as $K = \{A \mid\sim P, A \mid\sim \neg P, B \mid\sim Q\}$ is out of the scope of Pearl's semantics, and nothing can be said about its consequences. As a difference with respect to Pearl's approach we are able to consider such K, we just derive that A is impossible, without concluding that K is inconsistent or trivial, in the sense that everything follows from it. Moreover both 0-entailment and 1-entailment fail to validate:

$$\emptyset \models_{0-ent/1-ent} A \mid\sim \top \text{ whenever} \vdash_{PC} \neg A$$

which is valid in any KLM logic, whence in rational closure (as well as in our semantics). However two definitions should make apparent the relations with our semantics and rational closure. If we consider a K such that $\forall A \mid\sim B \in K, K \not\models_R A \mid\sim \bot$, we get an obvious correspondence between our *canonical* models (which will contain worlds for very possible propositional interpretation) and models of Pearl's semantics. The correspondence preserves $FIMS$ minimality, so that we get immediately:

Proposition 8 *$K \models_{1-ent} A \mid\sim B$ iff $A \mid\sim B$ holds in any $FIMS$-minimal* canonical *model of K.*

By Theorem 3, we therefore obtain $K \models_{1-ent} A \mid\sim B$ iff $A \mid\sim B \in \bar{K}$. This is not a surprise, the correspondence between 1-entailment and rational closure was already observed by Pearl in (Pearl 1990; Pearl and Goldszmidt 1990). However, it only works for knowledge bases with the strong consistency assumption as above.

Conclusions and Future Works

We have provided a semantic reconstruction of the known rational closure within modal logic. We have provided two minimal model semantics, based on the idea that preferred rational models are those one in which the height of the worlds is minimized. We have then shown that adding suitable possibility assumptions to a knowledge base, these two minimal model semantics correspond to rational closure.

The correspondence between the proposed minimal model semantics and rational closure suggests the possibility of defining variants of rational closure by varying the three ingredients underlying our approach, namely: (i) the properties of the preference relation $<$: for instance just preorder, or multi-linear (Giordano et al. 2010), or weakly-connected (observe that $\mathbf{P}$ is complete with respect to any of the three classes); (ii) the comparison relation on models: for instance based on the heights of the worlds or on the inclusion between the relations $<$, or on negated boxed formulas satisfied by a world, as in the logic $\mathbf{P}_{min}$; (iii) the choice between fixed or variable interpretations. The systems obtained by various combinations of the three ingredients are largely unexplored and may give rise to useful nonmonotonic logics. We finally intend to extend our approach to richer languages, notably in the context of nonmonotonic description logics.

References

Booth, R., and Paris, J. 1998. A note on the rational closure of knowledge bases with both positive and negative knowledge. *Journal of Logic, Language and Information* 7:165–190.

Boutilier, C. 1994. Conditional logics of normality: a modal approach. *Artificial Intelligence* 68(1):87–154.

Casini, G., and Straccia, U. 2010. Rational closure for defeasible Description Logics. In Janhunen, T., and Niemelä, I., eds., *Proceedings of JELIA 2010 (12th European Conference on Logics in Artificial Intelligence)*, volume 6341 of *LNAI*, 77–90. Helsinki, Finland: Springer.

Friedman, N., and Halpern, J. Y. 2001. Plausibility measures and default reasoning. *Journal of the ACM* 48(4):648–685.

Giordano, L.; Gliozzi, V.; Olivetti, N.; and Pozzato, G. L. 2010. A nonmonotonic extension of KLM preferential logic P. In Fermüller, C. G., and Voronkov, A., eds., *Proceedings of LPAR 2010 (17th Conference on Logic for Programming, Artificial Intelligence, and Reasoning)*, volume 6397 of *ARCoSS LNCS*, 317–332. Yogyakarta, Indonesia: Springer-Verlag.

Kraus, S.; Lehmann, D.; and Magidor, M. 1990. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence* 44(1-2):167–207.

Lehmann, D., and Magidor, M. 1992. What does a conditional knowledge base entail? *Artificial Intelligence* 55(1):1–60.

Pearl, J., and Goldszmidt, M. 1990. On the relation between rational closure and System-Z. In *Proceedings of NMR 1990 (3rd International Workshop on Non-Monotonic Reasoning)*.

Pearl, J. 1990. System Z: A natural ordering of defaults with tractable applications to nonmonotonic reasoning. In Parikh, R., ed., *Proceedings of TARK (3rd Conference on Theoretical Aspects of Reasoning about Knowledge)*, 121–135. Pacific Grove, CA, USA: Morgan Kaufmann.

Spohn, W. 1988. Ordinal conditional functions: A dynamic theory of epistemic states. In Harper, W., and Skyrms, B., eds., *Causation in Decision, Belief Change, and Statistics*, 105–134. Dordrecht: Kluwer.