

A preferential proposal for contextual reasoning

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Abstract

In recent years disciplines such as psychology, cognitive sciences and philosophy of mind have proposed alternative models to the *classical view* of conceptualization, that is traditionally centered on the role of *definitions*. Classical logic and classical set theory are no longer considered adequate tools for the formalization of the conceptual structures introduced by such new proposals. In this paper, starting from a recent work by Michael Freund, we present a logical formalization for a theory of concepts where notions such as *stereotypical properties* and *context* play a central role.

Introduction

In the study of concepts and conceptualization, the so-called classical view depicts concepts as represented in our mind by means of *definitions*: a list of properties that are individually necessary and jointly sufficient in order to recognize an item as an occurrence of a certain concept. This identification between concepts and definitions results in sharp categorizations (every item either falls or does not fall in a definitive way under a given concept), that can be formalized by means of classical set-theory and classical mathematical logic.

Since the 70's, especially thanks to the work of Eleanor Rosch (see e.g. (Rosch 1975)), a series of drawbacks of the classical view have become evident, since most concepts, under certain conditions, can reveal fuzzy boundaries and borderline cases. The experimental investigation over human conceptualization has revealed a subtle structure, where notions as 'typical properties' and 'degree of membership' play a fundamental role; hence, there has been a flourishing of new theories about the structure of concepts and their role in cognitive processes. The notion of *stereotype* acquires a key-role in the formation of concepts and in commonsense reasoning, where by 'stereotype' or 'prototype' we refer to an individual that satisfies some key properties, typically (but not necessarily) true of the members of a certain class. Hence, a concept is connected to a set of properties that can be ascribed to its *most typical* instantiations, not necessarily to *every* item that falls under such a concept. For example, the stereotype of a natural category such as 'bird' can be thought as a set of properties describing an individual that we consider to be particularly representative of the very

concept of a bird, as: little dimensions, covered with feathers, with wings, with a beak, flying, nesting on trees, laying eggs, singing. . . . In this case, the properties that we associate to the concept of bird are appropriate for identifying a robin or some other little tree-bird, but not all such properties are necessary for deciding if an individual is a bird (think of atypical birds as penguins, ostriches, etc.).

The role of the stereotypes in the formation and use of concepts (see (Murphy 2002) for an overview) has forced to abandon the classical definitional characterization of categories to the benefit of more elaborated characterizations, such that the members of a category can be arranged with respect to their 'resemblance' to the category's typical members: if an item satisfies or not a typical property of a concept, it falls, respectively, more or less under such a concept. More recently, new approaches to the study of concepts have taken into consideration also the role that each concept plays in a more comprehensive picture, taking into account also our knowledge about the world in general, and the way we interact with it in different situations (see again (Murphy 2002) for an overview). From such a point of view, a concept has no fixed connotation, but its meaning depends on the particular situation at hand, that can force the agent to focus his attention on particular properties, ignoring others. Some new theories embracing such a perspective about concepts are Barsalou's *ad-hoc categorization* (Barsalou 1983) or Sperber and Wilson's *Relevance Theory* (Sperber and Wilson 1995); according to such proposals, the conceptual schemes that an agent uses in its interactions cannot be identified without considering also the particular situations at hand, since every kind of interaction focuses the agent's attention on particular kinds of properties and information.

For example, consider a piano. You can have different kinds of interactions with it: if you are a musician, you see the piano principally as a musical instrument, and your main concern will be about the quality of its sound; otherwise, if you have to carry it up to the fourth floor, the piano will be considered primarily as a heavy object, and your attention will be focused on properties such as its weight and its fragility. In every situation the representation we have of an item varies: different contexts (i.e. different goals, different kinds of interactions. . .) result into an alteration of the importance our cognition ascribes to the properties taking part in the characterization of an item.

In what follows we propose a semantical formalization of concepts that takes under consideration also the role of stereotypes and contexts. We will assume an hypothesis about conceptualization that, in particular, Barsalou has inferred from its experiments: independently of the different contexts, people have a very general, complete and stable (and often socially shared) amount of information associated to every concept. The contextual variation in the meaning of a concept depends on which portion of such background knowledge the agent considers contextually relevant, retrieving it from his memory in the particular situation at hand. As Barsalou, we will distinguish between *categories*, classes defined on the base of the stable background information stored in the agent's memory, and *concepts*, the contextual representation of a category, defined only by the information the agent takes under consideration in the particular case. For example, we will have a relatively stable *category* 'piano', defined by a huge amount of information about pianos stored in our memory, and different, 'on the fly', *concepts* 'piano', depending on the situations at hand, for example whether we are going to play the piano or to carry it up the stairs, as in the above example.

"[...] on one hand, we may have a stable concept BIRD in our minds which gives access to a range of information about the category of birds (e.g. they have feather, they sing, they fly, etc.). On the other hand, only a certain subset of these features of birds may be accessible to us at one particular point. When this selective processing results in a representation which denotes a subset (or superset) of the general category of birds we can say that a new concept has been constructed ad hoc. In this way, a single encoded concept, say BIRD, can be used as a starting point to construct a wide array of ad hoc concepts, say BIRD*, BIRD**, BIRD***, on different occasions, each one with a different denotation." [(Vega Moreno 2007), p.21]

The starting point for the present proposal is in a recent work by Michael Freund, (Freund 2008), that introduces a qualitative formalization of vague concepts in a preferential semantic framework. We shall modify it in order to obtain a more 'flexible' framework, appropriate for the formalization of the role of contexts in conceptualization. From (Freund 2008) we take two basic ideas:

- The main elements for the definition of a concept C : a set of properties Δ_C that members of C typically, but not necessarily, satisfy, an associated salience order \geq_Q defined over Δ_C , and the complexity level function l defined over the concepts.
- The idea of the composition of concepts, presented in the last section.

Regarding the two above points, we have avoided some theoretical simplifications made by Freund (for example that every concept has a defined, fixed extension, or that at the base of our conceptual system are sharp, not-vague elementary concepts) and we have changed slightly the definition of the preorder defining a complex concept and the one of the construction of a composed concept.

The work is structured as follows: in the next section we shall present a semantical model for dealing with categories, *i.e.* our general background information about classes. Then, we shall introduce a formal notion of context and the contextual characterization of concepts. In the succeeding sections

we shall see how our model deals with problems typically faced by the theory of concepts, such as similarity, extension, and composition of concepts.

Default information: categories.

In this paragraph we present a possible way to deal with stereotypical, vague concepts. This structure shall represent the general, context-independent and exhaustive information the agent has about a particular domain, that is, we want this structure to represent the *categories* the agent has in mind, and in what follows we shall refer to it as the *default knowledge* of the agent.

Our structure is composed of a set of individuals and a set of categories. The set of individuals is represented by a finite domain \mathcal{D} of individuals ($\mathcal{D} = \{a_1, \dots, a_n\}$), and let \mathcal{P} be a finite set of categories ($\mathcal{P} = \{P_1, \dots, P_m\}$). We want our model to be appropriate for the expression of vague concepts, with no sharp boundaries. Hence, we do not associate to a concept P any definite extension; instead, we define how P applies to our individuals, the elements of the domain \mathcal{D} , by means of a *preference relation* (or *typicality relation*) \preceq_P . Preference relations are binary relations typically used in semantics for the formalization of defeasible reasoning (see e.g. (Kraus, Lehmann, and Magidor 1990)), in order to indicate which situations an agent considers more or less typical; here we apply them to our domain of items \mathcal{D} , in order to indicate which individuals an agent considers as more or less typical in the domain w.r.t. a given category.

A preference relation \preceq_P is a preorder (i.e. a transitive, reflexive relation) over \mathcal{D} ($\preceq_P: \mathcal{D} \times \mathcal{D}$), where ' $a \preceq_P b$ ' has to be read as ' a is at least as typical as b w.r.t. concept P '. We shall indicate by \prec_P and \equiv_P , respectively, the strict and the equivalence parts of the relation \preceq_P ($a \prec_P b$ if and only if $a \preceq_P b$ and $b \not\preceq_P a$, while $a \equiv_P b$ if and only if $a \preceq_P b$ and $b \preceq_P a$).

We assume that each agent has a set of *elementary categories*, that is, a set of categories whose definition does not refer to any other category (admittedly, this is a strong theoretical assumption, compatible, for example, with an empiricist characterization of cognition). Hence, we partition the set of categories \mathcal{P} into a set \mathcal{P}_E , the set of *elementary categories*, and a set \mathcal{P}_C , the set of *complex categories*. Every category P in \mathcal{P}_E , not depending for its characterization on any other category, is directly associated to a preorder \preceq_P over \mathcal{D} .

Instead, a complex category Q in \mathcal{P}_C is defined referring to other categories: we associate to it a pair $\langle \Delta_Q; \geq_Q \rangle$. $\Delta_Q = \{P_1, \dots, P_m\}$ is a finite set of categories (the prototypical properties associated to the elements of Q), and \geq_Q is a *salience relation* over Δ_Q ($\geq_Q: \Delta_Q \times \Delta_Q$), where ' $R \geq_Q S$ ' has to be interpreted as saying 'The property R is at least as salient as the property S with respect to the characterization of the category Q '.

\geq_Q is a *modular* order, i.e. it satisfies, besides reflexivity and transitivity, the property of completeness:

- *Completeness*: for every $R \in \Delta_Q$ and $S \in \Delta_Q$, $R \geq_Q S$ or $S \geq_Q R$.

As in (Freund 2008), we impose that every category is built from ‘simpler’ ones, that is, every concept P has a *complexity level* $l(P)$ defined in the following way:

- If $P \in \mathcal{P}_E$, then $l(P) = 0$.
- If $P \in \mathcal{P}_C$, then $l(P) = \max\{l(Q) \mid Q \in \Delta_P\} + 1$.

That is, we impose that in order for a *category system* (The set of categories \mathcal{P}) to be acceptable, the complexity level function l must be definable. We can define the typicality order \preceq_Q associated to the concept Q in a lexicographic way.

Definition 1 (Typicality Relation). *Given a pair $\langle \Delta_Q; \geq_Q \rangle$, associated to a complex category Q , we define a relation \preceq_Q over \mathcal{D} as*

$a \preceq_Q b$ if and only if, for any category $R \in \Delta_Q$,

- *$a \preceq_R b$; or*
- *there is a category $S \in \Delta_Q$, with $S >_Q R$, such that $a \prec_S b$.*

A lexicographic composition of the preorders is the most natural method to obtain a preferential relation ordering the elements of \mathcal{D} w.r.t. a category Q in such a way that the respective salience of the properties listed in Δ_Q is respected. It is easy to see that, assumed that Δ_Q is a finite set, the preference relation associated to a complex category Q is a preorder.

Proposition 1. *For every category $Q \in \mathcal{P}$, the relation \preceq_Q is a preorder.*

Proof. Check items 1. and 5. in (Andréka, Ryan, and Schobbens 2002), Theorem 4.1. \square

Now we have a set of objects \mathcal{D} and a set of properties (categories) \mathcal{P} such that every P in \mathcal{P} is defined by means of a (reflexive and transitive) preference relation \preceq_P over \mathcal{D} , relation that expresses the typicality of the objects in \mathcal{D} with respect to a concept P . We can identify the set of the most typical elements of the category P as

$$\min_{\preceq_P}(\mathcal{D}) = \{a \in \mathcal{D} \mid \text{for every } b \in \mathcal{D}, b \not\prec_P a\}$$

The set $\min_{\preceq_P}(\mathcal{D})$ contains the most typical samples, the prototypes, of the category P . The structure just defined represents the background information to which the agent can draw the pieces of information that are relevant in a particular context.

Example 2. Suppose we are treating the complex category *bird*, that is defined by a set of categories $\Delta_{bird} = \{animal, covered\ with\ feathers, has\ wings, has\ beak, flies\}$. Such properties are arranged by a salience relation \geq_{bird} into three layers (here and in what follows the items grouped between vertical bars are equivalent w.r.t. the considered order):

$$animal \quad >_{bird} \quad \left| \begin{array}{c} covered\ with\ feathers \\ has\ wings \\ has\ beak \end{array} \right| >_{bird} \quad flies$$

Let $\mathcal{D} = \{a, b, c, d, e\}$ be our domain, where a is a robin, b a penguin, c is a cat, d is a shoe, and e is a bat.

Each of the categories in Δ_{bird} order the domain in the following way:

$$\left| \begin{array}{c} a \\ b \\ c \\ e \end{array} \right| \prec_{animal} d \quad ; \quad \left| \begin{array}{c} a \\ b \end{array} \right| \prec_{feathers} \left| \begin{array}{c} c \\ d \\ e \end{array} \right| ;$$

$$\left| \begin{array}{c} a \\ b \\ e \end{array} \right| \prec_{wings} \left| \begin{array}{c} c \\ d \end{array} \right| ;$$

$$\left| \begin{array}{c} a \\ b \end{array} \right| \prec_{beak} \left| \begin{array}{c} c \\ d \\ e \end{array} \right| ; \quad \left| \begin{array}{c} a \\ e \end{array} \right| \prec_{flies} \left| \begin{array}{c} b \\ c \\ d \end{array} \right|$$

Using the lexicographic procedure, we obtain the following ordering \preceq_{bird} over \mathcal{D} :

$$a \prec_{bird} b \prec_{bird} e \prec_{bird} c \prec_{bird} d$$

That is, the robin a is a more typical exemplar of the category *bird* than the penguin b , which is more typical than the bat e ; then there is the cat c , and, finally, the shoe d , that is not even an animal.

Remark 1. In case we want to assume that every elementary category is characterized by modular orders, instead of simple preorders, also modularity is preserved through a lexicographic construction of the typicality orders.

Proposition 3. *If the typicality orders associated to the elementary categories in \mathcal{P}_E are modular, then, for every category $Q \in \mathcal{P}$, the relation \preceq_Q is modular.*

To prove the statement is sufficient to prove that also completeness is preserved in the lexicographic construction (see (Andréka, Ryan, and Schobbens 2002), theorem 4.1, item 6). So, what follows is applicable also if we want to consider only modular typicality orders.

Contextual information: concepts.

The preferential model presented above represents agent’s default information about categories. An agent, equipped with such background knowledge, refers to it in order to retrieve relevant information to interact with the environment and other agents in specific situations.

Above, every complex category Q is defined by means of a set Δ_Q of simpler concepts and by a salience order \geq_Q . Such information models the general comprehension an agent has of a category, and in every particular situation the agent retrieves and uses only portions of such default information, depending on the properties the agent considers as the most salient ones at the moment.

Hence, we define a *context* as the information the agent considers relevant in the particular situation at hand, that is, what the agent focuses his attention on. A *context* \mathcal{C} shall be defined by means of three constituents:

$$\mathcal{C} = \langle \mathcal{D}^{\mathcal{C}}, \mathcal{P}^{\mathcal{C}}, \geq^{\mathcal{C}} \rangle$$

where:

- $\mathcal{D}^{\mathcal{C}} \subseteq \mathcal{D}$. The set $\mathcal{D}^{\mathcal{C}}$ represents the set of individuals we are considering at the moment.
- $\mathcal{P}^{\mathcal{C}} \subseteq \mathcal{P}$. The set $\mathcal{P}^{\mathcal{C}}$ represents the properties that we take into consideration in the context \mathcal{C} , our contextual *concepts*.

As in the case of categories, we assume the set $\mathcal{P}^{\mathcal{C}}$ to be partitioned into the set of *elementary concepts*, $\mathcal{P}_E^{\mathcal{C}}$, and a set of *complex concepts*, $\mathcal{P}_C^{\mathcal{C}}$. The set of elementary concepts represents

those concepts that the agent treats as given, not questionable, in the particular situation. We do not assume a perfect correspondence between elementary concepts and elementary categories (i.e. $\mathcal{P}_E^{\mathcal{C}} = \mathcal{P}_E \cap \mathcal{P}^{\mathcal{C}}$). Obviously, a concept corresponding to an elementary category cannot be considered a complex concept in a contextual situation, since its nature is ‘essentially’ an elementary one, but we can assume a complex category to be treated as an elementary concept in a given context ($P \in \mathcal{P}_C$, but $P \in \mathcal{P}_E^{\mathcal{C}}$): in such a case the agent is simply assuming such categories as primitive, firmly acquired notions, that cannot be put under discussion in the situation at hand. Hence, we impose that $(\mathcal{P}^{\mathcal{C}} \cap \mathcal{P}_E) \subseteq \mathcal{P}_E^{\mathcal{C}}$, and consequently $\mathcal{P}_C^{\mathcal{C}} \subseteq \mathcal{P}_C$.

- $\geq^{\mathcal{C}}$ is a salience relation over the set of properties $\mathcal{P}^{\mathcal{C}}$ ($\geq^{\mathcal{C}}: \mathcal{P}^{\mathcal{C}} \times \mathcal{P}^{\mathcal{C}}$). As the salience order that we associate to every complex category, $\geq^{\mathcal{C}}$ is a modular order, and it indicates on which properties the attention of an agent is focused in the particular situation. that is, ‘ $P \geq^{\mathcal{C}} Q$ ’ is interpreted as ‘In the context \mathcal{C} , the concept P is at least as salient as the concept Q ’.

Hence, a context tells us what the agent is taking under consideration in a particular situation: which items and which properties.

For every context \mathcal{C} , we associate to every concept P in $\mathcal{P}^{\mathcal{C}}$ a preferential order $\preceq_P^{\mathcal{C}}$, appropriate for the context. If $P \in \mathcal{P}_E^{\mathcal{C}}$, then the agent uses the concept P in an ‘uncritical’ way, and consequently it associates to it its default preference order \preceq_P , defined by its background knowledge; we simply restrict \preceq_P to the contextual domain, that is

- $\preceq_P^{\mathcal{C}} = \preceq_P \cap (\mathcal{D}^{\mathcal{C}} \times \mathcal{D}^{\mathcal{C}})$

Otherwise, if $P \in \mathcal{P}_C^{\mathcal{C}}$, we associate to every complex concept Q a pair $\langle \Delta_Q^{\mathcal{C}}; \geq_Q^{\mathcal{C}} \rangle$, where

- $\Delta_Q^{\mathcal{C}} = \Delta_Q \cap \mathcal{P}^{\mathcal{C}}$. That is, of all the properties associated to the category Q , we associate to the concept Q in the context \mathcal{C} just those properties taken under consideration in \mathcal{C} (if $\Delta_Q \cap \mathcal{P}^{\mathcal{C}} = \emptyset$, Q must be treated as an elementary concept).
- The salience relation associated to a concept Q in \mathcal{C} , the relation $\geq_Q^{\mathcal{C}}$, is generated from the contextual relation $\geq^{\mathcal{C}}$, restricting it to the members of the set $\Delta_Q^{\mathcal{C}}$ (i.e. $\geq_Q^{\mathcal{C}} = \geq^{\mathcal{C}} \cap (\Delta_Q^{\mathcal{C}} \times \Delta_Q^{\mathcal{C}})$). That is

$$R \geq_Q^{\mathcal{C}} S \text{ iff } R \in \Delta_Q^{\mathcal{C}}, S \in \Delta_Q^{\mathcal{C}}, \text{ and } R \geq^{\mathcal{C}} S.$$

Hence, we have a contextual model where we can associate to every concept P in $\mathcal{P}^{\mathcal{C}}$ a typicality order $\preceq_P^{\mathcal{C}}$ with the same procedures adopted for the background knowledge.

Definition 2 (Contextual typicality relation). *Given a context $\mathcal{C} = \langle \mathcal{D}^{\mathcal{C}}, \mathcal{P}^{\mathcal{C}}, \geq^{\mathcal{C}} \rangle$, the typicality relation $\preceq_P^{\mathcal{C}}$ associated to a concept $P \in \mathcal{P}^{\mathcal{C}}$ is defined in the following way:*

- If $P \in \mathcal{P}_E^{\mathcal{C}}$, then the agent associates to the concept P the contextual restriction of its default preference order \preceq_P ($\preceq_P^{\mathcal{C}} = \preceq_P \cap (\mathcal{D}^{\mathcal{C}} \times \mathcal{D}^{\mathcal{C}})$).
- If $P \in \mathcal{P}_C^{\mathcal{C}}$, then the agent associates to the concept P a preference relation $\preceq_P^{\mathcal{C}}$ obtained in the usual lexicographic way, that is, given the pair $\langle \Delta_P^{\mathcal{C}}; \geq_P^{\mathcal{C}} \rangle$, $a \preceq_P^{\mathcal{C}} b$ if and only if, for any concept $R \in \Delta_P^{\mathcal{C}}$,
 - $a \preceq_R^{\mathcal{C}} b$; or
 - there is a concept $S \in \Delta_P^{\mathcal{C}}$, with $S >_P^{\mathcal{C}} R$, such that $a \prec_S^{\mathcal{C}} b$.

As in the case of categories, in every context \mathcal{C} , for every concept $P \in \mathcal{P}^{\mathcal{C}}$, we define a preference relation $\preceq_P^{\mathcal{C}}$ that is a preorder (or a modular order, if such are the elementary concepts). An individual is a typical sample of a concept P in a context \mathcal{C} iff it is in the set $\min_{\preceq_P^{\mathcal{C}}}(\mathcal{D}^{\mathcal{C}})$, where the function $\min_{\preceq_P^{\mathcal{C}}}$ is defined as the function \min_{\preceq_P} in the preceding section.

Example 4. Assume two contexts, \mathcal{C}' and \mathcal{C}'' . In \mathcal{C}' we reason about biological species in a naive way, while in \mathcal{C}'' we treat them in a more sophisticated way. The two contexts have the same domain: $\mathcal{D}^{\mathcal{C}'} = \mathcal{D}^{\mathcal{C}''} = \{t, s, d, o\}$, where t is a tuna, s is a shark, d is a dolphin, and o is an octopus. In our background knowledge, we associate to the category fish various kinds of information, but the context makes us consider only a subset of such information. Assume that in \mathcal{C}' the concept *fish* is defined by means of the following \mathcal{C}' -elementary concepts $\Delta_{fish}^{\mathcal{C}'} = \{\text{lives in water, is covered with scales, has fins}\}$, with *lives in water* more salient than the other two. These concepts are associated with the following preference relations over the domain:

$$\left| \begin{array}{c} \equiv \text{water} \\ s \\ t \\ d \\ o \end{array} \right| ; \quad t \prec_{scales} \left| \begin{array}{c} s \\ d \\ o \end{array} \right| ; \quad \left| \begin{array}{c} s \\ t \\ d \end{array} \right| \prec_{fins} o$$

Hence, our naive concept *fish* corresponds to the ordering $\preceq_{fish}^{\mathcal{C}'}$:

$$t \prec_{fish}^{\mathcal{C}'} \left| \begin{array}{c} d \\ s \end{array} \right| \prec_{fish}^{\mathcal{C}'} o$$

Moving to the context \mathcal{C}'' , the concept *fish* is defined by means of the following \mathcal{C}'' -elementary concepts $\Delta_{fish}^{\mathcal{C}''} = \{\text{lives in water, vertebrate, has the gills, lays eggs}\}$, where the salience relation is

$$\left| \begin{array}{c} \text{water} \\ \text{vertebrate} \end{array} \right| >_{fish}^{\mathcal{C}''} \text{gills} >_{fish}^{\mathcal{C}''} \text{lays eggs}$$

The preference orders associated with these concepts are

$$\left| \begin{array}{c} \equiv \text{water} \\ s \\ t \\ d \\ o \end{array} \right| ; \quad \left| \begin{array}{c} s \\ t \\ d \end{array} \right| \prec_{vertebrate} o ; \quad \left| \begin{array}{c} t \\ s \end{array} \right| \prec_{gills} \left| \begin{array}{c} o \\ d \end{array} \right|$$

$$\left| \begin{array}{c} t \\ o \end{array} \right| \prec_{eggs} s \prec_{eggs} d$$

From this preferential relations, we obtain the following ordering representing the concept *fish* in \mathcal{C}'' :

$$t \prec_{fish}^{\mathcal{C}''} s \prec_{fish}^{\mathcal{C}''} d \prec_{fish}^{\mathcal{C}''} o$$

As can be seen, focusing only on certain properties in the contextual characterization of a concept, we obtain different preference relations ordering the domain: in the more naive context \mathcal{C}' the dolphin and the shark are in the same relative position, are ‘equally fishy’, while in a more technical context, the shark results strictly more typical as a fish than a dolphin.

Now we have a basic structure for modeling the contextual variation of the meaning of a category. In the following paragraphs we shall see some of the problems typically associated to the study of concepts, and how they could be treated within our approach.

Similarity

Similarity between items, and between items and concepts, is an important ingredient of cognition. We can define a similarity measure between items, taking under consideration which interests the agent has in a particular context. First we shall define a notion of similarity between items with respect to a specific concept, and then a measure of relevance for the items relative to a given context.

Assume a context $\mathfrak{C} = \langle \mathcal{D}^{\mathfrak{C}}, \mathcal{P}^{\mathfrak{C}}, \geq^{\mathfrak{C}} \rangle$, and consider a concept $P \in \mathcal{P}^{\mathfrak{C}}$ with its associated preference relation $\preceq_P^{\mathfrak{C}}$, and two items a and b in $\mathcal{D}^{\mathfrak{C}}$. We can define a measure of distance between a and b with respect to the concept P in the context \mathfrak{C} . If $a \prec_P^{\mathfrak{C}} b$ or $b \prec_P^{\mathfrak{C}} a$, then there are *paths* between a and b , that is, finite sequences of items, $\langle a, c_1, \dots, c_n, b \rangle$, with $n \geq 0$, such that $a \prec_P^{\mathfrak{C}} c_1 \prec_P^{\mathfrak{C}} \dots \prec_P^{\mathfrak{C}} c_n \prec_P^{\mathfrak{C}} b$ or $b \prec_P^{\mathfrak{C}} c_n \prec_P^{\mathfrak{C}} \dots \prec_P^{\mathfrak{C}} c_1 \prec_P^{\mathfrak{C}} a$.

Let $p_P^{\mathfrak{C}}$ be a function that, given two items a and b such that either $a \prec_P^{\mathfrak{C}} b$ or $b \prec_P^{\mathfrak{C}} a$, gives back the longest path(s) between them.

Using the function $p_P^{\mathfrak{C}}$, we can define a distance function $d_P^{\mathfrak{C}}$ between two objects a and b in the context \mathfrak{C} . Given two objects a and b , we can have three cases.

- $a \prec_P^{\mathfrak{C}} b$ or $b \prec_P^{\mathfrak{C}} a$;
- $a \equiv_P^{\mathfrak{C}} b$;
- $a \not\prec_P^{\mathfrak{C}} b$ and $b \not\prec_P^{\mathfrak{C}} a$.

We consider two items a and b to be *comparable* just if we have the first two cases, that is if $a \preceq_P^{\mathfrak{C}} b$ or $b \preceq_P^{\mathfrak{C}} a$. If $a \not\prec_P^{\mathfrak{C}} b$ and $b \not\prec_P^{\mathfrak{C}} a$, then we consider the two items not to be comparable with respect to P in \mathfrak{C} (obviously, if we are dealing with modular orders two individuals are always comparable). If there is comparability, then we can define a measure of distance.

Definition 3 (Distance function $d_P^{\mathfrak{C}}$). *Given two items a and b , we define a partial distance-function $d_P^{\mathfrak{C}}$ between a and b in the following way:*

- If $a \prec_P^{\mathfrak{C}} b$ or $b \prec_P^{\mathfrak{C}} a$, then $d_P^{\mathfrak{C}}(a, b) = (|p_P^{\mathfrak{C}}(a, b)| - 1)$ (where $|p_P^{\mathfrak{C}}(a, b)|$ is the cardinality of the path(s) in $p_P^{\mathfrak{C}}(a, b)$).
- If $a \equiv_P^{\mathfrak{C}} b$, then $d_P^{\mathfrak{C}}(a, b) = 0$.
- if $a \not\prec_P^{\mathfrak{C}} b$ and $b \not\prec_P^{\mathfrak{C}} a$, then $d_P^{\mathfrak{C}}(a, b)$ has no value (the two items are not comparable with respect to P).

With the function $d_P^{\mathfrak{C}}$ we can define a *similarity measure* between items with respect to a concept P in a context \mathfrak{C} .

In order to define it, we need also a function $l^{\mathfrak{C}}(P)$, that, given a concept P , gives back the maximum distance between two elements of the domain with respect to the paths in the relation $\prec_P^{\mathfrak{C}}$.

$$l^{\mathfrak{C}}(P) = \max\{d_P^{\mathfrak{C}}(a, b) | a \in \mathcal{D}^{\mathfrak{C}}, b \in \mathcal{D}^{\mathfrak{C}}\}$$

Now we can define a measure of similarity between items.

Definition 4 (Similarity between items). *Given two items a and b , we define a partial function $s_P^{\mathfrak{C}}$ of similarity between a and b in the following way:*

- If $d_P^{\mathfrak{C}}(a, b)$ has a value, then $s_P^{\mathfrak{C}}(a, b) = \frac{l^{\mathfrak{C}}(P) - d_P^{\mathfrak{C}}(a, b)}{l^{\mathfrak{C}}(P)}$.
- If $d_P^{\mathfrak{C}}(a, b)$ has no value, then $s_P^{\mathfrak{C}}(a, b)$ is not defined too.

If $d_P^{\mathfrak{C}}(a, b)$ is defined, then $s_P^{\mathfrak{C}}(a, b)$ has a value between 0 and 1, where 1 says that a and b are maximally similar with respect to P in \mathfrak{C} , and we have such a value if and only if $a \equiv_P^{\mathfrak{C}} b$, while we have the value 0 in case a and b are maximally distant with respect to P in \mathfrak{C} .

Defined a similarity measure between items, we can define also a similarity measure between an item and a concept in a context. We identify the distance between an item and a concept with the distance between that item and a prototypical exemplar of that concept.

Definition 5 (Similarity between an item and a concept). *Given an item a and a concept P , we define a function $s^{\mathfrak{C}}$ of similarity between a and P in the following way:*

$$s^{\mathfrak{C}}(a, P) = \max\{s_P^{\mathfrak{C}}(a, b) | b \in \min_{\preceq_P^{\mathfrak{C}}}(\mathcal{D}^{\mathfrak{C}})\}$$

That is, given the set $\min_{\preceq_P^{\mathfrak{C}}}(\mathcal{D}^{\mathfrak{C}})$, representing the most typical exemplars of the concept P in the context \mathfrak{C} , we take under consideration the maximal similarity between a and an element of $\min_{\preceq_P^{\mathfrak{C}}}(\mathcal{D}^{\mathfrak{C}})$.

Note that $s^{\mathfrak{C}}(a, P)$ is a total function, with a defined value for every pair a, P , while the similarity between items is represented by a partial function, since two items may not be comparable (if our typicality relations are preorders, and not modular orders).

Example 5. Refer to Example 4. We can see how the similarity value of an item with respect to a concept can vary from context to context. In the context \mathfrak{C}' , describing a naive characterization of fishes, we have that a shark s and a dolphin d are equally distant from, and consequently equally similar to, a tuna t , that represents a typical instance of fish:

$$s_{fish}^{\mathfrak{C}'}(s, t) = s_{fish}^{\mathfrak{C}'}(d, t) = \frac{l^{\mathfrak{C}'}(fish) - d_{fish}^{\mathfrak{C}'}(d, t)}{l^{\mathfrak{C}'}(fish)} = \frac{2 - 1}{2} = \frac{1}{2}$$

Given that the only element of $\min_{\preceq_{fish}^{\mathfrak{C}'}}(\mathcal{D}^{\mathfrak{C}'})$ is t , we have

$$s^{\mathfrak{C}'}(s, fish) = s^{\mathfrak{C}'}(d, fish) = \frac{1}{2}$$

Otherwise, in the context \mathfrak{C}'' , where the concept fish is characterized in a slightly more scientific way, we have a different value of similarity, with the shark s being more similar to a typical fish than the dolphin d :

$$s_{fish}^{\mathfrak{C}''}(s, fish) = s_{fish}^{\mathfrak{C}''}(s, t) = \frac{l^{\mathfrak{C}''}(fish) - d_{fish}^{\mathfrak{C}''}(s, t)}{l^{\mathfrak{C}''}(fish)} = \frac{3 - 1}{3} = \frac{2}{3}$$

$$s_{fish}^{\mathfrak{C}''}(d, fish) = s_{fish}^{\mathfrak{C}''}(d, t) = \frac{l^{\mathfrak{C}''}(fish) - d_{fish}^{\mathfrak{C}''}(d, t)}{l^{\mathfrak{C}''}(fish)} = \frac{3 - 2}{3} = \frac{1}{3}$$

Obviously, we can also define a *default similarity measure* between items, defined exactly as above, but at the level of categories (that is, using the domain \mathcal{D} and the default typicality relations \preceq_P).

Eventually, using the preferential relation $\preceq_{\mathcal{C}}$ generated by a context \mathcal{C} we can think of a distance function $d^{\mathcal{C}}$ between items with respect to the context, in order to obtain, with the same procedure used for similarity, a *relevance function* $r^{\mathcal{C}}$ that tells us how relevant an item is in a given context. Given a context $\mathcal{C} = \langle \mathcal{D}^{\mathcal{C}}, \mathcal{P}^{\mathcal{C}}, \geq^{\mathcal{C}} \rangle$, the associated relevance relation $\preceq_{\mathcal{C}}$ can be defined simply treating \mathcal{C} as a concept defined by $\langle \mathcal{P}^{\mathcal{C}}, \geq^{\mathcal{C}} \rangle$ on $\mathcal{D}^{\mathcal{C}}$.

The steps are the same as for the similarity measure. Given the relevance relation $\preceq_{\mathcal{C}}$ over $\mathcal{D}^{\mathcal{C}}$, first we can define the related function $p^{\mathcal{C}}$ that, given two items a and b , gives back the longest path(s) between a and b with respect to the relation $\prec_{\mathcal{C}}$, and then a distance function $d^{\mathcal{C}}$ between items.

From the function $d^{\mathcal{C}}$ we can define a partial function $s^{\mathcal{C}}$ representing contextual similarity between items, expressing how much two items can be considered similar in a context, independently from any particular concept. The definitions are identical to those of $p^{\mathcal{C}}$, $d^{\mathcal{C}}$, and $s^{\mathcal{C}}$, just consider the preorder $\prec_{\mathcal{C}}$ instead of \prec_P .

Since the relation $\preceq_{\mathcal{C}}$ tells us which items are more or less relevant in a given context, we can think of $\min_{\preceq_{\mathcal{C}}}(\mathcal{D}^{\mathcal{C}})$ as the set of the most relevant items in the context \mathcal{C} . We can measure the relevance of an item a in \mathcal{C} considering how similar a is to the set $\min_{\preceq_{\mathcal{C}}}(\mathcal{D}^{\mathcal{C}})$.

Definition 6 (Relevance of an item in a context). *Given an item a , we define a relevance function $r^{\mathcal{C}}$ in the following way:*

$$r^{\mathcal{C}}(a) = \max\{s^{\mathcal{C}}(a, b) | b \in \min_{\preceq_{\mathcal{C}}}(\mathcal{D}^{\mathcal{C}})\}$$

That is, given the set $\min_{\preceq_{\mathcal{C}}}(\mathcal{D}^{\mathcal{C}})$, representing the most relevant items in the context \mathcal{C} , we take under consideration the maximal similarity between a and an element of $\min_{\preceq_{\mathcal{C}}}(\mathcal{D}^{\mathcal{C}})$.

Example 6. Consider an agent whose goal is to plant a nail in the wall, and that needs an appropriate tool. Such a need could lead to the creation of a context \mathcal{C} , where the set of properties could be $\mathcal{P}^{\mathcal{C}} = \{\text{robust}, \text{heavy}, \text{can be grasped}\}$, three equally salient properties an item should have in order to be an appropriate tool for planting a nail. Let our contextual domain be $\mathcal{D}^{\mathcal{C}} = \{h, p, s, c, r\}$, where h is a hammer, p is a pillow, s is a shoe, c a clog, and r a slipper.

We start with the following preference relations:

$$\begin{array}{l} \left| \begin{array}{l} h \\ c \end{array} \right| \prec_{\text{robust}} \left| \begin{array}{l} s \\ r \end{array} \right| \\ \left| \begin{array}{l} h \\ c \end{array} \right| \prec_{\text{heavy}} \left| \begin{array}{l} p \\ r \end{array} \right| \\ h \prec_{\text{grasp}} \left| \begin{array}{l} c \\ s \\ r \end{array} \right| \prec_{\text{grasp}} p \end{array}$$

Such orders define, in the usual lexicographic way, the following contextual preference relation:

$$h \prec^{\mathcal{C}} c \prec^{\mathcal{C}} s \prec^{\mathcal{C}} r \prec^{\mathcal{C}} p$$

Now, assume that it is summertime and the agent wants to take a walk on the seaside, and he needs to choose appropriate footwear. Suppose the set of relevant properties shall be $\mathcal{P}^{\mathcal{C}'} = \{\text{wearable}, \text{comfortable}, \text{socially accepted}\}$, with $\text{wearable} \succ^{\mathcal{C}'} \text{socially accepted} \succ^{\mathcal{C}'} \text{comfortable}$. These three concepts are associated with the following preference orders:

$$\begin{array}{l} \left| \begin{array}{l} c \\ s \\ r \end{array} \right| \prec_{\text{wearable}} \left| \begin{array}{l} p \\ h \end{array} \right| ; \\ \left| \begin{array}{l} r \\ c \end{array} \right| \prec_{\text{comfortable}} s \prec_{\text{comfortable}} \left| \begin{array}{l} p \\ h \end{array} \right| ; \\ \left| \begin{array}{l} s \\ c \end{array} \right| \prec_{\text{socially accepted}} r \prec_{\text{socially accepted}} \left| \begin{array}{l} p \\ h \end{array} \right| \end{array}$$

Hence, the new preference order becomes:

$$c \prec^{\mathcal{C}'} s \prec^{\mathcal{C}'} r \prec^{\mathcal{C}'} \left| \begin{array}{l} p \\ h \end{array} \right|$$

That is, talking about footwear for the beach, the clogs are the most relevant objects of our domain.

We can see how the relevance value of an object can vary in a radical way from one context to another. In context \mathcal{C} we have that the hammer h is in $\min_{\preceq_{\mathcal{C}}}(\mathcal{D}^{\mathcal{C}})$. So we can calculate that h has maximum relevance in \mathcal{C} , that is

$$r^{\mathcal{C}}(h) = 1$$

Otherwise, moving to \mathcal{C}' , we have

$$r^{\mathcal{C}'}(h) = s^{\mathcal{C}'}(h, c) = \frac{l^{\mathcal{C}'} - d^{\mathcal{C}'}(h, c)}{l^{\mathcal{C}'}} = \frac{3 - 3}{3} = 0$$

That is, the hammer h is not relevant at all, if we are reasoning about footwear.

Extension of concepts

Until now we have treated concepts as typicality orders over a domain. In this way we have been able to define a series of relational properties of the items, such as how typical an item is with respect to a concept, how similar two items are, or how relevant an item is with respect to a context. However, we have not dealt with the issue of defining the ‘borders’ of the concepts, their extensions, that is, which items we consider as falling under a concept and which we do not. In order to model inferential procedures we need our agents to be able to decide if an item falls under a given concept or not. Notwithstanding, we want to be able to treat the extension of a concept as a contextual matter, appropriate also for the treatment of vagueness; hence, if we ascribe to a concept a fixed extension, as in (Freund 2008), we would lose the contextual nature of concepts we wanted to formalize.

In the classical definitional characterization, we associate to every concept P a set of ‘essential properties’ such that only the items satisfying all of them can be considered as an instantiation of P . Here we proceed in an analogous way, but allowing the set of properties to vary from one context to another.

Again, we start considering first the agent’s background knowledge, his categories, and then the contextual characterization: given a concept, we will define its *default extension* w.r.t. categories, then we will specify how to modify it

according to the particular context. Given a category P , let $Ext(P)$ indicate the set of items falling under P (its extension).

First of all, recall that we have a set of elementary categories \mathcal{P}_E , each of them defined simply by a typicality relation \preceq_P . Hence, the extension of an elementary concept P cannot be defined by referring to some list of ‘essential properties’ an item should satisfy to fall under P . Now we present a possible formalization of the extension of elementary concepts, but such a proposal wants just to point out that in our model it is possible to treat a contextual variation of extensions, but more subtle methods to deal with such an issue are surely possible.

To define the default extension of an elementary concept we associate a value between 0 and 1 to it: such a value represents the *similarity threshold* (st_P) an item has to exceed in order to be considered as falling under P . For example, if we associate to the property red a similarity threshold $st_{red} = 0.8$, we define the extension of the category *red* over the domain \mathcal{D} as

$$a \in Ext(red) \text{ if and only if } s(a, red) > 0.8$$

where $s(a, red)$ is the similarity measure between a and the most typical exemplars of the category *red*.

For the complex categories in \mathcal{P}_C , since they are defined by means of other categories, we assume that there are properties the agent considers essential for the definition of a complex category. Hence, we associate to every category Q a set of properties Γ_Q , with $\Gamma_Q \subseteq \Delta_Q$. The satisfaction of the properties in Γ_Q is treated as a necessary and sufficient condition in order to consider an item a as falling under Q .

$$a \in Ext(Q) \text{ if and only if } a \in Ext(P) \text{ for every } P \in \Gamma_Q$$

Intuitively, the properties in Γ_Q should be the most salient properties in Δ_Q , but formally it is not necessary to impose such a constraint.

Hence, starting from the elementary categories and moving up to higher complexity levels, we can define the default extension $Ext(P)$ over the domain \mathcal{D} for every category $P \in \mathcal{P}$.

Using the default extensions, we can define a procedure to define the extension of a concept P with respect to a context \mathcal{C} ($Ext^{\mathcal{C}}(P)$). Recall that every context \mathcal{C} is associated to a domain $\mathcal{D}^{\mathcal{C}}$ and a set of concepts $\mathcal{P}^{\mathcal{C}}$, that is partitioned in a set of contextually elementary concepts, $\mathcal{P}_E^{\mathcal{C}}$, and a set of contextually complex concepts, $\mathcal{P}_C^{\mathcal{C}}$. When we look at a concept $P \in \mathcal{P}^{\mathcal{C}}$, we can identify three possible cases.

- $P \in \mathcal{P}_E^{\mathcal{C}}$ and $P \in \mathcal{P}_E$, that is, the concept P is not only contextually elementary, but is ‘genuinely’ elementary (it represents an elementary category). In such a case we have stated that P is characterized by a preference relation $\preceq_P^{\mathcal{C}} = \preceq_P \cap (\mathcal{D}^{\mathcal{C}} \times \mathcal{D}^{\mathcal{C}})$. Then we apply the similarity threshold st_P to the new relation $\preceq_P^{\mathcal{C}}$. This allows us to ‘modulate’ the extension of an elementary concept considering the domain we are working in.

$$\text{If } P \in \mathcal{P}_E^{\mathcal{C}} \text{ and } P \in \mathcal{P}_E, \text{ then } a \in Ext^{\mathcal{C}}(P) \text{ if and only if } s^{\mathcal{C}}(a, P) > st_P.$$

For instance, take into consideration the concept *red*, and suppose that it is associated to a similarity threshold $st_{red} = 0.8$. Assume the item a is an orange item. If we are working in a context with a domain composed of few ‘reddish’ items and many green or blue items, the item a will be contextually very similar to the most typical instances of red, and, consequently, it will be likely that the item shall overcome the similarity threshold of 0.8. On the contrary, if we are working in a domain composed only of ‘reddish’ objects, the similarity measure of a with respect to the concept *red* will diminish its value, since there will be many more shadings of red between a and prototypical examples of red. So, it will be more difficult for a to reach the threshold and be considered as a red object. That is, if we are considering many objects that are very similar to the typical instances of an elementary concept, then our capacity of discrimination becomes more subtle.

- $P \in \mathcal{P}_E^{\mathcal{C}}$ and $P \in \mathcal{P}_C$, that is, the concept P is contextually treated as an elementary concept, but the corresponding category has a complex nature. We have previously established that if we treat in a particular context a complex category as an elementary concept, we use it with its default characterization, that is we refer to its characterization at the level of categories. Hence it applies to the items in the context exactly as it applies to the same items in the default model. Consequently, not only the preference relation associated to P is defined as the restriction of the default preference relation in the domain of the context ($\preceq_P^{\mathcal{C}} = \preceq_P \cap (\mathcal{D}^{\mathcal{C}} \times \mathcal{D}^{\mathcal{C}})$), as defined in Section 3, but also its extension:

$$\text{If } P \in \mathcal{P}_E^{\mathcal{C}} \text{ and } P \in \mathcal{P}_C, \text{ then } Ext^{\mathcal{C}}(P) = Ext(P) \cap \mathcal{D}^{\mathcal{C}}.$$

- $P \in \mathcal{P}_C^{\mathcal{C}}$, that is, the concept P is contextually complex (and consequently it corresponds also to a complex category). In such a case, we have a set of essential properties Γ_P associated to the category P . So, we define $Ext^{\mathcal{C}}(P)$ considering a set of essential properties $\Gamma_P^{\mathcal{C}}$, obtained simply restricting Γ_P to the properties considered in the context, that is, $\Gamma_P^{\mathcal{C}} = \Gamma_P \cap \mathcal{P}^{\mathcal{C}}$.

$$\text{If } P \in \mathcal{P}_C^{\mathcal{C}}, \text{ then } a \in Ext^{\mathcal{C}}(P) \text{ if and only if } a \in Ext^{\mathcal{C}}(Q) \text{ for every } Q \in \Gamma_P^{\mathcal{C}}.$$

Example 7. Consider Example 4. Assume that, as a default, we have the following properties defining the membership to the category *fish*: $\Gamma_{fish} = \{lives\ in\ water, is\ vertebrate, has\ fins, has\ gills\}$, with $Ext(lives\ in\ water) = \{t, s, d, o\}$, $Ext(is\ vertebrate) = \{t, s, d\}$, $Ext(has\ fins) = \{t, s, d, \}$, and $Ext(has\ gills) = \{t, s\}$.

In the first context \mathcal{C}' , we have $\Gamma_{fish}^{\mathcal{C}'} = \{lives\ in\ water, has\ fins\}$, with *lives in water* and *has fins* treated as elementary concepts. Consequently, in such a naive context, we obtain that both the shark s and the dolphin d fall under the concept *fish*, since both of them live in water and have fins. On the contrary, if we move to the slightly more elaborate context \mathcal{C}'' , we have $\Gamma_{fish}^{\mathcal{C}''} = \{lives\ in\ water, is\ vertebrate, has\ gills\}$ (again, each of them treated as elementary concepts), and consequently now the shark s falls under the concept *fish*, but not the dolphin d , that does not have gills.

As we have just seen, it is possible to modulate the extension of every concept with respect to a particular context, defining the new extensions from the elementary concepts up to more complex ones.

Since there is the possibility of such contextual variations of extensions, the same two concepts can be *compatible* or not depending on the context, that is, in some context there could be at least an item falling under both of them, while in other contexts their extensions are disjoint.

Definition 7 (Compatibility). *Two concepts P and Q are compatible in \mathcal{C} if and only if $Ext^{\mathcal{C}}(P) \cap Ext^{\mathcal{C}}(Q) \neq \emptyset$.*

Composition of concepts.

A typical problem in the formalization of concepts is to account for their composition. By ‘composed concept’ we refer to a new concept obtained by specifying a main concept by means of a secondary one. For example, take concepts as *not-flying birds* or *toy-horses*, where the concepts *bird* and *toy* are specified by means of the concepts *not-flying* and *horse*, respectively.

Refining the proposal in (Freund 2008), we can use contexts in order to combine concepts. To express the concept obtained by combining two concepts P and Q , we write $P * Q$, where Q is the main property and P works as a specification. Following Freund, in order for a composition to be definable, we need the property P to be applicable to some elements of the principal concept Q , that is, the two concepts have to be compatible ($Ext(P) \cap Ext(Q) \neq \emptyset$).

Such a request can possibly be not satisfied at the level of categories. For example, presumably an agent does not have an intersection between the extensions of the categories *toy* and *horse* ($Ext(toy) \cap Ext(horse) = \emptyset$). Notwithstanding, *toy-horse* can be considered a commonsense concept, indicating a puppet shaped as a horse; hence, a combination as *horse * toy* can be conceived only at a contextual level, in every context \mathcal{C} such that the concepts *toy* and *horse* are compatible ($Ext(toy) \cap Ext(horse) \neq \emptyset$). So, the combination of concepts represents, in general, a contextual procedure, that we can apply only in those contexts in which the concepts we are focusing on are compatible.

Assumed that in a context \mathcal{C} two concepts P and Q are compatible, i.e. $Ext^{\mathcal{C}}(P) \cap Ext^{\mathcal{C}}(Q) \neq \emptyset$, we have to define how to combine such concepts in the composed concept $P * Q$. Regarding the extension, simply state that $Ext^{\mathcal{C}}(P * Q) = Ext^{\mathcal{C}}(P) \cap Ext^{\mathcal{C}}(Q)$. The problem is in the definition of the typicality order $\preceq_{P*Q}^{\mathcal{C}}$. First of all, the sets of the essential and typical properties of $P * Q$ are obtained simply unifying the respective sets of P and Q , that is, $\Delta_{P*Q}^{\mathcal{C}} = \Delta_P^{\mathcal{C}} \cup \Delta_Q^{\mathcal{C}}$ and $\Gamma_{P*Q}^{\mathcal{C}} = \Gamma_P^{\mathcal{C}} \cup \Gamma_Q^{\mathcal{C}}$.

If P is an elementary concept in the context \mathcal{C} , just impose, trivially, that $\Delta_P^{\mathcal{C}} = \Gamma_P^{\mathcal{C}} = \{P\}$. Given that $a \in Ext^{\mathcal{C}}(P * Q)$ if and only if $a \in Ext^{\mathcal{C}}(R)$ for every $R \in \Gamma_{P*Q}^{\mathcal{C}}$, it is immediate to see that, imposing $\Gamma_{P*Q}^{\mathcal{C}} = \Gamma_P^{\mathcal{C}} \cup \Gamma_Q^{\mathcal{C}}$, we obtain exactly $Ext^{\mathcal{C}}(P * Q) = Ext^{\mathcal{C}}(P) \cap Ext^{\mathcal{C}}(Q)$.

We have to define the salience order $\geq_{P*Q}^{\mathcal{C}}$ over $\Delta_{P*Q}^{\mathcal{C}}$. In defining such an order, we want the properties connected to concept Q to be in general more relevant than the properties

connected to the concept P , since Q is the principal concept, but we want also the properties in $\Gamma_{P*Q}^{\mathcal{C}}$ to be more prominent than the properties in the set $\Delta_{P*Q}^{\mathcal{C}} - \Gamma_{P*Q}^{\mathcal{C}}$. Hence we define the salience order $\geq_{P*Q}^{\mathcal{C}}$ in the following way.

Definition 8 (Salience relation of a composed concept.). *$R \geq_{P*Q}^{\mathcal{C}} S$ if and only if one of the following holds:*

- $R \in \Gamma_{P*Q}^{\mathcal{C}}$ and $S \in \Gamma_{P*Q}^{\mathcal{C}}$
- $R \in \Gamma_{P*Q}^{\mathcal{C}}$ and $S \in \Delta_{P*Q}^{\mathcal{C}} - \Gamma_{P*Q}^{\mathcal{C}}$
- $R \in \Delta_Q^{\mathcal{C}} - \Gamma_{P*Q}^{\mathcal{C}}$, $S \in \Delta_Q^{\mathcal{C}} - \Gamma_{P*Q}^{\mathcal{C}}$ and $R \geq_Q^{\mathcal{C}} S$
- $R \in \Delta_Q^{\mathcal{C}} - \Gamma_{P*Q}^{\mathcal{C}}$ and $S \in \Delta_P^{\mathcal{C}} - (\Delta_Q^{\mathcal{C}} \cup \Gamma_{P*Q}^{\mathcal{C}})$
- $R \in \Delta_P^{\mathcal{C}} - (\Delta_Q^{\mathcal{C}} \cup \Gamma_{P*Q}^{\mathcal{C}})$, $S \in \Delta_P^{\mathcal{C}} - (\Delta_Q^{\mathcal{C}} \cup \Gamma_{P*Q}^{\mathcal{C}})$ and $R \geq_P^{\mathcal{C}} S$

It is easy to show that $\geq_{P*Q}^{\mathcal{C}}$ is a modular order. Being the proof easy, we have omitted it here for space reasons.

Proposition 8. *$\geq_{P*Q}^{\mathcal{C}}$ is a modular order.*

Defined the concept $P * Q$ by means of the pair $\langle \Delta_{P*Q}^{\mathcal{C}}, \geq_{P*Q}^{\mathcal{C}} \rangle$, we can define a typicality order $\preceq_{P*Q}^{\mathcal{C}}$ in the usual, lexicographic way.

Example 9. Assume that the categories *toy* and *horse* are defined by the sets of properties

$$\Gamma_{toy} = \{inanimate\ object, \ can\ be\ handled\ by\ kids, \ kids\ enjoy\ playing\ with\ it\}$$

$$\Gamma_{horse} = \{animal, \ equine, \ has\ four\ legs, \ has\ hoofs, \ has\ mane, \ has\ elongated\ snout, \ taller\ than\ a\ person\}$$

In the default model, there is no item that can fall under both of the concepts. The situation could change if we consider contextual characterizations. For example, assume we are in a toy-shop. Our concept *toy* is associated to its default essential properties, which represent also the contextual typical properties ($\Delta_{toy}^{\mathcal{C}} = \Gamma_{toy}^{\mathcal{C}} = \Gamma_{toy}$). Instead, the context allows us to have just the following properties for *horse*:

$$\Delta_{horse}^{\mathcal{C}} = \Gamma_{horse}^{\mathcal{C}} = \{has\ four\ legs, \ has\ hoofs, \ has\ mane, \ has\ elongated\ snout\}$$

The domain is represented by the set $\mathcal{D}^{\mathcal{C}} = \{h, g, t\}$, where h is a piece of plastic shaped as a horse, g is a piece of plastic shaped as a giraffe, and t is a piece of plastic shaped as a truck. We have $\Gamma_{horse*toy}^{\mathcal{C}} = \Gamma_{toy} \cup \Gamma_{horse}$ and $\Delta_{horse*toy}^{\mathcal{C}} = \Gamma_{horse*toy}^{\mathcal{C}}$. The typicality orders associated to each property are:

$$\begin{array}{c} \left| \begin{array}{c} \equiv_{inanimate} \\ h \\ g \\ t \end{array} \right| ; \left| \begin{array}{c} \equiv_{handled} \\ h \\ g \\ t \end{array} \right| ; \left| \begin{array}{c} \equiv_{enjoy} \\ h \\ g \\ t \end{array} \right| ; \\ \left| \begin{array}{c} h \\ g \end{array} \right| \prec_{fourlegs}^{\mathcal{C}} t ; \\ \left| \begin{array}{c} h \\ g \end{array} \right| \prec_{hoofs}^{\mathcal{C}} t ; \quad h \prec_{mane}^{\mathcal{C}} g \prec_{mane}^{\mathcal{C}} t ; \quad \left| \begin{array}{c} h \\ g \end{array} \right| \prec_{snout}^{\mathcal{C}} t \end{array}$$

The resulting typicality relation is:

$$h \prec_{horse*toy}^{\mathcal{C}} g \prec_{horse*toy}^{\mathcal{C}} t$$

Conclusions and further work.

Concerning the logical characterization of contexts, we are aware of two main kinds of approaches: on one hand the works of Giunchiglia and al. (Ghidini and Giunchiglia 2001), or the work of Attardi and Simi (Attardi and Simi 1995), and on the other hand works as McCarthy's (McCarthy 1993). The former approaches describe a notion of context that is quite different from ours, since their 'contexts' refer to the different 'points of view' of the agents, *i.e.* the different pieces of information the agent have access to in a given situation (a context is usually a partial description of the world), with a particular attention towards the interaction of different points of view, focusing particularly on multi-agent systems, but with an underlying monotonic form of reasoning. On the other hand our model deals with the way an agent 'modulates' its own conceptual organization w.r.t. the particular situation at hand. McCarthy's work is more general, addressing foundational issues, surely to be taken under consideration. However, we are not aware of any work proposing a formalization of context in order to deal with the notion of vagueness.

The present model could work well as a semantical base to formalize more than one aspect of uncertain reasoning:

- Fuzzy reasoning. We could point to the formalization of a qualitative characterization of fuzzy reasoning. Moreover, it is known that context plays a big role in the vagueness that often our concepts show, but by now it has not been considered.
- Non-monotonic preferential reasoning. The semantical tool we have used (preferential orders) are fundamental in the non-monotonic reasoning field. The present model has been created with a non-monotonic consequence relation in mind.
- Similarity-based reasoning. An option to be considered, since we can implement similarity measures in our model.

Moreover, the notion of context, besides being theoretically interesting per se, could be attractive from the point of view of multi-agent systems and goal-oriented reasoning.

Obviously, this is just a first step, since most of the work still needs to be done. We need to define the behaviour of classical propositional connectives (\neg , \wedge , \vee); regarding such a point, we agree with Freund [(Freund 2008), p.5] that the definition of the behaviour of the classical operators should be secondary to the definition of a solid framework characterizing the behaviour of concepts, since, from a cognitive point of view, it is not obvious that the negation of a concept or the disjunction and conjunction of concepts should be again considered concepts. Most important, we are working on the definition of a consequence relation (and, possibly, on its characterization also from the inferential point of view), in order to implement the forms of reasoning addressed above (Freund has proposed in (Freund 2009) a first attempt in this direction w.r.t. his own semantic models).

The challenge is interesting from more than one point of view, since the potential implementation of more than one form of uncertain reasoning (primarily, (qualitative) fuzzy and non-monotonic) poses not only technical problems, but

also theoretical ones about their distinction (already widely investigated) and their possible integration.

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